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THE ORTHOGONAL TITS QUADRANGLES

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This paper is dedicated to the memory of Vaughan Jones

Abstract. We show that every 4-plump razor-sharp normal Tits quadrangle X is uniquely determined by a non-degenerate quadratic space whose Witt index m is at least 2. If this Witt index is finite, then X is the Tits quadrangle arising from the corresponding building of type B_m or D_m by a standard construction.

1. Introduction

A generalized polygon is the same thing as a spherical building of rank 2. Tits observed that the spherical buildings of rank 2 that arise from absolutely simple algebraic groups all satisfy a property he called the Moufang condition. The classification of Moufang polygons was given in [**9**]. It says that all Moufang polygons (and indeed, by [**8**], all irreducible spherical buildings of rank at least 3) arise as the fixed point geometry of a Galois group acting on the spherical building associated with a split simple algebraic group (or by certain variations on this theme involving algebraic structures of infinite dimension).

The notion of a Tits polygon was introduced in [**2**]. A Tits polygon is a bipartite graph Γ in which for each vertex v, the set Γ_v of vertices adjacent to v is endowed with a symmetric relation we call "opposite at v" satisfying certain axioms. A Moufang polygon is the same thing as a Tits polygon all of whose local opposition relations are trivial.

Let $\mathcal{P}$ denote the set of pairs (Δ, T), where Δ is a spherical building of type M satisfying the Moufang condition and T is a Tits index of absolute type M and relative rank 2. Every pair (Δ, T) in $\mathcal{P}$ gives rise by a simple construction to a Tits polygon X and a natural action of the stabilizer of T in $\mathrm{Aut}(\Delta)$ on X. We call the Tits polygons that arise in this way the Tits polygons of index type. Moufang polygons are all Tits polygons of index type; this is the case that not just the relative rank but also the absolute rank of T is 2.

For every irreducible spherical building Δ of rank at least 2, there exist Tits indices T such that $(\Delta, T) \in \mathcal{P}$. Thus the theory of Tits polygons allows us to regard a spherical building of arbitrary rank at least 2 as a rank 2 structure to which the methods developed in [**9**] can be applied.

With a few exceptions, Tits polygons of index type satisfy a condition we call dagger-sharp. This is a natural condition on the action of the stabilizer of an apartment on the corresponding root groups. It is trivially satisfied by all Moufang polygons. Tits n-gons exist for every value of n (as was observed in [**2**, 1.2.33]), but by [**2**, 1.6.14], dagger-sharp Tits n-gons exist only for $n = 3$, 4, 6 and 8. In other

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words, the only dagger-sharp Tits polygons that can exist are Tits triangles, Tits quadrangles, Tits hexagons and Tits octagons.

In [**4**, 5.11 and 5.12], we showed that all dagger-sharp Tits triangles are of index type (or a variation defined over a simple associative ring that is infinite dimensional over its center), in [**3**, 7.7], we showed that all dagger-sharp Tits hexagons are of index type and in [**6**], we showed that all dagger-sharp Tits octagons are, in fact, Moufang octagons. This leaves only the case $n = 4$. As was the case with Moufang polygons, Tits polygons exist in the greatest variety and their classification presents the greatest difficulties in this case.

Much as in the classification of Moufang quadrangles, dagger-sharp Tits quadrangles are either indifferent, reduced or wide (as defined in 4.3) and every reduced dagger-sharp Tits quadrangle is either normal (as defined in 4.5) or not. In [**6**], we showed that all dagger-sharp indifferent quadrangles are Moufang quadrangles. The wide dagger-sharp Tits quadrangles can be studied inductively as extensions (in an appropriate sense) of a reduced dagger-sharp Tits quadrangle. In [**5**], we showed that the wide dagger-sharp Tits quadrangles that are extensions of orthogonal Tits quadrangles (subject to certain restriction which we would like to eliminate) are precisely the Tits quadrangles of index type associated with exceptional algebraic groups of type E_6, E_7, E_8 and F_4.

In this paper, we study normal Tits quadrangles. This case presented difficulties which we were able to overcome only by replacing the dagger-sharp condition with a mildly stronger, but at least equally natural, condition (defined in 2.21) that we call razor-sharp. (We do not, however, know any examples of dagger-sharp normal Tits quadrangles that are not, in fact, razor-sharp.)

The main result of this paper is the classification of the normal razor-sharp Tits quadrangles. We show that these Tits quadrangles are all uniquely determined by a non-degenerate quadratic space (K, L, q) over a field K. If the Witt index of q is finite, these Tits quadrangles are of index type related to an orthogonal group.

We conjecture that all razor-sharp Tits quadrangles are of index type (or a variation involving algebraic structures of infinite rank). It remains only to consider Tits quadrangles that are reduced but not normal and to complete the case of wide Tits quadrangles. Since razor-sharp implies dagger-sharp, a proof of this conjecture would be the last step in a classification of all razor-sharp Tits polygons.

Let k be an integer at least 3. We say that a Tits polygon is k-plump if for each vertex v, the valency $|\Gamma_v|$ of v is not too small in an appropriate sense. All Tits polygons of index type corresponding to a pair (Δ, T) in $\mathcal{P}$ are k-plump if the field of definition of Δ contains at least k elements (by [**2**, 1.2.7]). It should be mentioned that in all the classification results mentioned above, it is assumed that all the Tits n-gons under consideration are 5-plump (see 5.93).

This paper is organized as follows. In Section 2, we give the definition of a Tits polygon and assemble all the results and definitions from [**2**] that we require. In Section 3, we review the basic properties of the orthogonal Tits quadrangles and in 3.11 we state our main result. In Section 4, we prove some properties shared by all Tits quadrangles. Finally, in Section 5, we focus on normal Tits quadrangles and give the proof of 3.11.

* * *

Conventions 1.1. Let G be a group. We denote by G^* the set $G\backslash\{1\}$ (or $G\backslash\{0\}$ if G is additive). As in **[9]**, we set $a^b = b^{-1}ab$ and

$$[a, b] = a^{-1}b^{-1}ab$$

for all $a, b \in G$. With these definitions, we have
(i) $[ab, c] = [a, c]^b \cdot [b, c]$ and
(ii) $[a, bc] = [a, c] \cdot [a, b]^c$.
for all $a, b, c \in G$.

Conventions 1.2. If i, j are indices of some variable, we denote by $[i, j]$ the interval of integers from i to j if $i \leq j$ and interpret $[i, j]$ to be the empty set if $j < i$, as in 2.7, for example.

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2. Tits Polygons

Tits polygons were introduced in **[2]**. In this section, we give the definition and assemble all the results and definitions from **[2]** that we will require.

Definition 2.1. A *dewolla* is a triple

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V}),$$

where:
(i) Γ is a bipartite graph with vertex set V and $|\Gamma_v| \geq 3$ for each $v \in V$, where Γ_v denotes the set of vertices adjacent to v.
(ii) For each $v \in V$, $\equiv_v$ is an anti-reflexive symmetric relation on Γ_v. We say that vertices $u, w \in V$ are *opposite at* v if $u, w \in \Gamma_v$ and $u \equiv_v w$. A path $(w_0, w_1, \ldots, w_m)$ in Γ is called *straight* if w_{i-1} and w_{i+1} are opposite at w_i for all $i \in [1, m-1]$.
(iii) There exists $n \geq 3$ and a non-empty set $\mathcal{A}$ of circuits of length $2n$ such that every path contained in a circuit in $\mathcal{A}$ is straight.

The parameter n is called the *level* of X. The automorphism group $\text{Aut}(X)$ is the subgroup of $\text{Aut}(\Gamma)$ consisting of all the elements of $\text{Aut}(\Gamma)$ that preserve both the set $\mathcal{A}$ and the set of all straight paths in Γ. A *root* of the dewolla X is a straight path of length n in Γ.

Definition 2.2. A *Tits n-gon* is a dewolla

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$$

of level n for some $n \geq 3$ such that Γ is connected and the following axioms hold:
(i) For all $v \in V$ and all $u, w \in \Gamma_v$, there exists $z \in \Gamma_v$ that is opposite both u and w at v.
(ii) For each straight path $\delta = (w_0, \ldots, w_k)$ of length $k \leq n-1$, δ is the unique straight path of length at most k from w_0 to w_k.
(iii) For each root $\alpha = (w_0, \ldots, w_n)$ of X, the group U_α acts transitively on the set of vertices opposite w_{n-1} at w_n, where U_α is the pointwise stabilizer of

$$\Gamma_{w_1} \cup \Gamma_{w_2} \cup \cdots \cup \Gamma_{w_{n-1}}$$

in $\mathrm{Aut}(X)$. The group U_α is called the *root group* associated with the root α. A *Tits polygon* is a Tits n-gon for some $n \geq 3$. A Tits n-gon is called a *Tits triangle* if $n = 3$, a *Tits quadrangle* if $n = 4$, etc.

By [**2**, 1.3.12], $\mathcal{A}$ is the set of all circuits in Γ of length at most $2n$ containing only straight paths. Thus, $2n$ is, roughly speaking, the "straight girth" of Γ.

Notation 2.3. We will say that a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ is *Moufang* if all the relations $\equiv_v$ are trivial, i.e. if all paths in Γ are straight. If X is Moufang, then by [**2**, 1.2.3], Γ is a Moufang n-gon and $\mathcal{A}$ is the set of its apartments. Conversely, if Γ is a Moufang n-gon, $\mathcal{A}$ is the set of its apartments and $\equiv_v$ is the trivial relation on Γ_v for every v in the vertex set V, then by [**2**, 1.2.2], $(\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ is a Tits n-gon.

Notation 2.4. Let $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ be a Tits n-gon for some $n \geq 3$. A *coordinate system* for X is a pair $(\gamma, i \mapsto w_i)$ where γ is an element of $\mathcal{A}$ and $i \mapsto w_i$ is a surjection from $\mathbb{Z}$ to the vertex set of γ such that for each i, the image of the sequence $(i, i+1, \ldots, i+n)$ is a root of X. For each coordinate system $(\gamma, i \mapsto w_i)$, we denote by U_i the root group associated with the root $(w_i, w_{i+1}, \ldots, w_{i+n})$ for each i and call the map $i \mapsto U_i$ the *root group labeling* associated with $(\gamma, i \mapsto w_i)$. Note that $w_i = w_j$ and $U_i = U_j$ whenever i and j have the same image in $\mathbb{Z}_{2n}$. From now on, we fix a Tits n-gon $X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v\in V})$ and a coordinate system $(\gamma, i \mapsto w_i)$ of X, we let $i \mapsto U_i$ be the corresponding root group labeling and we let $G = \mathrm{Aut}(X)$.

Proposition 2.5. *Let*

$$U_{[k,m]} = \begin{cases} U_kU_{k+1}\cdots U_m & \text{if } k \leq m \text{ and} \\ 1 & \text{otherwise.} \end{cases}$$

Then the following hold:

(i) $[U_i, U_j] \subset U_{[i+1,j-1]}$ *for all* i, j *such that* $i < j < i+n$. *In particular,* $[U_i, U_{i+1}] = 1$ *for all* i.

(ii) *The product map* $U_1 \times U_2 \times \cdots \times U_n \to U_{[1,n]}$ *is bijective.*

All these assertions hold if all the subscripts are shifted by an arbitrary constant.

Proof. This holds by [**2**, 1.3.36(ii) and (iii)]. □

Notation 2.6. By 2.5(i), $U_{[i,j]}$ is a subgroup of G for all i, j such that $1 \leq i \leq j \leq n$. Notice that all of these subgroups fix the adjacent vertices w_n and w_{n+1}. We call the $(n+1)$-tuple

$$(U_{[1,n]}, U_1, \ldots, U_n)$$

a *root group sequence* of X.

Notation 2.7. Suppose that $i < j < i+n$ and that $[a_i, a_j] = a_{i+1}a_{i+2}\cdots a_{j-1}$ with $a_k \in U_k$ for all $k \in [i, j]$. It follows from 2.5(ii) that for each $k \in [i+1, j-1]$, a_k is uniquely determined by $[a_i, a_j]$. We denote this element a_k by $[a_i, a_j]_k$.

Notation 2.8. Let

$$U_i^\sharp = \{a \in U_i \mid w_{i+n+1}^a \text{ is opposite } w_{i+n+1} \text{ at } w_{i+n}\}$$

for each i. By [**2**, 1.4.3], we have $U_i^\sharp \neq \emptyset$ and by [**2**, 1.4.8], we have

$$U_i^\sharp = \{a \in U_i \mid w_{i-1}^a \text{ is opposite } w_{i-1} \text{ at } w_i\}$$

for each i.

Proposition 2.9. *For each $i \in \mathbb{Z}$, there exist unique maps κ_γ and λ_γ from $U_i^\sharp$ to $U_{i+n}^\sharp$ such that for each $a \in U_i^\sharp$, the product*

$$\mu_\gamma(a) := \kappa_\gamma(a) \cdot a \cdot \lambda_\gamma(a) \tag{2.10}$$

interchanges the vertices w_{i+n-1} and w_{i+n+1}. For each $a \in U_i^\sharp$, the element $\mu_\gamma(a)$ fixes the vertices w_i and w_{i+n} and interchanges the vertices w_{i+j} and w_{i-j} for all $j \in \mathbb{Z}$ and

$$U_k^{\mu_\gamma(a)} = U_{2i+n-k} \tag{2.11}$$

for all $k \in \mathbb{Z}$. In particular, $U_k^{\mu_\gamma(a)} = U_{n+2-k}$ for all k if $i = 1$ and $U_k^{\mu_\gamma(a)} = U_{n-k}$ for all k if $i = n$.

Proof. This holds by [**2**, 1.4.4 and 1.4.8]. □

Proposition 2.12. *Let $a \in U_i^\sharp$ for some i. Then the following hold:*

(i) *$a^{-1} \in U_i^\sharp$ and $\mu_\gamma(a^{-1}) = \mu_\gamma(a)^{-1}$.*
(ii) *$\kappa_\gamma(a^{-1}) = \lambda_\gamma(a)^{-1}$ and $\lambda_\gamma(a^{-1}) = \kappa_\gamma(a)^{-1}$.*
(iii) *$\mu_\gamma(a^g) = \mu_\gamma(a)^g$ for all g mapping γ to itself.*

Proof. This holds by [**2**, 1.4.3 and 1.4.13]. □

Proposition 2.13. *Let H denote the pointwise stabilizer of γ in* $\mathrm{Aut}(X)$. *Then $C_H(\langle U_i, U_{i+1}\rangle) = C_H(\langle U_{i+1}, U_{i+n}\rangle) = 1$ for all i.*

Proof. This holds by [**2**, 1.4.19(ii)]. □

Proposition 2.14. *Suppose that $[a_1, a_n^{-1}] = a_2 \cdots a_{n-1}$ with $a_i \in U_i$ for each $i \in [1, n]$. Then the following hold:*

(i) *$a_2 = a_n^{\mu_\gamma(a_1)}$ if $a_1 \in U_1^\sharp$ and $a_1 = a_{n-1}^{\mu_\gamma(a_n)}$ if $a_n \in U_n^\sharp$.*
(ii) *$[a_2, \lambda_\gamma(a_1)^{-1}] = a_3 \cdots a_{n-1}a_n$ if $a_1 \in U_1^\sharp$ and $[\kappa_\gamma(a_n), a_{n-1}^{-1}] = a_1a_2 \cdots a_{n-2}$ if $a_n \in U_n^\sharp$.*

All these assertions hold if all the subscripts are shifted by an arbitrary constant.

Proof. This holds by [**2**, 1.4.16]. □

Proposition 2.15. *If $a \in U_1$ and $U_n^{ab} = U_2$ for some $b \in U_{n+1}$, then $a \in U_1^\sharp$ and $b = \lambda_\gamma(a)$.*

Proof. This holds by [**2**, 1.4.27(i)]. □

Definition 2.16. Let $k \geq 3$. As in [**2**, 1.4.21], we call X *k-plump* if for all $v \in V$, and for every subset Ω of Γ_v of cardinality at most k, there exists a vertex that is opposite u at v for all $u \in \Omega$. Thus k-plump implies $(k-1)$-plump, and "2-plump" is simply 2.2(i).

Proposition 2.17. *If X is* 3*-plump, then for all i, every element of U_i is the product of at most two elements of $U_i^\sharp$.*

Proof. This holds by [**2**, 1.4.23]. □

Proposition 2.18. *Let E be the edge set of* Γ. *Then* $\mathrm{Aut}(X)$ *acts transitively on the set* $\{(\delta, e) \in \mathcal{A} \times E \mid \ e \subset \delta\}$.

Proof. This holds by [**2**, 1.3.13]. □

Notation 2.19. Let $G^\dagger$ denote the subgroup of $\mathrm{Aut}(X)$ generated by all the root groups of X, let H be as in 2.13 and let $H^\dagger = H \cap G^\dagger$.

Proposition 2.20. *Let* $H_i = \langle mm' \mid m, m' \in \mu_\gamma(U_i^\sharp)\rangle$ *for all i and let $H^\dagger$ be as in* 2.19. *Then H_1 and H_n normalize each other and if X is* 4*-plump, then* $H^\dagger = H_1H_n$.

Proof. The first claim holds by 2.12(iii) and the second claim by [**2**, 1.5.28]. □

Definition 2.21. Let H, $H^\dagger$, H_1 and H_n be as in 2.19 and 2.20. The subgroup H normalizes U_i for each i. We say that X is *sharp* if for each i, every nontrivial H-invariant normal subgroup of U_i contains elements of $U_i^\sharp$, where $U_i^\sharp$ is as in 2.8. We say that X is *dagger-sharp* if for each i, every nontrivial $H^\dagger$-invariant normal subgroup of U_i contains elements of $U_i^\sharp$. Finally, we say that X is *razor-sharp* if for each i, every nontrivial H_i-invariant normal subgroup of U_i contains an element of $U_i^\sharp$. Note that razor-sharp implies dagger-sharp implies sharp.

Remark 2.22. It follows from 2.12(iii) and 2.18 that the definitions in 2.21 do not depend on the choice of the coordinate system $(\gamma, i \mapsto w_i)$ in 2.4.

Remark 2.23. Let H_i for each i be as in 2.20. We say that X is *razor-sharp at* U_i for some i if every non-trivial H_i-invariant normal subgroup of U_i contains an element of $U_i^\sharp$. Let i and j be two integers and if n is even, suppose that i and j have the same parity. It follows from 2.9 that there is an element g of G stabilizing the apartment γ such that $U_i^g = U_j$. By 2.12(iii), we also have $H_i^g = H_j$. Thus X is razor-sharp at U_i if and only if it is razor-sharp at U_j. It follows that X is razor-sharp if and only if it is razor-sharp at U_1 and at U_n.

Definition 2.24. Two vertices of Γ are called *opposite* if there is a root (as defined in 2.1) that starts at the one and ends at the other.

Proposition 2.25. *Suppose x and y are opposite vertices as defined in* 2.24 *and that z is an arbitrary vertex adjacent to y. Then there exists a unique root from x to z that passes through y.*

Proof. This holds by [**2**, 1.3.16 and 1.3.18]. □

3. Orthogonal Tits Quadrangles

We introduce orthogonal Tits quadrangles in 3.2 and formulate our main result in 3.11.

Notation 3.1. We will denote by *quadratic space* a triple (K, L, q) where K is a field, L is a vector space over K and q is a quadratic form on L.

Notation 3.2. Let $\Lambda = (K, L, q)$ be a quadratic space and let f be the bilinear form associated with q. We assume that Λ is *non-degenerate*, i.e. that if $q(v) = 0$ and $f(v, L) = 0$ for some $v \in L$, then $v = 0$. A Tits quadrangle X is *orthogonal of* *type* Λ if for some coordinate system $(\gamma, i \mapsto w_i)$ of X with associated root group sequence

$$(U_+, U_1, \ldots, U_4)$$

as defined in 2.6, there exist isomorphisms x_i from the additive group of K to U_i for $i = 1$ and 3 and x_j from the additive group of L to U_j for $j = 2$ and 4 such that

$$\begin{aligned} [x_2(w), x_4(u)^{-1}] &= x_3(f(w,u)) \text{ and} \\ [x_1(t), x_4(u)^{-1}] &= x_2(tu)x_3(q(u)t) \end{aligned} \tag{3.3}$$

for all $u, w \in L$ and all $t \in K$ and $[U_1, U_3] = 1$.

Remark 3.4. Suppose that $\Lambda = (K, L, q)$ is a non-degenerate quadratic space of Witt index m. Let $V = K^4 \oplus L$, let Q be the quadratic form on V given by

$$Q(t_1, t_2, t_3, t_4, v) = t_1t_2 + t_3t_4 + q(v)$$

for all $(t_1, t_2, t_3, t_4, v) \in V$. Assume now that m is finite and let Δ be the spherical building associated with the quadratic space (K, V, Q). Then Δ is a building of type X_{m+2}, where $X = D$ if q is hyperbolic and $X = B$ if it is not. Let Π be the Tits index of absolute type X_{m+2} and relative rank 2 in which the first two nodes are circled. Finally, we let X_Λ denote the Tits quadrangle obtained by applying [**2**, 1.2.12 and 1.2.28] to the pair (Δ, Π). Then by [**5**, 6.3], X_Λ is an orthogonal Tits quadrangle of type Λ.

Remark 3.5. There exists, in fact, a Tits quadrangle of type Λ for every non-degenerate quadratic space Λ even if its Witt index is not finite. This is shown in [**7**] by considering the thick, non-degenerate polar space (as defined in [**1**, 7.4.1]) associated with the quadratic form Q defined in 3.4.

Remark 3.6. Note that it is allowed in 3.4 that the bilinear form f belonging to q is identically zero. In this case, $[U_2, U_4] = 1$ by (3.3), so X_Λ is indifferent as defined in 4.3, and q is anisotropic (since Λ is non-degenerate). Thus $U_i^* = U_i^\sharp$ for all i by [**5**, 6.4(i)–(ii)] and therefore X_Λ is a Moufang quadrangle by [**2**, 1.4.15] in this case.

Proposition 3.7. *Let X_Λ and $\Lambda = (K, L, q)$ be as in* 3.4. *Then X_Λ is* 4*-plump if and only if* $|K| \geq 4$.

Proof. This holds by [**5**, 3.4]. □

Proposition 3.8. *For every non-degenerate quadratic space $\Lambda = (K, L, q)$ with $|K| \geq 4$, there is at most one Tits quadrangle of type Λ up to isomorphism.*

Proof. This holds by [**5**, 6.10]. (As we observe in 5.93 below, the hypothesis $|K| > 4$ in [**5**, 6.10] can be replaced by $|K| \geq 4$. Note, too, that in the penultimate line of the proof of [**5**, 6.10], 2.7 should be replaced by 2.20.) □

Proposition 3.9. *Orthogonal Tits quadrangles of type Λ and Λ' are isomorphic if and only if the quadratic spaces Λ and Λ' are similar.*

Proof. This holds by [**5**, 6.8]. □

Proposition 3.10. *Let $\Lambda = (K, L, q)$ and X_Λ be as in* 3.4. *Then X_Λ is is razor-sharp unless Λ is a hyperbolic plane.*

Proof. For each $w \in L$ such that $q(w) \neq 0$, we denote by π_w the reflection $u \mapsto u - f(u,w)q(w)^{-1}w$, where f is as in 3.2. Let $e \in L$ and suppose that $e \neq 0$ but $q(e) = 0$. Since Λ is non-degenerate, there exists $e' \in L$ such that $f(e,e') \neq 0$. Thus $B := \langle e, e' \rangle$ is a hyperbolic plane. Thus there exists $d \in B$ such that $q(d) = 0$ and $f(d,e) = 1$ (and hence $B = \langle e, d \rangle$). Let $a = d + e$. Then $q(a) = 1$ and the reflection π_a interchanges $\langle d \rangle$ and $\langle e \rangle$. Suppose that $L \neq B$. Then we can choose $b \in B^\perp$ such that $q(b) \neq 0$. The reflection π_b acts trivially on B and hence the product $\pi_b\pi_a$ maps $\langle e \rangle$ to $\langle d \rangle$. It follows that there is no non-trivial additive subgroup of L that is invariant under the group

$$\langle \pi_u\pi_v \mid u, v \in L,\ q(u) \neq 0, q(v) \neq 0\}$$

on which q vanishes identically.

Let U_1, U_4 and x_4 be as in 3.2 and let H_4 be as in 2.20. By [**5**, 6.4(ii)], $x_4(a) \in U_4^\sharp$ for $a \in L$ if and only if $q(a) \neq 0$. Hence by [**5**, 6.4(vi)(c)] and the conclusion of the previous paragraph, every H_4-invariant subgroup of U_4 contains elements of $U_4^\sharp$. By [**5**, 6.4(i)], $U_1^\sharp = U_1^*$ (see 1.1). By 2.23, therefore, X_Λ is razor-sharp. □

The following is our main result.

Theorem 3.11. *Let X be a Tits quadrangle and suppose that X is* 3*-plump as defined in* 2.16*, that X is normal as defined in* 4.5 *below and that X is razor-sharp as defined in* 2.21*. Then there exists a non-degenerate quadratic space $\Lambda = (K, L, q)$ such that X is orthogonal of type Λ.*

Remark 3.12. If two Tits quadrangles are normal and 3-plump, then their product (in a suitable sense) is a Tits quadrangle that is normal and 3-plump but not razor-sharp. More interesting examples of 3-plump normal Tits quadrangles that are not razor-sharp can be constructed from suitable quadratic spaces over commutative rings; see [**3**] and [**10**].

4. Arbitrary Tits Quadrangles

Before focusing on the proof of 3.11, we assemble some results about arbitrary Tits quadrangles derived from [**9**, Chapter 21].

Let

$$X = (\Gamma, \mathcal{A}, \{\equiv_v\}_{v \in V})$$

be an arbitrary Tits quadrangle, where V is the vertex set of Γ, let $(\gamma, i \mapsto w_i)$ be a coordinate system of X and let $i \mapsto U_i$ be the corresponding root group labeling as defined in 2.4.

Definition 4.1. Let $V_i = [U_{i-1}, U_{i+1}]$ and let $Y_i = C_{U_i}(U_{i-2})$ for all i. Note that by 2.5(i), $V_i \subset U_i$.

Remark 4.2. By [**5**, 4.9], $Y_i = C_{U_i}(U_{i+2})$ and thus $[Y_i, U_{i-2}] = [Y_i, U_{i+2}] = 1$ for all i and the definition of Y_i in 4.1 is equivalent to the definition of Y_i given in [**5**, 4.6].

Definition 4.3. We call X *indifferent* if $U_i = Y_i$ for all i, *reduced* if $U_i = Y_i$ for some but not all i and *wide* if $U_i \neq Y_i$ for all i.

Remark 4.4. By [**5**, 5.3], $\mu_\gamma(a_i) \in Y_{i+4}a_iY_{i+4}$ and hence

$$[\mu_\gamma(a_i), U_{i-2}] = [\mu_\gamma(a_i), U_{i+2}] = 1$$

for all i and all $a_i \in Y_i \cap U_i^\sharp$.

It follows from (2.11) that if $U_i = Y_i$ for some i, then $U_j = Y_j$ for all j of the same parity as i. Note, too, that if $(\gamma, i \mapsto w_i)$ is replaced by its opposite and $i \mapsto U_i$ by the root group labeling corresponding to this new coordinate system, then U_i is replaced by U_{n+1-i}, Y_i by Y_{n+1-i} and V_i by V_{n+1-i} for all i. Thus if X is reduced, we can assume that $U_i = Y_i$ for all odd i.

Definition 4.5. The Tits quadrangle X is *normal* if $U_i = Y_i$ for all odd i (so X is reduced or indifferent) and the group H_1 defined in 2.20 normalizes the set $[U_1, a_4^{-1}]_2$ for all $a_4 \in U_4^\sharp$.

Remark 4.6. It follows from 2.18 that the definitions in 4.3 and 4.5 do not depend on the choice of the coordinate system $(\gamma, i \mapsto w_i)$.

Theorem 4.7. *Suppose that X is sharp as defined in* 2.21 *and let V_i and Y_i for all i be as in* 4.1*. If X is indifferent or reduced, then U_i is abelian for all i. If X is wide, then either $1 \neq V_i \subset Y_i$ and U_i is abelian for all i or, after replacing $(\gamma, i \mapsto w_i)$ by its opposite if necessary, the following hold:*
(i) *$1 \neq [U_i, U_i] \subset V_i \subset Y_i \subset Z(U_i)$ for all i odd and*
(ii) *$Y_i = 1$ and U_i is abelian but $V_i \neq 1$ for all i even.*

Proof. This holds by [**5**, 4.8]. □

For the rest of this section, we assume X is sharp and that $Y_i \neq 1$ for all odd i and let

$$Y_i^\sharp = Y_i \cap U_i^\sharp. \tag{4.8}$$

Since X is sharp, the set $Y_i^\sharp$ is non-empty.

Proposition 4.9. *Let $a_1 \in U_1^\sharp$ and $a_4 \in U_4^\sharp$. Then the map $u_1 \mapsto [u_1, a_4^{-1}]_3$ is an isomorphism from U_1 to U_3 and the map $u_4 \mapsto [a_1, u_4^{-1}]_2$ is an isomorphism from U_4 to U_2. These assertions remain valid if all the indices are shifted by a constant.*

Proof. By 2.14(i), $[u_1, a_4^{-1}]_3^{\mu_\gamma(a_4)} = u_1$ and $[a_1, u_4^{-1}]_2 = u_4^{\mu_\gamma(a_1)}$ for all $u_1 \in U_1$ and all $u_4 \in U_4$. The claim follows. □

Proposition 4.10. *Let $a_1 \in U_1$ and $a_4 \in U_4$. Then the map $u_1 \mapsto [u_1, a_4]_2$ is an homomorphism from Y_1 to U_2 and the map $u_4 \mapsto [a_1, u_4]_2$ is an homomorphism from U_4 to U_2.*

Proof. This holds by 1.1. □

Proposition 4.11. $N_{U_1}(U_{[3,4]}) = 1$ *and* $N_{U_4}(U_{[1,2]}) = 1$.

Proof. By 2.14(i), $N_{U_1}(U_{[3,4]}) \cap U_1^\sharp = \emptyset$ and $N_{U_4}(U_{[1,2]}) \cap U_4^\sharp = \emptyset$. The claims hold, therefore, since X is sharp and the subgroup H defined in 2.13 normalizes $N_{U_1}(U_{[3,4]})$ and $N_{U_4}(U_{[1,2]})$. □

Proposition 4.12. *Let $h = \mu_\gamma(a_1)^2$ for some $a_1 \in Y_1^\sharp$, where $Y_i^\sharp$ is as in* (4.8)*. Then:*

(i) $a_i^h = a_i^{-1}$ *for* $i = 2$ *and* 4 *and for all* $a_i \in U_i$.
(ii) $[h, Y_1] = [h, U_3] = 1$.

Proof. Choose $a_1 \in Y_1^\sharp$ and $a_4 \in U_4$ and let $h = \mu_\gamma(a_1)^2$ and $b_4 = a_4^h$. Let $a_2 = a_4^{\mu_\gamma(a_1)}$. By (2.11), $a_2 \in U_2$ and the element h normalizes U_i and Y_i for all i. By 2.12(i), $\mu_\gamma(a_1^{-1}) = \mu_\gamma(a_1)^{-1}$. Hence

$$a_2 = a_4^{\mu_\gamma(a_1)} = b_4^{h^{-1}\mu_\gamma(a_1)} = b_4^{\mu_\gamma(a_1)^{-1}} = b_4^{\mu_\gamma(a_1^{-1})},$$

so

$$[a_1, a_4^{-1}]_2 = a_2 = [a_1^{-1}, b_4^{-1}]_2$$

by two applications of 2.14(i). By 4.10, therefore, $[a_1, b_4^{-1}]_2 = a_2^{-1}$. By 4.9, it follows that $[a_1, (a_4b_4)^{-1}]_2 = a_2a_2^{-1} = 1$ and thus $a_4b_4 = 1$. We conclude that h inverts every element of U_4. Since h commutes with $\mu_\gamma(a_1)$ and $U_4^{\mu_\gamma(a_1)} = U_2$, h inverts every element of U_2 as well. Thus (i) holds.

By 4.4, $[\mu_\gamma(a_1), U_3] = 1$. Therefore $[h, U_3] = 1$. Choose $d_1 \in Y_1$ and $d_4 \in U_4$ and let $d_2 = [d_1, d_4]_2$. By two applications of (i), we have

$$d_2^{-1} = d_2^h = [d_1^h, d_4^h]_2 = [d_1^h, d_4^{-1}]_2.$$

By two applications of 4.10, we have $[d_1^h, d_4]_2 = d_2$ and then $[d_1^{-1}d_1^h, d_4]_2 = 1$. Since d_4 is arbitrary, it follows that $d_1^{-1}d_1^h \in N_{U_1}(U_{[3,4]})$, so by 4.11, $d_1 = d_1^h$. Hence $[h, Y_1] = 1$. Thus (ii) holds. □

Proposition 4.13. $\kappa_\gamma(a_4) = \lambda_\gamma(a_4)$ *for all* $a_4 \in U_4^\sharp$.

Proof. Choose $a_4 \in U_4^\sharp$ and let $u_0 = \kappa_\gamma(a_4)$ and $v_0 = \lambda_\gamma(a_4)$. Choose $a_1 \in Y_1^\sharp$ and let $a_k = [a_1, a_4^{-1}]_k$ for $k = 2$ and 3. By 2.14(i), $a_3 \in Y_3^\sharp$ and by 2.14(ii), $[u_0, a_3^{-1}] = a_1a_2$. Conjugating $[a_1, a_4^{-1}] = a_2a_3$ by $\mu_\gamma(a_1)^2$ and applying 4.12, we obtain $[a_1, a_4] = a_2^{-1}a_3$. By 2.14(ii) again, this implies that $[\kappa_\gamma(a_4^{-1}), a_3^{-1}] = a_1a_2^{-1}$. By 2.12(ii), $\kappa_\gamma(a_4^{-1}) = \lambda_\gamma(a_4)^{-1} = v_0^{-1}$. It follows by 4.9 that $[u_0v_0^{-1}, a_3^{-1}]_2 = [u_0, a_3^{-1}]_2 \cdot [v_0^{-1}, a_3^{-1}]_2 = a_2a_2^{-1} = 1$ and hence $u_0v_0^{-1} = 1$. □

Proposition 4.14. *Let* $a_4 \in U_4^\sharp$. *Then* $[Y_1^\sharp, \mu_\gamma(a_4)^2] = 1$ *and* $a_2^{\mu_\gamma(a_4)} = a_2^{-1}$ *for every* $a_2 \in [Y_1^\sharp, a_4^{-1}]_2$.

Proof. Choose $a_1 \in Y_1^\sharp$ and let $m = \mu_\gamma(a_4)$ and $v_0 = \lambda_\gamma(a_4)$. By (4.13), $m = v_0a_4v_0$. By 2.12(ii), $\kappa_\gamma(a_4^{-1}) = v_0^{-1}$. Let $a_k = [a_1, a_4^{-1}]_k$ for $k = 2$ and 3. Conjugating by $\mu_\gamma(a_1)^2$, we obtain $[a_1, a_4] = a_2^{-1}a_3$ by (4.12). By 2.14(ii), therefore, $[v_0, a_3^{-1}] = a_1a_2$ and $[v_0^{-1}, a_3^{-1}] = a_1a_2^{-1}$. By 2.14(i), $a_3^m = a_1$. By (2.11), $a_1^m \in U_3$ and $U_2^m = U_2$. Thus

$$\begin{aligned} a_3 = a_3^{a_4} = a_3^{v_0^{-1}mv_0^{-1}} &= ([v_0^{-1}, a_3^{-1}] \cdot a_3)^{mv_0^{-1}} \\ &= (a_1a_2^{-1}a_3)^{mv_0^{-1}} = (a_1^m(a_2^{-1})^m a_1)^{v_0^{-1}} \\ &= [v_0^{-1}, (a_1^m)^{-1}] \cdot a_1^m \cdot [v_0^{-1}, a_2^m] \cdot (a_2^{-1})^m a_1 \\ &\in U_{[1,2]}a_1^m \end{aligned}$$

by 2.5(i). By 2.5(ii), therefore, $a_3 = a_1^m$. It follows that $[a_1, m^2] = 1$ and that

$$[v_0^{-1}, (a_1^m)^{-1}] = [v_0^{-1}, a_3^{-1}] = a_1a_2^{-1}.$$

Therefore

$$\begin{aligned} a_3 &= [v_0^{-1}, (a_1^m)^{-1}] \cdot a_1^m \cdot [v_0^{-1}, a_2^m] \cdot (a_2^{-1})^m a_1 \\ &= a_1 a_2^{-1} a_3 \cdot [v_0^{-1}, a_2^m] \cdot (a_2^{-1})^m a_1 \in U_1 (a_2 a_2^m)^{-1} a_3 \end{aligned}$$

since $[U_1, a_3] \subset [U_1, Y_3] = 1$ and U_2 is abelian. By 2.5(ii), we conclude that $a_2^m = a_2^{-1}$. □

Proposition 4.15. *Let $a_2 \in U_2$, $a_4 \in U_4^\sharp$ and $v_0 = \lambda_\gamma(a_4)$. Then*

$$a_2^{\mu_\gamma(a_4)} a_2^{-1} = [[v_0, a_2^{-1}], a_4]_2 \in [Y_1, a_4^{-1}]_2 = [Y_1, a_4]_2 \tag{4.16}$$

and

$$[[v_0, a_2^{-1}], a_4]_3 = [a_2, a_4]^{-1}. \tag{4.17}$$

Proof. By 4.13, we have $\mu_\gamma(a_4) = v_0 a_4 v_0$. By 2.5(i), therefore,

$$\begin{aligned} a_2^{\mu_\gamma(a_4)} &= a_2^{v_0 a_4 v_0} = ([v_0, a_2^{-1}] \cdot a_2)^{a_4 v_0} \\ &= \big([v_0, a_2^{-1}] \cdot [[v_0, a_2^{-1}], a_4] \cdot a_2 \cdot [a_2, a_4]\big)^{v_0} \\ &\in U_{[1,2]} [[v_0, a_2^{-1}], a_4]_3 \cdot [a_2, a_4]. \end{aligned}$$

Since $a_2^{\mu_\gamma(a_4)} \in U_2$, it follows by 2.5(ii) that (4.17) holds. Thus

$$\begin{aligned} a_2^{\mu_\gamma(a_4)} &= ([v_0, a_2^{-1}] \cdot [[v_0, a_2^{-1}], a_4] \cdot a_2 \cdot [a_2, a_4])^{v_0} \\ &= \big([v_0, a_2^{-1}] \cdot [[v_0, a_2^{-1}], a_4]_2 \cdot a_2\big)^{v_0} \\ &\in U_1 [[v_0, a_2^{-1}], a_4]_2 \cdot a_2. \end{aligned}$$

By another application of 2.5(ii), we conclude that

$$a_2^{\mu_\gamma(a_4)} a_2^{-1} = [[v_0, a_2^{-1}], a_4]_2.$$

By 4.7, $[v_0, a_2^{-1}] \in V_1 \subset Y_1$. Conjugating by $\mu_\gamma(a_1)^2$ for an arbitrary $a_1 \in Y_1^\sharp$, we obtain $[Y_1, a_4]_2 = [Y_1, a_4^{-1}]_2$ by 4.12. Thus (4.16) holds. □

Proposition 4.18. *Let $h = \mu_\gamma(a_4)^2$ for some $a_4 \in U_4^\sharp$ and suppose that Y_1 is generated by $Y_1^\sharp$. Then $[h, Y_1] = [h, U_2] = [h, Y_3] = [h, U_4] = 1$.*

Proof. Let $a_1 \in Y_1^\sharp$, $a_4 \in U_4^\sharp$ and $h = \mu_\gamma(a_4)^2$. Then $[h, Y_1^\sharp] = 1$ by 4.14. Since $Y_1 = \langle Y_1^\sharp \rangle$, it follows that $[h, Y_1] = 1$ and, by 4.10, $[Y_1^\sharp, a_4^{-1}]_2 = [Y_1, a_4^{-1}]_2$. Thus $[h, U_2] = 1$ by 4.14 and (4.16). Hence $[h, \mu_\gamma(a_1)] = 1$ by 2.12(iii). It follows that

$$[h, U_4] = [h, U_2^{\mu_\gamma(a_1)}] = [h, U_2]^{\mu_\gamma(a_1)} = 1.$$

We also have

$$[h, Y_3] = [h, Y_1^{\mu_\gamma(a_4)}] = [h, Y_1]^{\mu_\gamma(a_4)} = 1$$

since $[h, \mu_\gamma(a_4)] = 1$. □

Proposition 4.19. *The following hold:*

(i) $[a_1, a_4^{-1}]_2^{\mu_\gamma(a_1)^{-1} \mu_\gamma(b_1)} = [b_1, a_4^{-1}]_2$ *for all $a_1, b_1 \in U_1^\sharp$ and all $a_4 \in U_4$.*

(ii) $[a_0, a_3^{-1}]_2^{\mu_\gamma(a_3) \mu_\gamma(b_3)^{-1}} = [a_0, b_3^{-1}]_2$ *for all $a_3, b_3 \in U_3^\sharp$ and all $a_0 \in U_0$.*

Proof. This follows from 2.14(i). □

Proposition 4.20. *Suppose that* $[a_1, a_4] = a_3$ *for some* $a_1 \in U_1$, $a_3 \in U_3$ *and* $a_4 \in U_4$. *Then* $a_3 \notin U_3^\sharp$.

Proof. Suppose that $a_3 \in U_3^\sharp$ and let w_1, w_7, w_8 be as in 2.4. The elements a_1 and a_4 are, of course, non-trivial. It follows that

$$(w_1, w_8, w_1^{a_4}, w_8^{a_1 a_4}, w_1^{a_4^{-1} a_1 a_4})$$

is a gallery from w_8 to

$$z := w_1^{a_4^{-1} a_1 a_4} = w_1^{[a_1, a_4]} = w_1^{a_3}.$$

Since $a_3 \in U_3^\sharp$, the sequence

$$(w_1, w_8, w_7, w_8^{a_3}, z)$$

is a root. It follows that w_1 and z are opposite vertices as defined in 2.24. Let y be a vertex opposite $w_8^{a_3}$ at z as defined in 2.1(ii). Thus

$$\beta := (w_8, w_7, w_8^{a_3}, z, y)$$

is also a root. Since w_1 and $w_1^{a_4}$ are both opposite w_7 at w_8, there exists $b \in U_\beta$ mapping $w_1^{a_4}$ to w_1. Since z lies on β and $v_1 := w_8^{a_1 a_4}$ is adjacent to z, the element b fixes v_1. Since v_1 is adjacent to $w_1^{a_4}$, it follows that v_1 is adjacent to w_1. Thus by 2.25, there exist vertices v_2, v_3 such that

$$(w_1, v_1, v_2, v_3, z)$$

is a root. Thus (v_1, v_2, v_3, z) and (v_1, z) are two straight paths from v_1 to z. By 2.2(ii), this is impossible. With this contradiction, we conclude that $a_3 \notin U_3^\sharp$. □

5. Normal Tits Quadrangles

We focus now on the proof 3.11. Our proof is derived from the arguments in [**9**, Chapter 23], but numerous modifications needed to be made.

We assume from now on that $U_i = Y_i$ for all odd i, that X is 3-plump as defined in 2.16, that X is normal as defined in 4.5 and that X is razor-sharp as defined in 2.23. Our goal is to produce a non-degenerate quadratic space $\Lambda = (K, L, q)$ and show that X is orthogonal of type Λ.

Proposition 5.1. *The following hold:*
(i) $[U_i, U_{i+2}] = 1$ *for all odd* i.
(ii) U_i *is abelian for all* i.
(iii) $[\mu_\gamma(U_i^\sharp), U_{i-2}] = [\mu_\gamma(U_i^\sharp), U_{i+2}] = 1$ *for all odd* i.
(iv) $[M, N] = 1$, *where*

$$M = \langle \mu_\gamma(a_1)\mu_\gamma(b_1) \mid a_1, b_1 \in U_1^\sharp \rangle \text{ and } N = \langle \mu_\gamma(a_3)\mu_\gamma(b_3) \mid a_3, b_3 \in U_3^\sharp \rangle$$

are the groups called H_1 *and* H_3 *in* 2.20.

Proof. By 4.2, 4.5 and 4.7, (i) and (ii) hold. By 2.9, $\mu_\gamma(U_i^\sharp) \subset \langle U_i, U_{i+4} \rangle$ for all i. Hence (iii) and (iv) follow from (i). □

Proposition 5.2. *Suppose that* $[a_1, a_4]_2 = 1$ *for some* $a_1 \in U_1$ *and some* $a_4 \in U_4^\sharp$. *Then* $a_1 = 1$.

Proof. Let N be as in 5.1(iv) and let V denote the subgroup of U_4 generated by the N-orbit of a_4. Since $[N, U_1] = 1$ (by 5.1(iii)), we have $[a_1, a_4^N]_2 = 1$. By 1.1(ii) and 2.5(i), it follows first that $[a_1, V]_2 = 1$ and then that $B := [a_1, V]$ is a subgroup of U_3. The subgroup B is N-invariant and by 4.20, $B \cap U_3^\sharp = \emptyset$. Since X is razor-sharp at U_3, it follows that $B = 1$. Since $a_4 \in U_4^\sharp$, we have

$$a_1^{\mu_\gamma(a_4)^{-1}} \in B$$

by 2.14(i). Therefore $a_1 = 1$. □

Proposition 5.3. *Let* $a_4 \in U_4^\sharp$. *Then* $[U_1, \mu_\gamma(a_4)^2] = 1$ *and* $a_2^{\mu_\gamma(a_4)} = a_2^{-1}$ *for every* $a_2 \in [U_1, a_4^{-1}]_2$.

Proof. By 2.17, $U_1 = \langle U_1^\sharp \rangle$. By 4.10, $[U_1, a_4^{-1}]_2$ is generated by $[U_1^\sharp, a_4^{-1}]_2$. Since $U_1 = Y_1$, the claims hold now by 4.14 and 5.1(ii). □

Proposition 5.4. *Let* $a_1 \in U_1^\sharp$. *Then* $\mu_\gamma(a_1)^{-1} \in \mu_\gamma(U_1^\sharp)$ *and*

$$M = \langle \mu_\gamma(a_1)\mu_\gamma(b_1) \mid b_1 \in U_1^\sharp \rangle,$$

where M *is as in* 5.1(iv).

Proof. The first claim holds by 2.12(i) and the second claim follows from the first. □

Notation 5.5. Choose $e_1 \in U_1^\sharp$ and let $S = \{\mu_\gamma(e_1)^{-1}\mu_\gamma(a_1) \mid a_1 \in U_1^\sharp\}$. By 5.4, we have $M = \langle S \rangle$.

Proposition 5.6. *Let* S *and* M *be as in* 5.5. *Then* $a_2^S = [U_1^\sharp, a_4^{-1}]_2$ *and*

$$\langle a_2^S \rangle = \langle a_2^M \rangle = [U_1, a_4^{-1}]_2$$

for all $a_2 \in U_2^\sharp$, *where* $a_4 = a_2^{\mu_\gamma(e_1)^{-1}}$.

Proof. Choose $a_2 \in U_2^\sharp$ and let $a_4 = a_2^{\mu_\gamma(e_1)^{-1}}$. By 2.14(i), $a_2 = [e_1, a_4^{-1}]_2$. By 4.19(i), therefore, $[U_1^\sharp, a_4^{-1}]_2 = a_2^S$. By 2.17, $\langle U_1^\sharp \rangle = U_1$. Therefore $\langle a_2^S \rangle = [U_1, a_4^{-1}]_2$ by 4.10. Since X is normal as defined in 4.5, we thus have $a_2^M \subset \langle a_2^S \rangle$. Hence $\langle a_2^M \rangle = \langle a_2^S \rangle$. □

Proposition 5.7. *Let* $a_2 \in U_2^\sharp$ *and let* $a_4 = a_2^{\mu_\gamma(e_1)^{-1}}$. *Then* $\mu_\gamma(a_4)$ *inverts every element of* $\langle a_2^M \rangle$.

Proof. This holds by 5.3 and 5.6. □

Proposition 5.8. *M is abelian.*

Proof. We first claim that $[M, M]$ acts trivially on $U_2^\sharp$. Let $a_2 \in U_2^\sharp$ and let $a_4 = a_2^{\mu_\gamma(e_1)^{-1}}$. By 5.6, we have $\langle a_2^M \rangle = [U_1, a_4^{-1}]_2$. Let $h \in M$ and let $h' = h^{\mu_\gamma(a_4)}$, so $h' \in N$, where N is as in 5.1(iv). By 5.7, $\mu_\gamma(a_4)$ inverts every element of $\langle a_2^M \rangle$. It follows that h and h' induce the same automorphism of $\langle a_2^M \rangle$. Hence $[h, M]$ and $[h', M]$ induce the same group of automorphisms on $\langle a_2^M \rangle$. By 5.1(iv), we have $[h', M] \subset [N, M] = 1$. Since h is arbitrary and $a_2 \in \langle a_2^M \rangle$, we deduce that $[M, M]$ fixes a_2. Since a_2 is arbitrary, we conclude that $[M, M]$ acts trivially on $U_2^\sharp$ as claimed. By 2.17, therefore, $[[M,M],U_2]=1$. By 5.1(iii), $[M,U_3]=1$. By 2.13, therefore, $[M,M]=1$. □

Proposition 5.9. $h^{\mu_\gamma(a_1)}=h^{-1}$ *for each* $h\in M$ *and each* $a_1\in U_1^\sharp$.

Proof. Choose $a_1,b_1\in U_1^\sharp$. By 4.12, $\mu_\gamma(a_1)^2\mu_\gamma(b_1)^{-2}$ centralizes both U_2 and U_3. Thus $\mu_\gamma(a_1)^2=\mu_\gamma(b_1)^2$ by 2.13. It follows that

$$h^{\mu_\gamma(a_1)}=\mu_\gamma(a_1)^{-2}\mu_\gamma(b_1)\mu_\gamma(a_1)=\mu_\gamma(b_1)^{-1}\mu_\gamma(a_1)=h^{-1}$$

for $h=\mu_\gamma(a_1)^{-1}\mu_\gamma(b_1)$. By 5.4, we have $M=\langle\mu_\gamma(a_1)\mu_\gamma(c_1)\mid c_1\in U_1^\sharp\rangle$. By 5.8, therefore, $h^{\mu_\gamma(a_1)}=h^{-1}$ for all $h\in M$. □

Proposition 5.10. $[a_1,a_4^{-1}]_2^{h^2}=[a_1^h,a_4^{-1}]_2$ *for all* $a_1\in U_1$, $a_4\in U_4$ *and* $h\in M$.

Proof. Choose $a_1\in U_1$, $a_4\in U_4$ and $h\in M$. Since U_1 is generated by $U_1^\sharp$, it suffices to assume that $a_1\in U_1^\sharp$ (by 4.10). With this assumption, we have

$$h^2=(h^{-1})^{\mu_\gamma(a_1)}h=\mu_\gamma(a_1)^{-1}\mu_\gamma(a_1)^h=\mu_\gamma(a_1)^{-1}\mu_\gamma(a_1^h)$$

by 2.12(i) and (iii) and 5.9 and therefore $[a_1,a_4^{-1}]_2^{h^2}=[a_1^h,a_4^{-1}]_2$ by 4.19(i). □

Proposition 5.11. $a_2^M=a_2^N$ *for all* $a_2\in U_2^\sharp$, *where* N *is as in* 5.1(iv).

Proof. Let $a_2\in U_2^\sharp$. Choose $a_1,b_1\in U_1^\sharp$ and let $h=\mu_\gamma(a_1)^{-1}\mu_\gamma(b_1)$. By 2.14(i), $a_2=[a_1,b_4^{-1}]_2$ for $b_4=a_2^{\mu_\gamma(a_1)^{-1}}\in U_4^\sharp$. Thus $a_2^h=[b_1,b_4^{-1}]_2$ by 4.19(i). Let $b_0=\kappa_\gamma(b_4)$. By 2.14(ii), $[b_0,a_3^{-1}]_2=a_2$ for $a_3=[a_1,b_4^{-1}]_3$ and $[b_0,b_3^{-1}]_2=a_2^h$ for $b_3=[b_1,b_4^{-1}]_3$. By 2.14(i), $a_3^{\mu_\gamma(b_4)}=a_1$ and $b_3^{\mu_\gamma(b_4)}=b_1$. Thus a_3 and b_3 both lie in $U_3^\sharp$. Let $h'=\mu_\gamma(a_3)\mu_\gamma(b_3)^{-1}$. Then $h'=\mu_\gamma(a_3)\mu_\gamma(b_3^{-1})\in N$ and by 4.19(ii),

$$a_2^{h'}=[b_0,a_3^{-1}]_2^{h'}=[b_0,b_3^{-1}]_2=a_2^h.$$

It follows that $a_2^M\subset a_2^N$ for each $a_2\in U_2^\sharp$. Conjugating by $\mu_\gamma(a_4)$ for an arbitrary $a_4\in U_4^\sharp$, we deduce that $a_2^N\subset a_2^M$ and therefore $a_2^M=a_2^N$ for all $a_2\in U_2^\sharp$. □

Notation 5.12. The group M centralizes U_3 (by 5.1(iv)) and hence acts faithfully on U_2 (by 2.13). The group M can thus be considered a subset of $\mathrm{End}(U_2)$. Let K denote the subring of $\mathrm{End}(U_2)$ generated by M. Thus $1\in K$ and by 5.8, K is commutative. Note that K is generated additively by M and $M\subset K^\times$.

Notation 5.13. Let L be an additive group isomorphic to U_2 and choose an isomorphism x_2 from L to U_2. Let $(t,a)\mapsto ta$ be the map from $K\times L$ to L such that $x_2(ta)$ is the image of $x_2(a)$ under the endomorphism $t\in K\subset\mathrm{End}(U_2)$. This map makes L into a module over K. Note that if $ta=0$ for some $t\in K$ and for all $a\in L$, then $t=0$. This holds because $K\subset\mathrm{End}(U_2)$. Let x_4 denote the isomorphism from L to U_4 given by

$$x_4(a)=x_2(-a)^{\mu_\gamma(e_1)} \tag{5.14}$$

for all $a\in L$ and let $L^\sharp=x_2^{-1}(U_2^\sharp)$. Thus $x_i(L^\sharp)=U_i^\sharp$ for $i=2$ and 4. Since U_4 is generated by $U_4^\sharp$, the group L is generated as an additive group by $L^\sharp$. By 4.12(i), we have

$$x_4(a)^{\mu_\gamma(e_1)}=x_2(a) \tag{5.15}$$

for all $a\in L$.

Proposition 5.16. $[e_1, x_4(a)^{-1}]_2 = x_2(a)$ *for all* $a \in L$.

Proof. This holds by 2.14(i) and (5.15). □

Notation 5.17. For each $b_1 \in U_1^\sharp$, let $\psi(b_1)$ be the element of $K^\times$ induced by $\mu_\gamma(e_1)^{-1}\mu_\gamma(b_1)$ and let K_1 denote the additive subgroup of K generated by $\psi(U_1^\sharp)$. Thus, in particular,

$$[b_1, x_4(a)^{-1}]_2 = x_2(\psi(b_1)a) \tag{5.18}$$

for all $b_1 \in U_1^\sharp$ and all $a \in L$ by 4.19(i) and 5.16, and $1 = \psi(e_1) \in K_1$.

Proposition 5.19. *There exists a unique isomorphism* x_1 *from* K_1 *to* U_1 *such that* $x_1(1) = e_1$ *and*

$$[x_1(t), x_4(a)^{-1}]_2 = x_2(ta) \tag{5.20}$$

for all $t \in K_1$ *and all* $a \in L$.

Proof. Let $t \in K_1$ and $a \in L$. Then there exist $b_1, \ldots, b_1' \in U_1^\sharp$ such that

$$t = \psi(b_1) + \cdots + \psi(b_1').$$

Let $a_1 = b_1 \cdots b_1'$. Then a_1 depends on t and the choice of $b_1, \ldots, b_1'$ and

$$[a_1, x_4(a)^{-1}]_2 = [b_1, x_4(a)^{-1}]_2 \cdots [b_1', x_4(a)^{-1}]_2$$

(by 4.10) and by (5.18), the expression on the right hand side is the image of $x_2(a)$ under t. Hence $[a_1, x_4(a)^{-1}]_2 = x_2(ta)$. Since a is arbitrary, it follows by 4.11 that the element a_1 is the unique element of U_1 satisfying this identity for all $a \in L$. Therefore a_1 is independent of the choice of $b_1, \ldots, b_1'$. We conclude that there exists a unique homomorphism x_1 from K_1 to U_1 such that (5.20) holds. If $t \neq 0$ for some $t \in K_1$, then $ta \neq 0$ for some $a \in L$, so by (5.20), $x_1(t) \neq 0$. Hence x_1 is injective. Now let a_1 be an arbitrary element of U_1. Then $a_1 = b_1 \cdots b_1'$ for some $b_1, \ldots, b_1' \in U_1^\sharp$ (by 2.17) and $a_1 = x_1(t)$ for $t = \psi(b_1) + \cdots + \psi(b_1')$. Hence x_1 is surjective. □

Proposition 5.21. *Let* $s \in K$ *and* $a \in L^\sharp$. *Then there exists* $t \in K_1$ *such that* $sa = ta$.

Proof. We have $x_2(sa) \in \langle x_2(a)^M \rangle = [U_1, x_4(a)^{-1}]_2$ by 5.6 and 5.12. The claim holds, therefore, by (5.20). □

Proposition 5.22. *Let* $t \in K_1$ *and* $a \in L^\sharp$. *If* $ta = 0$, *then* $t = 0$.

Proof. If $ta = 0$, then by 5.19, $[x_1(t), x_4(a)^{-1}]_2 = 1$. By 5.2, therefore, $x_1(t) = 0$ and hence $t = 0$. □

We emphasize that in 5.21 and 5.22, a must be an element of $L^\sharp$ (not L) and in 5.22, t must be an element of K_1 (not K).

Notation 5.23. Let $K_1^\sharp = x_1^{-1}(U_1^\sharp)$. Thus $1 \in K_1^\sharp$ since $x_1(1) = e_1$ by 5.19 and $e_1 \in U_1^\sharp$ by 5.5. Note that by (5.18) and (5.20), $x_1(\psi(b_1)) = b_1$ for all $b_1 \in U_1^\sharp$. Thus $\psi(U_1^\sharp) \in K_1^\sharp$. By 5.17, therefore, K_1 is generated (additively) by $K_1^\sharp$. Let

$$\alpha_t = \mu_\gamma(x_1(1))^{-1}\mu_\gamma(x_1(t))$$

for each $t \in K_1^\sharp$. By 5.5, we have

$$M = \langle \alpha_t \mid t \in K_1^\sharp \rangle. \tag{5.24}$$

Note, too, that by 5.5 and 5.12, K is generated by $K_1^\sharp$ as a ring.

Proposition 5.25. *Let $t \in K_1^\sharp$. Then $x_1(s)^{\alpha_t} = x_1(t^2 s)$ and $x_2(a)^{\alpha_t} = x_2(ta)$ for all $s \in K_1$ and all $a \in L$.*

Proof. Choose $s \in K_1$ and $t \in K_1^\sharp$. Then

$$x_2(ta) = [x_1(t), x_4(a)^{-1}]_2 = [x_1(1), x_4(a)^{-1}]_2^{\alpha_t} = x_2(a)^{\alpha_t} \tag{5.26}$$

for all $a \in L$ by 4.19(i) and (5.20). By 5.19, $U_1 = x_1(K_1)$. Thus $x_1(s)^{\alpha_t} = x_1(r)$ for some $r \in K_1$. It follows from 5.10, (5.20) and (5.26) that

$$x_2(ra) = [x_1(s)^{\alpha_t}, x_4(a)^{-1}]_2 = [x_1(s), x_4(a)^{-1}]_2^{\alpha_t^2} = x_2(st^2 a)$$

for all $a \in L$. Since K operates faithfully on L (by 5.12), it follows that $r = st^2$. □

Proposition 5.27. *For each $t \in K_1^\sharp$, let φ_t be the map from K to itself given by $\varphi_t(s) = t^2 s$ for each $s \in K$. Then $\varphi_t(K_1) = K_1$ and the restriction of φ_t to K_1 is an (additive) automorphism of K_1. In particular, $t^2 = \varphi_t(1) \in K_1$ for all $t \in K_1^\sharp$.*

Proof. This follows from 5.25. □

Proposition 5.28. *$K_1^\sharp \subset K^\times$.*

Proof. Let $t \in K_1^\sharp$. By 5.27, there exists $r \in K_1$ such that $t^2 r = 1$. Hence $t \in K^\times$. □

Proposition 5.29. *$t^{-1} \in K_1^\sharp$ and $\alpha_t^{-1} = \alpha_{t^{-1}}$ for each $t \in K_1^\sharp$.*

Proof. Let $t \in K_1^\sharp$. By 5.28, $t \in K^\times$ and by 5.25, $x_1(t)^{\alpha_t^{-1}} = x_1(t^{-1})$. Since $x_1(t) \in U_1^\sharp$, it follows that $x_1(t^{-1}) \in U_1^\sharp$. Hence $t^{-1} \in K_1^\sharp$. By 2.13 and 5.25, it follows that $\alpha_t^{-1} = \alpha_{t^{-1}}$. □

Proposition 5.30. *The set $\{\alpha_t \mid t \in K_1^\sharp\}$ is closed under inverses and generates M as a monoid.*

Proof. The first claim holds by 5.28 and 5.29 and the second claim follows by (5.24). □

Proposition 5.31. *$K_1^* \subset K^\times$.*

Proof. Let t be a non-zero element of K_1 and let $I = Kt$. Then $t \in I$ and by 5.25 and 5.30, $x_1(I \cap K_1)$ is M-invariant. Since X is razor-sharp at U_1, it follows that $I \cap K_1^\sharp \neq 0$. By 5.28, therefore, $ts \in K^\times$ for some $s \in K$. Hence $t \in K^\times$. □

Proposition 5.32. *Let K_0 denote the subring of K generated by $\{t^2 \mid t \in K_1^\sharp\}$. Then $K_0 \subset K_1 = K_0 K_1$ and K_0 is a field.*

Proof. By 5.25, 5.27 and 5.30, $K_0 K_1 \subset K_1$ (and hence $K_0 \subset K_1 = K_0 K_1$ since $1 \in K_0 \cap K_1$) and every ideal of K_0 is M-invariant. Let I be a non-zero ideal of K_0. Since X is razor-sharp at U_1, there exists $t \in I \cap K_1^\sharp$. By 5.28, $t \in K^\times$ and by

5.29, $t^{-1} \in K_1^\sharp$. Since $t^{-1} = (t^{-1})^2 \cdot t$ and $t \in K_0$, it follows that $t^{-1} \in K_0$. Hence $t \in K_0^\times$. It follows that $I = K_0$. Thus K_0 is a field. □

From now on, we denote by 2 the element $1 + 1$ in K_1.

Proposition 5.33. *Either* $2 \in K^\times$ *or* $2 = 0$.

Proof. This holds by 5.31. □

Proposition 5.34. *Suppose that* $2 = 0$ *and let* $I = \{t \in K \mid t^2 = 0\}$. *Then* $K = I \cup K^\times$.

Proof. Let K_0 be as in 5.32. If $t \in K_1^\sharp$, then $t^2 \in K_0$. Since $2 = 0$, the map $t \mapsto t^2$ is an endomorphism of K whose kernel is I. Since K is generated by $K_1^\sharp$ as a ring (as was observed in 5.23), the image of this endomorphism is K_0. Thus if t is an element of K not in I, then t^2 is a non-zero element of K_0 and hence $t \in K^\times$ by 5.32. □

Proposition 5.35. *Suppose that* $2 = 0$. *Then* $K = K_1$ *and* K *is a field.*

Proof. Let I be as in 5.34 and suppose that s is a non-zero element of I. By 5.12, $sa \neq 0$ for some non-zero $a \in L$. Since L is generated by $L^\sharp$, we can assume that $a \in L^\sharp$. By 5.21, $sa = ta$ for some $t \in K_1$. Since $sa \neq 0$, we have $t \neq 0$. By 5.31, therefore, $t \in K^\times$. We have $s^2 = 0$ and hence $sta = s^2a = 0$. This implies, however, that $sa = 0$. We conclude that $I = 0$. By 5.34, therefore, K is a field.

Now suppose that s is an arbitrary element of $K^\times$ and let a be an arbitrary element of $L^\sharp$. By another application of 5.21, there exists $t \in K_1$ such that $sa = ta$. Thus $(s - t)a = 0$. Since $a \neq 0$, it follows that $s - t \notin K^\times$. Hence $s = t \in K_1$. Thus $K = K_1$. □

We note that in the case that $2 \neq 0$ it will take us until 5.87 to reach the conclusions in 5.35.

Proposition 5.36. $\mu_\gamma(b_4)^2 = 1$ *for all* $b_4 \in U_4^\sharp$.

Proof. We have assumed that $Y_1 = U_1$ and by 2.17, $U_1 = \langle U_1^\sharp \rangle$. The claim holds, therefore, by 2.13 and 4.18. □

Proposition 5.37. *For all* $b \in L^\sharp$, *the subgroup* $[M, \mu_\gamma(x_4(b))]$ *centralizes* U_2.

Proof. Let $b \in L^\sharp$ be arbitrary. By (5.20), we have

$$[U_1, x_4(b)^{-1}]_2 = \{x_2(tb) \mid t \in K_1\}. \tag{5.38}$$

By 4.15 and 5.19, therefore, there exists a function $a \mapsto p_a$ from L to K_1 such that

$$x_2(a)^{\mu_\gamma(x_4(b))} = x_2(a + p_a b) \tag{5.39}$$

for all $a \in L$. Since

$$x_2(a + c)^{\mu_\gamma(x_4(b))} = x_2(a)^{\mu_\gamma(x_4(b))} x_2(c)^{\mu_\gamma(x_4(b))}$$

for all $a, c \in L$, the map $a \mapsto p_a b$ is additive. Let

$$g_t(a) = p_{ta} - tp_a \tag{5.40}$$

for all $t \in K_1^\sharp$ and all $a \in L$. By (5.24), 5.25 and (5.39), $[M, \mu_\gamma(x_4(b))]$ centralizes U_2 if and only if for each $t \in K_1^\sharp$, the map $a \mapsto g_t(a)b$ is identically zero. Thus our goal is to show that for all $t \in K_1^\sharp$, the map $a \mapsto g_t(a)b$ is identically zero. Since this map is additive and L is generated by $L^\sharp$, it suffices to show that $g_t(a)b = 0$ for all $t \in K_1^\sharp$ and all $a \in L^\sharp$.

Choose $t \in K_1^\sharp$ and let α_t be as in 5.23. Then

$$\begin{aligned} x_2(a)^{\mu_\gamma(x_4(b))\alpha_t} &= x_2(a + p_a b)^{\alpha_t} \\ &= x_2(ta + tp_a b) \end{aligned} \tag{5.41}$$

for all $a \in L$. Let

$$\beta_t = \alpha_t^{\mu_\gamma(x_4(b))}. \tag{5.42}$$

By (5.24), we have $\beta_t \in M^{\mu_\gamma(x_4(b))} = N$. By 5.36 and (5.41), we have

$$\begin{aligned} x_2(a)^{\beta_t} &= x_2(ta + tp_a b)^{\mu_\gamma(x_4(b))} \\ &= x_2\big(ta + (tp_a + p_{ta} + p_{tp_a b})b\big) \end{aligned} \tag{5.43}$$

for all $a \in L$. By 5.7, $\mu_\gamma(x_4(b))$ inverts every element of $\langle x_2(b)^M \rangle$. Thus by (5.39),

$$x_2(-ub) = x_2(ub)^{\mu_\gamma(x_4(b))} = x_2(ub + p_{ub} b)$$

for all $u \in K$. Hence $ub + p_{ub} b = -ub$ and therefore

$$p_{ub} b = -2ub \tag{5.44}$$

for all $u \in K$. Thus, in particular,

$$p_{tp_a b} b = -2tp_a b. \tag{5.45}$$

for all $a \in L$. By (5.40) and (5.44), we have

$$g_t(ub)b = p_{tub} b - tp_{ub} b = 0 \tag{5.46}$$

for all $u \in K$ and by (5.43) and (5.45), we have

$$x_2(a)^{\beta_t} = x_2(ta + g_t(a)b) \tag{5.47}$$

for all $a \in L$ and all $t \in K_1^\sharp$.

Now let a be an arbitrary non-zero element of $L^\sharp$. By 5.1(iv) and (5.24), $[\alpha_u, \beta_t] \in [M, N] = 1$ for all $u \in K_1^\sharp$. By (5.47), therefore, $wg_t(a)b = g_t(wa)b$ whenever w is the product of elements in $K_1^\sharp$. Since K is generated additively by the set of all such products (by 5.12), we have $wg_t(a)b = g_t(wa)b$ for all $w \in K$. By 5.6 and 5.11,

$$x_2(a)^{\beta_t} = x_2(a)^{\alpha_u} = x_2(ua)$$

for some $u \in K_1^\sharp$. By (5.47), it follows that

$$ta + g_t(a)b = ua.$$

Let $w = u - t$. Then $w \in K_1$ and

$$wa = g_t(a)b. \tag{5.48}$$

Hence $w^2 a = wg_t(a)b = g_t(wa)b = g_t(g_t(a)b)b$. By (5.46), therefore, $w^2 a = 0$. Hence $w \notin K^\times$. By 5.31, therefore, $w = 0$. By 5.48, we conclude that $g_t(a)b = 0$. □

Notation 5.49. Choose $e_4 \in U_4^\sharp$. Let $x_3(t) = x_1(t)^{\mu_\gamma(e_4)}$ for all $t \in K_1$ and let $\beta_t = \alpha_t^{\mu_\gamma(e_4)}$ for each $t \in K_1^\sharp$. Thus x_3 is an isomorphism from K_1 to U_3 and by 5.36, we have $x_1(t) = x_3(t)^{\mu_\gamma(e_4)}$ for all $t \in K_1$.

Proposition 5.50. *Let $t \in K_1^\sharp$. Then $x_3(t)^{\alpha_t} = x_3(t)$ and $x_4(a)^{\alpha_t} = x_4(t^{-1}a)$ for all $s \in K_1$ and all $a \in L$.*

Proof. By 5.1(iii), $[\alpha_t, U_3] = 1$. Choose $a \in L$. By 5.13, $U_4 = x_4(L)$, so $x_4(a)^{\alpha_t} = x_4(b)$ for some $b \in L$. Conjugating the identity $[x_1(1), x_4(a)^{-1}]_2 = x_2(a)$ by α_t, we find that $[x_1(t^2), x_4(b)^{-1}]_2 = x_2(ta)$ by 5.25. By (5.20), it follows that $t^2b = ta$. By 5.28, $t \in K^\times$. Hence $b = t^{-1}a$. □

Proposition 5.51. *Let $a \in L$, $u \in K_1$ and $t \in K_1^\sharp$ and let β_t be as in* 5.49*. Then*

$$x_1(u)^{\beta_t} = x_1(u),\ x_2(a)^{\beta_t} = x_2(ta),\ x_3(u)^{\beta_t} = x_3(t^2u),\quad \textit{and } x_4(a)^{\beta_t} = x_4(ta).$$

Proof. Choose $a \in L$, $u \in K$ and $t \in K_1^\sharp$. We have $[\beta_t, U_1] = 1$ by 5.1(iii). By 5.37,

$$x_2(a)^{\beta_t} = x_2(a)^{\alpha_t} = x_2(ta).$$

We have $x_4(a)^{\beta_t} = x_4(b)$ for some $b \in L$. Conjugating $[x_1(1), x_4(a)^{-1}]_2 = x_2(a)$ by β_t, we obtain $[x_1(1), x_4(b)^{-1}]_2 = x_2(ta)$. Therefore $b = ta$. Finally, we have

$$x_3(u)^{\beta_t} = x_3(u)^{\mu_\gamma(e_4)\alpha_t\mu_\gamma(e_4)} = x_1(u)^{\alpha_t\mu_\gamma(e_4)} = x_3(t^2u)$$

by 5.25 and 5.49. □

Definition 5.52. Let $h\colon K_1 \times L \to K_1$ and $f\colon L \times L \to K_1$ be the functions defined so that

$$[x_1(t), x_4(a)^{-1}]_3 = x_3(h(t,a))$$

and

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a,b)) \tag{5.53}$$

for all $t \in K_1$ and $a, b \in L$. Let $q(a) = h(1,a)$ for all $a \in L$.

Proposition 5.54. *The function f is symmetric and bi-additive and for all $s, t \in K_1$ and all $a, b \in L$,*

$$h(s+t,a) = h(s,a) + h(t,a) \tag{5.55}$$

and

$$h(t,a+b) = h(t,a) + h(t,b) + f(ta,b). \tag{5.56}$$

In particular,

$$q(a+b) = q(a) + q(b) + f(a,b) \tag{5.57}$$

for $a, b \in L$.

Proof. Choose $a, b, c \in L$ and $s, t \in K_1$. Recall that $[U_1, U_3] = 1$. By 1.1(i),

$$\begin{aligned} x_2((s+t)a)x_3(h(s+t,a)) &= [x_1(s+t), x_4(a)^{-1}] \\ &= [x_1(s), x_4(a)^{-1}] \cdot [x_1(t), x_4(a)^{-1}] \\ &= x_2(sa+ta)x_3(h(s,a)+h(t,a)) \end{aligned}$$

and thus (5.55) holds. By 1.1(ii),

$$\begin{aligned} x_2(t(a+b))x_3(h(t,a+b)) &= [x_1(t), x_4(a+b)^{-1}] \\ &= [x_1(t), x_4(b)^{-1}] \cdot [x_1(t), x_4(a)^{-1}]^{x_4(b)^{-1}} \\ &= x_2(t(a+b))x_3\big(h(t,a) + h(t,b) + f(ta,b)\big) \end{aligned}$$

and

$$[x_2(a+b), x_4(c)^{-1}] = [x_2(a), x_4(c)^{-1}] \cdot [x_2(b), x_4(c)^{-1}].$$

Therefore (5.56) holds and

$$f(a+b,c)=f(a,c)+f(b,c).$$

Setting $t=1$ in (5.56), we obtain (5.57) and by (5.57), f is symmetric. □

Proposition 5.58. *$f(ta,b)=f(a,tb)$ for all $a,b\in L$ and all $t\in K$.*

Proof. Let $t\in K_1^\sharp$ and $a,b\in L$. Conjugating the identity $[x_2(a),x_4(b)^{-1}]=x_3(f(a,b))$ by α_t, we conclude that $f(ta,t^{-1}b)=f(a,b)$ by 5.25 and 5.50. Hence $f(a,tb)=f(ta,t^{-1}\cdot tb)=f(ta,b)$. Since f is bi-additive (by 5.54) and K is generated as a ring by $K_1^\sharp$, it follows that $f(a,tb)=f(ta,b)$ for all $t\in K$. □

Proposition 5.59. *Let $t\in K_1^\sharp$. Then $h(s,ta)=h(t^2s,a)$ for all $s\in K_1$ and all $a\in L$.*

Proof. By 5.25 and 5.50, it suffices to conjugate the identity $[x_1(s),x_4(ta)^{-1}]_3=x_3(h(s,ta))$ by α_t. □

Proposition 5.60. *$h(t,a)=h(t,-a)$ for all $t\in K_1$ and all $a\in L$*

Proof. By 2.12(i), $-1\in K_1^\sharp$. It thus suffices to set $t=-1$ in 5.59. □

Proposition 5.61. *Let $x_5(t)=x_1(t)^{\mu_\gamma(e_1)}$ for all $t\in K_1$. Then*

$$[x_2(a),x_5(t)]=x_3(-h(t,a))x_4(-ta)$$

for all $a\in L$ and all $t\in K_1$.

Proof. In light of 5.1(iii), (5.14) and (5.15), conjugating the identity

$$[x_1(t),x_4(-a)^{-1}]=x_2(-ta)x_3(h(t,-a))$$

by $\mu_\gamma(e_1)$ yields

$$[x_5(t),x_2(a)]=x_4(ta)x_3(h(t,-a)).$$

The claim holds, therefore, by 5.60. □

Proposition 5.62. *Suppose that either $2=0$ or $2q(a)=f(a,a)$ for all $a\in L$. Then $K_1^\sharp=\{t\in K_1\cap K^\times\mid t^{-1}\in K_1\}$.*

Proof. By 5.28 and 5.29, $K_1^\sharp\subset\{t\in K_1\cap K^\times\mid t^{-1}\in K_1\}$. We only need to show the other inclusion holds. Let t be an element of $K_1\cap K^\times$ such that $t^{-1}\in K_1$ and suppose that

$$h(t,a)=h(t^{-1},ta) \tag{5.63}$$

for all $a\in L$. By 5.61 and (5.63), we have

$$[x_2(ta),x_5(t^{-1})]=x_3(-h(t^{-1},ta))x_4(-a)=x_3(-h(t,a))x_4(-a)$$

for all $a\in L$. Therefore

$$\begin{aligned}x_4(a)^{x_1(t)x_5(t^{-1})}&=\big([x_1(t),x_4(a)^{-1}]\cdot x_4(a)\big)^{x_5(t^{-1})}\\&=\big(x_2(ta)x_3(h(a,t))x_4(a)\big)^{x_5(t^{-1})}\\&=x_2(ta)\cdot[x_2(ta),x_5(t^{-1})]\cdot x_3(h(a,t))x_4(a)\\&=x_2(ta)\in U_2\end{aligned}$$

for all $a \in L$. By 2.15, it follows that $t \in K_1^\sharp$. It thus suffices to show that the identity (5.63) holds for all $a \in L$.

Let $a \in L$. Suppose first that $2 \neq 0$. Since $2 \in K_0$, it follows (by 5.32) that $2 \in K_0^\times$ and thus

$$2^{-1} \in K_0 \subset K_1 \tag{5.64}$$

by 5.32. By (5.55), $2h(2, a) = 4h(1, a) = 4q(a)$ and $2h(2^{-1}, 2a) = h(1, 2a) = q(2a)$. By (5.57), $q(2a) = 2q(a) + f(a, a)$. By hypothesis, $f(a, a) = 2q(a)$. Thus $q(2a) = 4q(a)$. It follows that (5.63) holds with 2 in place of t. Hence $2 \in K_1^\sharp$.

By 5.59 and the conclusion of the previous paragraph, we have $h(t, 2a) = h(4t, a)$ if $2 \neq 0$ (where t continues to be an arbitrary element of $K_1 \cap K^\times$ such that $t^{-1} \in K_1$). By 5.52, $h(t, a) = 0$ if $t = 0$ or $a = 0$, so $h(t, 2a) = h(4t, a)$ also if $2 = 0$. By (5.55) and (5.56), it follows that

$$4h(t, a) = h(t, 2a) = 2h(t, a) + f(ta, a) \tag{5.65}$$

whether or not $2 \neq 0$. Therefore $f(ta, a) = 2h(t, a)$. Since K_1 is generated by $K_1^\sharp$ and $K_1^\sharp \subset K_1 \cap K^\times$, it follows that K_1 is generated by $\{t \in K_1 \cap K^\times \mid t^{-1} \in K_1\}$. By 5.54 and (5.55), the maps $w \mapsto f(wa, a)$ and $w \mapsto 2h(w, a)$ from K_1 to K_1 are additive. Hence

$$f(wa, a) = 2h(w, a) \tag{5.66}$$

for all $w \in K_1$. By 2.17, we can assume that there exist $s, u \in K_1^\sharp$ such that $t = s+u$. Note that $sut^{-1} = 2^{-1}\big(t - t^{-1}(s^2 + u^2)\big)$ and that $2^{-1}\big(t - t^{-1}(s^2 + u^2)\big) \in K_1$ by 5.32 and (5.64). Thus $sut^{-1} \in K_1$. Therefore

$$\begin{aligned} h(t^{-1}, ta) &= h(t^{-1}, sa) + h(t^{-1}, ua) + f(t^{-1}sa, ua) && \text{by (5.56)} \\ &= h(t^{-1}, sa) + h(t^{-1}, ua) + f(sut^{-1}a, a) && \text{by 5.58} \\ &= h(t^{-1}, sa) + h(t^{-1}, ua) + 2h(sut^{-1}, a) && \text{by 5.66} \\ &= h(t^{-1}s^2, a) + h(t^{-1}u^2, a) + 2h(sut^{-1}, a) && \text{by 5.59} \\ &= h\big(t^{-1}(s^2 + u^2 + 2su), a\big) = h(t, a) && \text{by (5.55).} \end{aligned}$$

Thus (5.63) holds. □

It will take us until 5.86 to show that the hypothesis $2q(a) = f(a, a)$ for all $a \in L$ if $2 \neq 0$ in 5.62, in fact, holds.

Notation 5.67. Let ε be the inverse image in L of e_4 under the map $a \mapsto x_4(a)$, where e_4 is as in 5.49.

Proposition 5.68. *Let ε be as in* 5.67*. Then $q(\varepsilon) = 1$ and*

$$x_2(a)^{\mu_\gamma(e_4)} = x_2(a - f(a, \varepsilon)\varepsilon)$$

for all $a \in L$.

Proof. Choose $a \in L$ and let $v_0 = \kappa_\gamma(e_4)$, where κ_γ is as in 2.9. Then $v_0 \in U_0$, so there exists $u \in K_1$ such that $x_1(u) = [v_0, x_2(a)^{-1}]$. By 2.14(i), (5.20) and 5.49,

$$[x_1(u), e_4^{-1}] = [x_1(u), x_4(\varepsilon)^{-1}] = x_2(u\varepsilon)x_3(u).$$

Conjugating with $\mu_\gamma(e_1)^2$, we have

$$[x_1(u), e_4] = x_2(-u\varepsilon)x_3(u)$$

by 4.12. By (5.53), we have $[x_2(a), e_4] = x_3(-f(a,\varepsilon))$. Thus

$$\begin{aligned} x_2(a)^{\mu_\gamma(e_4)} = x_2(a)^{v_0 e_4 \lambda_\gamma(e_4)} &= \big(x_1(u)x_2(a)\big)^{e_4\lambda_\gamma(e_4)} \\ &= \big(x_1(u)x_2(a-u\varepsilon)x_3(u-f(a,\varepsilon))\big)^{\lambda_\gamma(e_4)}. \end{aligned}$$

Since $U_2^{\mu_\gamma(e_4)} = U_2$, it follows now by 2.5(i) and 2.5(ii) that $u = f(a,\varepsilon)$ and

$$x_2(a)^{\mu_\gamma(e_4)} = x_2(a - u\varepsilon).$$

Since $[x_1(1), x_4(\varepsilon)^{-1}]_3 = x_3(1)$, we have $q(\varepsilon) = 1$ by 5.52. □

Notation 5.69. Let $\bar{a} = f(a,\varepsilon)\varepsilon - a$ for all $a \in L$.

Proposition 5.70. *The map $a \mapsto \bar{a}$ is a K-linear map from L to itself whose square is the identity.*

Proof. By 5.68, $\mu_\gamma(e_4)$ induces the map $x_2(a) \mapsto x_2(-\bar{a})$ on U_2 and by (5.24) and 5.25, M induces the group generated by $\{x_2(a) \mapsto x_2(ta) \mid t \in K_1^\sharp\}$ on U_2. Thus by 5.37, the map $a \mapsto \bar{a}$ is K-linear and by 5.36, its square is the identity. □

Proposition 5.71. *$f(ta,\varepsilon)\varepsilon = tf(a,\varepsilon)\varepsilon$ for all $t \in K$ and all $a \in L$.*

Proof. This holds by 5.70. □

Notation 5.72. Let $x_0(a) = x_4(\bar{a})^{\mu_\gamma(e_4)}$ for all $a \in L$. By 5.70, the map $a \mapsto x_0(a)$ is an isomorphism from L to U_0 and by 5.36 and 5.70 that $x_4(a)^{\mu_\gamma(e_4)} = x_0(\bar{a})$ for all $a \in L$.

Proposition 5.73. *Let $a \in L^\sharp$. Then $q(a) \in K_1^\sharp \subset K^\times$, $\kappa_\gamma(x_4(a)) = x_0(q(a)^{-1}a)$ and $\big(h(t,a) - tq(a)\big)a = 0$ for all $t \in K_1$.*

Proof. Let $t \in K_1$ and $a \in L$. Since $\mu_\gamma(e_4)^2 = 1$, we have

$$x_2(ta - f(ta,\varepsilon)\varepsilon)^{\mu_\gamma(e_4)} = x_2(ta)$$

by 5.68. By 5.71, we have

$$\begin{aligned} [x_1(t), x_4(a - f(a,\varepsilon)\varepsilon)^{-1}]_2 &= x_2(t(a - f(a,\varepsilon)\varepsilon)) \\ &= x_2(ta - f(ta,\varepsilon)\varepsilon). \end{aligned}$$

Conjugating this equation by $\mu_\gamma(e_4)$, we thus obtain $[x_3(t), x_0(a)]_2 = x_2(ta)$ and hence

$$[x_0(a), x_3(t)^{-1}]_2 = x_2(ta) \tag{5.74}$$

for all $t \in K_1$ and all $a \in L$.

Now choose $t \in K_1$ and $a \in L^\sharp$ and let a' denote the unique element of $L^\sharp$ such that $\kappa_\gamma(x_4(a)) = x_0(a')$. By 2.14(ii) applied to

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(h(t,a)), \tag{5.75}$$

we have

$$[x_0(a'), x_3(h(t,a))^{-1}]_2 = x_2(ta).$$

By (5.74), therefore,

$$h(t,a)a' = ta. \tag{5.76}$$

Setting $t = 1$ in (5.76), we obtain $q(a)a' = a$. By 2.14(i) applied to (5.75), we have

$$x_3(h(1,a))^{\mu_\gamma(x_4(a))} = x_1(1).$$

Since $1 \in K_1^\sharp$ (by 5.23), it follows that $q(a) = h(1,a) \in K_1^\sharp$. By 5.28, $K_1^\sharp \subset K^\times$. Hence $q(a) \in K^\times$ and $a' = q(a)^{-1}a$. By (5.76), therefore, $h(t,a)a = tq(a)a$. □

Proposition 5.77. *Let $a \in L$ and $b \in L^\sharp$. Then*

$$x_2(a)^{\mu_\gamma(x_4(b))} = x_2\big(a - q(b)^{-1}f(\bar{a},\bar{b})b\big) \tag{5.78}$$

and

$$tf(\bar{a},\bar{b})b = f(t\bar{a},\bar{b})b \tag{5.79}$$

for all $a \in L$ and all $t \in K$.

Proof. By 5.73, $q(b) \in K_1^\sharp \subset K^\times$ and $\kappa_\gamma(x_4(b)) = x_0(q(b)^{-1}b)$. Conjugating the identity

$$[x_2(a), x_4(c)^{-1}] = x_3(f(a,c))$$

by $\mu_\gamma(e_4)$, we obtain

$$[x_2(-\bar{a}), x_0(\bar{c})^{-1}] = x_1(f(a,c))$$

by 5.68 and 5.72. Thus

$$[x_0(c), x_2(a)^{-1}] = x_1(f(\bar{a},\bar{c}))$$

for all $a, c \in L$. Hence

$$[x_0(q(b)^{-1}b), x_2(a)^{-1}] = x_1(f(\bar{a}, q(b)^{-1}\bar{b}))$$

for all $a \in L$. Choose $a \in L$ and let $u = f(\bar{a}, q(b)^{-1}\bar{b})$. Then

$$\begin{aligned} x_2(a)^{\mu_\gamma(x_4(b))} = x_2(a)^{x_0(q(b)^{-1}b)x_4(b)\lambda_\gamma(x_4(b))} &= (x_1(u)x_2(a))^{x_4(b)\lambda_\gamma(e_4)} \\ &\in \big(x_1(u)x_2(a-ub)U_3\big)^{\lambda_\gamma(x_4(b))}. \end{aligned}$$

Since $U_2^{\mu_\gamma(x_4(b))} = U_2$, it follows now by 2.5(i) and 2.5(ii) that

$$x_2(a)^{\mu_\gamma(x_4(b))} = x_2(a - ub). \tag{5.80}$$

By 5.37, it follows that $tf(\bar{a}, q(b)^{-1}\bar{b})b = f(t\bar{a}, q(b)^{-1}\bar{b})b$ for all $t \in K_1^\sharp$. Since K is generated by $K_1^\sharp$, this identity holds for all $t \in K$. By 5.58, therefore, (5.79) holds, and by (5.79) and (5.80), (5.78) holds. □

Proposition 5.81. *If $a \in L^\sharp$, then $\bar{a} \in L^\sharp$.*

Proof. Let $a \in L^\sharp$. By 5.68 and 5.69, $x_2(a)^{\mu_\gamma(x_4(\varepsilon))} = x_2(-\bar{a})$. By 2.12(i), therefore, $x_2(\bar{a}) \in U_2^\sharp$ and hence $\bar{a} \in L^\sharp$. □

Proposition 5.82. *If $2 \neq 0$, then $q(t\varepsilon)\varepsilon = t^2\varepsilon$ for all $t \in K_1$.*

Proof. By 5.68 and 5.69, $x_2(\varepsilon)^{\mu_\gamma(x_4(\varepsilon))} = x_2(-\bar{\varepsilon})$. By 5.7 and (5.15), on the other hand, we have $x_2(\varepsilon)^{\mu_\gamma(x_4(\varepsilon))} = x_2(-\varepsilon)$. Thus $\varepsilon = \bar{\varepsilon}$. By 5.69 and 5.70, it follows that $f(\varepsilon,\varepsilon)\varepsilon = 2\varepsilon$ and $\overline{t\varepsilon} = t\bar{\varepsilon}$ for all $t \in K$.

Now let $t \in K_1^\sharp$. By 5.25, we have $x_2(t\varepsilon) = x_2(\varepsilon)^{\alpha_t}$. Thus $t\varepsilon \in L^\sharp$. We have $x_2(t\varepsilon)^{\mu_\gamma(x_4(t\varepsilon))} = x_2(-t\varepsilon)$ by 5.7. By (5.78) with $t\varepsilon$ in place of both a and b, it follows (since $\overline{t\varepsilon} = t\bar{\varepsilon}$) that

$$2q(t\varepsilon)t\varepsilon = f(t\varepsilon, t\varepsilon)t\varepsilon.$$

By 5.58 and 5.71, we have

$$f(r\varepsilon, s\varepsilon)\varepsilon = rsf(\varepsilon,\varepsilon)\varepsilon = 2rs\varepsilon \tag{5.83}$$

for all $r, s \in K$. Hence $f(t\varepsilon, t\varepsilon)\varepsilon = 2t^2\varepsilon$. Since $2t \in K^\times$ (by 5.28 and 5.33), we thus have $w\varepsilon = 0$ for $w = q(t\varepsilon) - t^2$. By 5.32, $w \in K_1$. Hence $w = 0$ by 5.22. Thus $q(t\varepsilon) = t^2$ for all $t \in K_1^\sharp$.

Now let t be an element of K_1 not in $K_1^\sharp$. By 2.17, there exist $r, s \in K_1^\sharp$ such that $t = r + s$. Thus

$$q(t\varepsilon) = q(r\varepsilon + s\varepsilon) = r^2 + s^2 + f(r\varepsilon, s\varepsilon)$$

by (5.57). By (5.83), it follows that $q(t\varepsilon)\varepsilon = (r^2 + s^2 + 2rs)\varepsilon = t^2\varepsilon$. □

Proposition 5.84. *Suppose that* $2 \neq 0$. *Then* $f(a, a) = 2q(a)$ *all* $a \in L^\sharp$.

Proof. Let $a \in L^\sharp$. Setting $a = b$ in (5.78), we have

$$x_2(a)^{\mu_\gamma(x_4(a))} = x_2(a - f(\bar{a}, \bar{a})q(a)^{-1}a).$$

On the other hand, $x_2(a)^{\mu_\gamma(x_4(a))} = x_2(-a)$ by 5.7. Hence $2q(a)a = f(\bar{a}, \bar{a})a$. Since $2q(a)$ and $f(\bar{a}, \bar{b})$ both lie in K_1, it follows from 5.22 that $2q(a) = f(\bar{a}, \bar{a})$. By 5.81, we can replace a by $\bar{a}$ in this equation to obtain

$$2q(\bar{a}) = f(a, a). \tag{5.85}$$

Next, we note that

$$q(a + \bar{a}) = q(a) + q(\bar{a}) + f(a, \bar{a})$$

by (5.57). By 5.69 and 5.82, we have $q(a + \bar{a})\varepsilon = q(f(a, \varepsilon)\varepsilon)\varepsilon = f(a, \varepsilon)^2\varepsilon$ and

$$f(a, \bar{a}) = -f(a, a) + f(a, f(a, \varepsilon)\varepsilon).$$

By (5.79),

$$f(a, f(a, \varepsilon)\varepsilon)\varepsilon = f(a, \varepsilon)^2\varepsilon.$$

By (5.85), it follows that

$$\big(q(a) + q(\bar{a})\big)\varepsilon = f(a, a)\varepsilon = 2q(\bar{a})\varepsilon$$

and hence

$$\big(q(a) - q(\bar{a})\big)\varepsilon = 0.$$

Since $q(a) - q(\bar{a})$ lies in K_1, we have $q(\bar{a}) = q(a)$ by 5.22. The claim holds, therefore, by (5.85). □

Proposition 5.86. *Suppose that* $2 \neq 0$. *Then* $2q(a) = f(a, a)$ *for all* $a \in L$.

Proof. By 5.84, $2q(a) = f(a, a)$ for all $a \in L^\sharp$. Since f is symmetric (by 5.54), we have

$$f(a + b, a + b) = f(a, a) + f(b, b) + 2f(a, b)$$

for all $a, b \in L$. By 2.17, every element of L is a sum of at most two elements of $L^\sharp$. The claim follows by (5.57). □

Proposition 5.87. $K = K_1$ *and* K *is a field.*

Proof. By 5.35, we can assume that $2 \neq 0$. Suppose that $K_1 = K_0$, where K_0 is as in 5.32. Thus K_0 is a field, $t^2 \in K_0$ for all $t \in K_1^\sharp$ and $K_0 \subset K_0K_1 = K_1$. Since K_0 is a subring of K and K_1 generates K as a ring, it follows that $K = K_1$. Therefore $K = K_0$, so K is a field. It thus suffices to show that $K_1 = K_0$. Note that since $2 \neq 0$ and $2 \in K_0$, we have $2 \in K_0^\times$.

Suppose first that $t^2 = 1$ for all $t \in K_1^\sharp$. Let t be an arbitrary element of K_1^* not in $K_1^\sharp$. By 2.17, there exist $u, v \in K_1^\sharp$ such that $t = u + v$. Since $u^2 = v^2 = 1$, we have $tuv = (u + v)uv = v + u = t$. Since $t \in K^\times$ (by 5.31), it follows that $uv = 1$ and hence $u = v$. Thus $t = 2u$ and hence $t^2 = 4 \in K_0$. Therefore $t^2 \in K_0$ for all $t \in K_1$. Hence $2t = (t + 1)^2 - t^2 - 1 \in K_0$ for all $t \in K_1$. Since $2 \in K_0^\times$, it follows that $t \in K_0$. Hence $K_1 = K_0$.

Next suppose that $t \in K_1^\sharp$ and $t^2 \neq 1$. Let $u = t + 1$ and $b = t^2 - 1$. Then $u(t - 1) = b \in K_0^\times$ and hence $u^{-1} = b^{-1}(t - 1) \in K_0K_1 = K_1$. By 5.62 and 5.86, it follows that $u \in K_1^\sharp$. Thus $u^2 \in K_0$. Since $t^2 \in K_0$ and $2 \in K_0^\times$, it follows that $t \in K_0$. We conclude that $t \in K_0$ whenever $t \in K_1^\sharp$ and $t^2 \neq 1$.

Finally, suppose there exist $s, t \in K_1^\sharp$ such that $t^2 = 1$ but $s^2 \neq 1$ and let $b = (s-t)(s+t)$. Then $b = s^2-t^2 \in K_0^\times$ and hence $(s+t)^{-1} = b^{-1}(s-t) \in K_0K_1 = K_1$. By 5.62 and 5.86, it follows that $s+t \in K_1^\sharp$. Hence $(s+t)^2 \in K_0$. By the conclusion of the previous paragraph, $s \in K_0$. Hence $2s \in K_0^\times$ and thus $t \in K_0$.

We conclude that in every case $K_1^\sharp \subset K_0$. Since K_1 is generated by $K_1^\sharp$, it follows that $K_0 = K_1$. □

Proposition 5.88. $K^* = K_1^\sharp$.

Proof. This holds by 5.62, 5.86 and 5.87. □

Proposition 5.89. *The map f is bilinear.*

Proof. By 5.54, f is symmetric and bi-additive. Let $a \in L$, $b \in L^\sharp$ and $t \in K$. By 5.81, $\bar{b} \in L^\sharp$. We can thus replace a and b by $\bar{a}$ and $\bar{b}$ in (5.79) to obtain $tf(a, b)\bar{b} = f(ta, b)\bar{b}$. Since K is a field, it follows that $tf(a, b) = f(ta, b)$ for all $a \in L$ and all $b \in L^\sharp$. Since L is generated by $L^\sharp$, this identity holds for all $a, b \in L$. □

Proposition 5.90. $q(ta) = t^2q(a)$ *for all* $t \in K$ *and all* $a \in L$.

Proof. Let $a \in L$ and let $t \in K^*$. By 5.88, $t \in K_1^\sharp$. Conjugating the identity

$$[x_1(1), x_4(a)^{-1}]_3 = x_3(q(a))$$

by β_t, we obtain $[x_1(1), x_4(ta)^{-1}]_3 = x_3(t^2q(a))$ by 5.51. Thus $q(ta) = t^2q(a)$. □

Proposition 5.91. (K, L, q) *is a quadratic space as defined in* 3.1 *and* f *is the associated bilinear form.*

Proof. This holds by (5.57), 5.87, 5.89 and 5.90. □

Proposition 5.92. *The identities*

$$[x_1(t), x_4(a)^{-1}] = x_2(ta)x_3(tq(a))$$

and

$$[x_2(a), x_4(b)^{-1}] = x_3(f(a, b))$$

hold for all $t \in K$ *and all* $a, b \in L$.

Proof. Let $t \in K$. Since K is a field and $K = K_1$ (by 5.87), it follows from 5.73 that $h(t, a) = tq(a)$ for all $a \in L^\sharp$. By (5.56) and 5.89, we have

$$h(t, a + b) = h(t, a) + h(t, b) + tf(a, b)$$

for all $a, b \in L$. Since L is generated by $L^\sharp$, it follows by (5.57) that $h(t, a) = tq(a)$ for all $a \in L$. The claims hold now by (5.20) and 5.52. □

By 5.91 and 5.92, X is orthogonal of type Λ for $\Lambda = (K, L, q)$. Suppose that $f(a, L) = q(a) = 0$ for some $a \in L$. Then $x_4(a) \in N_{U_4}(U_{[1,2]})$ by 5.92. Hence $a = 0$ by 4.11. Thus Λ is non-degenerate (as defined in 3.2). This concludes the proof of 3.11.

Remark 5.93. We observe that in the proof of 3.11, we needed to assume only that X is 3-plump (in order to be able to apply 2.17), but that the uniqueness result 3.7 requires that X be 4-plump. In an earlier version of [**2**], the results [**2**, 1.5.2, 1.5.19, 1.5.28 and 1.5.29] all contained the hypothesis that X is $(n+1)$-plump. Before publication, it was noticed that this hypothesis is unnecessary in the first two of these results and that 4-plump suffices in the remaining two. As a consequence, the hypothesis $|K| > 4$ in [**5**, 6.10, 7.4 and 8.2] can be replaced by $|K| \geq 4$. We note, too, that the proofs of [**6**, 1.1 and 1.2] remain valid if it is assumed only that X is 3-plump.

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