BANACH-STEINHAUS THEOREM FOR THE SPACE $\mathcal{P}$ OF ALL PRIMITIVES OF HENSTOCK-KURZWEIL INTEGRABLE FUNCTIONS

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Abstract. In this paper, it is shown how the Banach-Steinhaus theorem for the space $\mathcal{P}$ of all primitives of Henstock-Kurzweil integrable functions on a closed bounded interval, equipped with the uniform norm, can follow from the Banach-Steinhaus theorem for the Denjoy space by applying the classical Hahn-Banach theorem and Riesz representation theorem.

1. Introduction

The Banach-Steinhaus theorem is an important result in the field of functional analysis. The statement of the theorem is often given in various forms, one of which states that any family of continuous linear operators between Banach spaces is uniformly bounded provided that it is bounded pointwise. A *Banach space* is a complete normed linear space and a normed linear space X equipped with a norm $\|\cdot\|_X$ is *complete* if $\|x_n - x_m\|_X \to 0$ as $n, m \to \infty$ implies that there is $x \in X$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$, that is, every Cauchy sequence is convergent.

In [**2**, Theorem 11.6], the Banach-Steinhaus theorem for Sargent spaces, a special kind of normed linear spaces, is proved. More precisely, given a sequence $\{T_n\}$ of continuous linear operators from a Sargent space E to a normed linear space Z, if $\sup\{\|T_n(x)\|_Z : n \geq 1\} < +\infty$ for every $x \in E$, then $\sup\{\|T_n\| : n \geq 1\} < +\infty$. Here $\|T_n\|$ denotes the norm of T_n.

Note that every Banach space is a Sargent space but a Sargent space is not necessarily a Banach space. Furthermore, the Denjoy space is a special case of Sargent spaces [**2**]. The *Denjoy space* of $[a, b]$, or simply the Denjoy space when there is no ambiguity, denoted by $\mathcal{H}[a, b]$, is the space of all Henstock-Kurzweil integrable functions on a closed bounded interval $[a, b]$ equipped with the Alexiewicz norm as given below.

$$\|f\|_{\mathcal{H}[a,b]} := \sup\left\{\left|\int_b^x f(t)dt\right| : a \leq x \leq b\right\}$$

for every $f \in \mathcal{H}[a, b]$. Here f is a real-valued point function defined on $[a, b]$. Clearly, $\mathcal{H}[a, b]$ is a normed linear space. Note that two functions f and g in $\mathcal{H}[a, b]$ are regarded as identical if $f(x) = g(x)$ almost everywhere in $[a, b]$, that is f and g agree everywhere except perhaps in a set of measure zero.

Since the Denjoy space is a Sargent space (see [**2**, Example 11.3]), the Banach-Steinhaus theorem for $\mathcal{H}[a, b]$ can be obtained as a consequence of that for Sargent spaces. This is a useful result for the study of integrals which are nonabsolute, such as the Henstock-Kurzweil integral, because the Denjoy space is not a Banach space, and thus the Banach-Steinhaus theorem for $\mathcal{H}[a, b]$ can be considered as an extension of the classical version of the theorem. Integrals such as the Riemann integral and the Lebesgue integral are *absolute* in the sense that if f is integrable, then so is $|f|$, where $|f|$ denotes the absolute value of f. We refer the interested reader to [**3**] for two definitions of the Henstock-Kurzweil integral on the real line, and [**4**] for properties and results of the integral in a more abstract setting.

It is known that if T is a continuous linear functional on $\mathcal{H}[a, b]$, then

$$T(f) = \int_a^b f(x)g(x)dx$$

for every $f \in \mathcal{H}[a, b]$, where g is of bounded variation on $[a, b]$ (see, for example, [**2**, Theorem 12.7]). Following the work of Kurzweil and Jarnik [**1**], we find that functionals defined on a function space should be expressed in terms of the primitives F of integrable functions f rather than the functions f *per se*.

In this paper, we shall thus prove the Banach-Steinhaus theorem for the space $\mathcal{P}[a, b]$, or for brevity $\mathcal{P}$, of all primitives of Henstock-Kurzweil integrable functions defined on a closed bounded interval $[a, b]$, in which continuous linear operators are replaced with continuous linear functionals. This will be done by applying the classical Hahn-Banach theorem and Riesz representation theorem, and the Banach-Steinhaus theorem for $\mathcal{H}[a, b]$. An operator taking real values is called a *functional*.

2. Preliminaries

A functional T defined on a normed linear space X is *linear* if

$$T(x + y) = T(x) + T(y)$$

and

$$T(\lambda x) = \lambda T(x)$$

for all $x, y \in X$ and $\lambda \in \mathbb{R}$. It is *continuous* if $\|x_n - x\|_X \to 0$ as $n \to \infty$ implies that $T(x_n) \to T(x)$ as $n \to \infty$.

The norm of the functional T is defined as

$$\|T\| := \inf\{m \geq 0 : |T(x)| \leq m \|x\|_X\}.$$

The functional T is *bounded* if there exists $M \geq 0$ such that $|T(x)| \leq M \|x\|_X$ for every $x \in X$. It is well known that a linear functional on a normed linear space is bounded if and only if it is continuous.

We first state the Banach-Steinhaus theorem for $\mathcal{H}[a, b]$ which, as pointed out previously, follows from that for Sargent spaces because the Denjoy space is a Sargent space.

Theorem 1. *Let $\{T_n\}$ be a sequence of continuous linear functionals on $\mathcal{H}[a, b]$. If $\sup\{|T_n(f)| : n \geq 1\} < +\infty$ for every $f \in \mathcal{H}[a, b]$, then $\sup\{\|T_n\| : n \geq 1\} < +\infty$.*

We shall show that the Banach-Steinhaus theorem for $\mathcal{P}$ follows from the above theorem by applying the classical Hahn-Banach theorem and Riesz representation theorem.

The Hahn-Banach theorem is a useful tool in functional analysis which allows the extension of bounded linear functionals defined on a subspace of some linear space to the whole space. There are now many versions of the Hahn-Banach theorem and in this paper we shall use the following version which we state here for ease of discussion.

Theorem 2 (Hahn-Banach). *Suppose Y is a non-trivial proper subspace of a normed linear space X. Let T be a bounded linear functional on Y. Then there is a bounded linear functional T^* on X such that $\|T^*\| = \|T\|$ and $T^*(x) = T(x)$ if $x \in Y$.*

Let $C[a, b]$ denote the space of all continuous functions defined on $[a, b]$ with the uniform norm given by

$$\|F\|_{C[a,b]} = \sup_{a \leq x \leq b} |F(x)|$$

for every $F \in C[a, b]$. Since the primitive of a Henstock-Kurzweil integrable function defined on $[a, b]$ is continuous on $[a, b]$ (see [**2**, Corollary 3.8]), the space $\mathcal{P}$ is a proper subspace of $C[a, b]$ as a normed linear space (see [**2**, Theorem 5.7]). As a result, Theorem 2 holds for $X = C[a, b]$ and $Y = \mathcal{P}$.

The classical Riesz representation theorem, which we state below, tells us explicitly what functionals on $C[a, b]$, and in particular those on $\mathcal{P}$, are.

Theorem 3 (Riesz). *A functional T defined on $C[a, b]$ is linear and continuous if and only if*

$$T(F) = \int_a^b F(x) dg(x)$$

for every $F \in C[a, b]$ and for some function g of bounded variation on $[a, b]$ where the integral is in the sense of Riemann-Stieltjes. Furthermore, $\|T\| = V(g; [a, b])$.

Here $V(g; [a, b])$ denotes the *total variation* of the function g on $[a, b]$ and is defined as

$$V(g; [a, b]) = \sup \sum_{i=1}^n |g(t_i) - g(t_{i-1})|,$$

where the supremum is taken over partitions $a = t_0 < t_1 < \cdots < t_n = b$ of the interval $[a, b]$. We say that g is of *bounded variation* if $V(g; [a, b]) < +\infty$.

Since $\mathcal{P}$ is a proper subspace of $C[a, b]$, it follows readily from Theorems 2 and 3 that the following result holds.

Proposition 1. *If T is a continuous linear functional defined on $\mathcal{P}$, then there is a continuous linear functional T^* defined on $C[a, b]$ such that*

$$T^*(F) = \int_a^b F(x) dg(x)$$

for every $F \in C[a, b]$, where g is of bounded variation on $[a, b]$. Furthermore, $T^*(F) = T(F)$ for every $F \in \mathcal{P}$ and

$$\|T^*\| = \|T\| = V(g; [a, b]).$$

3. Main Results

We are now ready to prove that the Banach-Steinhaus Theorem for $\mathcal{P}$ follows from Theorem 1.

Theorem 4. *Let $\{T_n\}$ be a sequence of continuous linear functionals on $\mathcal{P}$. If* $\sup\{|T_n(F)| : n \geq 1\} < +\infty$ *for every $F \in \mathcal{P}$, then* $\sup\{\|T_n\| : n \geq 1\} < +\infty$.

Proof. By Proposition 1, for each n we have

$$T_n(F) = \int_a^b F(x)dg_n(x)$$

for every $F \in \mathcal{P}$, where g_n is of bounded variation on $[a, b]$ and the integral is in the Riemann-Stieltjes sense. Furthermore,

$$\|T_n\| = V(g_n; [a, b]).$$

For every $F \in \mathcal{P}$, let f be the function of which F is the primitive on $[a, b]$. By means of the integration by parts formula for the Henstock-Kurzweil integral (see [**2**, Corollary 12.2]), we have

$$\int_a^b f(x)g_n(x)dx = F(b)g_n(b) - F(a)g_n(a) - \int_a^b F(x)dg_n(x)$$

where $F(a) = 0$. For each n we define

$$S_n(f) = \int_a^b f(x)h_n(x)dx$$

for every $f \in \mathcal{H}[a, b]$, and $h_n(x) = -g_n(x)$ when $x \in [a, b)$ and $h_n(b) = 0$. Clearly, the functionals T_n and S_n are equivalent. Hence $\{S_n\}$ is a sequence of continuous linear functionals on $\mathcal{H}[a, b]$ and $\|S_n\| = \|T_n\|$. By the hypothesis that $\sup\{|T_n(F)| : n \geq 1\} < +\infty$ for every $F \in \mathcal{P}$, we also have $\sup\{|S_n(f)| : n \geq 1\} < +\infty$ for every $f \in \mathcal{H}[a, b]$. It follows from Theorem 1 that $\sup\{\|S_n\| : n \geq 1\} < +\infty$ and equivalently $\sup\{\|T_n\| : n \geq 1\} < +\infty$ as desired. $\square$

Note that in the statement of Theorem 4, it is not necessary to restrict the functionals to be a sequence. Take an arbitrary family K of continuous linear functionals on $\mathcal{P}$ which is pointwise bounded on $\mathcal{P}$. For each T in K define the functional S as in the proof of Theorem 4. This gives a pointwise bounded family of continuous linear functionals on the Denjoy space which is, therefore, uniformly bounded implying K is uniformly bounded.

We shall end this paper with a useful result which is an immediate consequence of Theorem 4.

Corollary 1. *Let $\{T_n\}$ be a sequence of continuous linear functionals on $\mathcal{P}$. If*

$$\lim_{n \to \infty} T_n(F) = T(F)$$

for every $F \in \mathcal{P}$, then T is also continuous.

Proof. By the hypothesis, we have $\sup\{|T_n(F)| : n \geq 1\} < +\infty$ for every $F \in \mathcal{P}$, and thus, it follows from Theorem 4 that the functional T is bounded and, therefore, continuous. $\square$

References

[1] J. Kurzweil and J. Jarnik, *A convergence theorem for Henstock-Kurzweil integral and its relation to topology*, Bull. Cl. Sci., VI. Sér., Acad. R. Belg. **8** (7–12) (1997), 217–230.

[2] P. Y. Lee, *Lanzhou Lectures on Henstock Integration*, Series in Real Analysis 2, World Scientific, Singapore, 1989.

[3] W. L. Ng and P. Y. Lee, *An alternative definition of the Henstock-Kurzweil integral using primitives*, N. Z. J. Math. **48** (2018), 121–128.

[4] W. L. Ng, *Nonabsolute Integral on Measure Spaces*, Series in Real Analysis 14, World Scientific, Singapore, 2018.

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