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MANIFOLD NEIGHBOURHOODS AND A CONJECTURE OF ADJAMAGBO

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Abstract. We verify a conjecture of P. Adjamagbo that if the frontier of a relatively compact subset V_0 of a manifold is a submanifold then there is an increasing family $\{V_r\}$ of relatively compact open sets indexed by the positive reals so that the frontier of each is a submanifold, their union is the whole manifold and for each $r \geq 0$ the subfamily indexed by (r, ∞) is a neighbourhood basis of the closure of the r^{th} set. We use smooth collars in the differential category, regular neighbourhoods in the piecewise linear category and handlebodies in the topological category.

Dedicated to the memory of Vaughan Jones

Vaughan Jones was a student in one of my classes, Mathematical Analysis, at the University of Auckland in 1972. Rumours of his discovery of what came to be known as the Jones polynomial were floating around at a conference I attended in Dubrovnik in 1985: it was exciting to be a New Zealander and to learn about this work! Alas, being on a waiting list for four months to fly to the International Congress of Mathematicians in 1990 was not enough to get me to witness his award of the Fields Medal in Kyoto in 1990. A year or two later I was able to negotiate an offer for him to be appointed part time as the first Distinguished Alumnus Professor of the University of Auckland, thereby beginning over a quarter of a century of Vaughan's wonderful gift to the mathematical community in New Zealand in the form of the summer workshops and his many other contributions. He attended all of these workshops and his mixture of down-to-earth friendliness towards all and his mathematical prowess were greatly appreciated by those who attended. Sorry, Vaughan, that I couldn't attend your wind-surfing schools nor you join me on my tramps in the hills but the times we spent together are amongst my best memories.

1. Introduction

Pascal Adjamagbo, [**1**], has proposed the following conjecture[1]:

> Given a relatively compact non-empty open subset V_0 of a connected manifold M^m such that the boundary of V_0 is a submanifold, there exists an increasing family $\langle V_r \rangle_{r\in[0,\infty)}$ of relatively compact open subsets of M the boundaries of which are submanifolds such that M is the union of the

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[1] I thank Adjamagbo for drawing my attention to this conjecture.

elements of the family, and that for any $r \in [0, \infty)$, the family $\langle V_s \rangle_{s>r}$ is a fundamental system of neighbourhoods of the closure of V_r.

In this paper we verify Adjamagbo's conjecture in the differential, piecewise linear and topological categories. In the topological category, Adjamagbo makes no assumptions regarding the tameness of the boundary manifold, so it could be wild at every point; see [**6**, Theorem 2.6.1] for example. In fact in both the piecewise linear and topological categories we do not need the boundary of V_0 to be a manifold for all of the rest of the conjecture to be satisfied. While our proof in the differential category does assume that the boundary of V_0 is a manifold we can dispense with that, too, by use of, for example, handlebodies as in the proof in the topological case.

All of our manifolds are assumed to be metrisable. Indeed, the positive answer to the question we address implies that the manifold is σ-compact and that in turn implies that the manifold is metrisable if Hausdorff. As in the conjecture we also assume our ambient manifold M to be connected, but when we talk of submanifolds, such as the boundary of a manifold, we do not demand connectedness. The connectedness assumption is not entirely necessary but in addressing Adjamagbo's conjecture we can deal with components of M separately so we may as well assume connectedness. When a metric on $\mathbb{R}^n$ is required it is assumed to be the usual euclidean metric.

The differential category is easier to deal with than the topological and piecewise linear categories but it also uses a technique that helps us in the other two. We consider this case in Section 2.

In Section 3 we verify the conjecture in the topological category. An important tool used in our proof is a handlebody, a special structure on (part of) a manifold. We also discuss aspects of handlebodies in Section 3.

In many ways the PL category is similar to the topological category. It is considered in Section 4.

There is some sort of continuity in the choice of the neighbourhoods required in the conjecture: for any $r \in [0, \infty)$, the family $\langle V_s \rangle_{s>r}$ is a fundamental system of neighbourhoods of the closure of V_r. However our proofs of the conjecture also allow jumps in the sense that there is no requirement that $V_r = \cup_{s<r} V_s$ for any r, though our construction ensures that this equality holds for most r. We explore this further in Section 5.

While Adjamagbo, like almost all manifold topologists, intended that the manifolds in the conjecture should be Hausdorff it should be pointed out that the conjecture fails in the non-Hausdorff context, even in dimension 1. Indeed, begin with the topological product $\mathbb{R} \times \mathbb{N}$, where the reals $\mathbb{R}$ and the positive integers $\mathbb{N}$ carry their usual topologies, and then define an equivalence relation $\sim$ on $\mathbb{R} \times \mathbb{N}$ by declaring $(x, m) \sim (y, n)$ if and only if $(x, m) = (y, n)$ or $x = y < 0$. Then the quotient $M = \mathbb{R} \times \mathbb{N}/\sim$ is a non-Hausdorff manifold consisting of a single copy of the interval $(-\infty, 0)$ with branches at points of the form $(0, n)$ leading to infinitely many copies of the interval $(0, \infty)$. The 'origins,' ie points of the form $(0, n)$, cannot be mutually separated by disjoint neighbourhoods. Letting $V_0 = (0, 1) \times \{1\}/\sim$, which is homeomorphic to the open interval $(0, 1)$, it follows that $\overline{V_0}$ is homeomorphic to the compact interval $[0, 1]$. However any neighbourhood of $\overline{V_0}$ must contain the set $(-\varepsilon, 0) \times \mathbb{N}/\sim$ for some $\varepsilon > 0$ and hence its closure contains all of the infinitely many 'origins' $(0, n)$ so cannot be compact.

2. Adjamagbo's Conjecture in the Differential Category

Firstly we deal with the smooth case.

Lemma 1. *Let M^m be a non-compact differentiable manifold and $C \subset M$ a compact subset. Then there is a sequence $\langle M_n \rangle_{n=1}^{\infty}$ satisfying:*

- *each M_n is a compact smooth submanifold with boundary of M;*
- *$M = \bigcup_{n=1}^{\infty} M_n$;*
- *$C \subset M_1$;*
- *each M_n is contained in the interior of M_{n+1};*
- *If C is also connected then so is each M_n.*

Proof. Choose a proper, smooth function $f : M \to [0, \infty)$ such that $f(C) = 0$. By Sard's Theorem the set of critical values has Lebesgue measure 0; in particular there is an unbounded increasing sequence of regular values. Without loss of generality we may assume that for each positive integer n, the integer n is itself a regular value of f. Then for each n the set $f^{-1}(n)$ is an $(m-1)$-submanifold of M. Set $M'_n = f^{-1}([0, n])$. In the case where C is connected, if M'_n is not connected then we take connected sums along the boundaries and within $f^{-1}([0, n+1))$ of the components of M'_n to obtain M_n; otherwise just set $M_n = M'_n$. □

Theorem 2 (Smooth Category)**.** *Let M be a smooth manifold and $V_0 \subset M$ a relatively compact non-empty open subset of M such that the boundary of V_0 is a submanifold of M. Then there exists an increasing family $\langle V_r \rangle_{r \in [0,\infty)}$ of relatively compact open subsets of M the boundaries of which are submanifolds such that M is the union of the elements of the family, and that for any $r \in [0, \infty)$, the family $\langle V_s \rangle_{s>r}$ is a fundamental system of neighbourhoods of the closure of V_r. If $\overline{V_0}$ is connected then so is each V_r.*

Proof. Set $M_0 = \overline{V_0}$, a submanifold with boundary of M.

Consider firstly the case where M is compact. The boundary ∂M_0 is collared in M so there is an embedding $e_0 : \partial M_0 \times [0, 1] \to M$ such that $e_0(x, 0) = x$ for each $x \in \partial M_0$ and $e_0(\partial M_0 \times [0, 1]) \cap M_0 = \partial M_0$. For each $r \in (0, 1)$ set $V_r = M_0 \cup e_0(\partial M_0 \times [0, r))$. For $r \geq 1$ set $V_r = M$. Then the family $\langle V_r \rangle_{r \in [0,\infty)}$ satisfies the requirements of the theorem.

Now consider the case where M is non-compact. Let M_n $(n \geq 1)$ be as in Lemma 1 and, as in the compact case, choose embeddings $e_n : \partial M_n \times [0, 1] \to \mathrm{Int}(M_{n+1})$ such that $e_n(x, 0) = x$ for each $x \in \partial M_n$ and $e_n(\partial M_n \times [0, 1]) \cap M_n = \partial M_n$. For each $n = 0, 1, \ldots$ set $V_n = \mathrm{Int}(M_n)$ and for each $r \in (0, 1)$ set $V_{n+r} = M_n \cup e_n(\partial M_n \times [0, r))$. Then the family $\langle V_r \rangle_{r \in [0,\infty)}$ satisfies the requirements of the theorem. □

3. Handlebody Neighbourhoods and Adjamagbo's Conjecture in the Topological Category

We begin by recalling some facts about handlebodies.

Definition 3. *Let M^m be a topological manifold with boundary and let $k \in \{0, 1, \ldots, m\}$. Suppose that $e : \mathbb{S}^{k-1} \times \mathbb{B}^{m-k} \to \partial M$ is an embedding and let M_e be the m-manifold obtained from the disjoint union of M and $\mathbb{B}^k \times \mathbb{B}^{m-k}$ by identifying*

$x \in \mathbb{S}^{k-1} \times \mathbb{B}^{m-k}$ with $e(x) \in \partial M$. Then we say that M_e is obtained from M by adding a k-handle of dimension m to M*, with the prefix k suppressed when we do not want to specify it. The image of $\mathbb{B}^k \times \mathbb{B}^{m-k}$ in M_e is called a* (k)-handle. *A* handlebody *is a manifold obtained inductively beginning at $\varnothing$ then successively adding a handle of dimension m to the output of the previous step: if infinitely many handles are added then we demand that the handles are locally finite. If a handlebody has been obtained by adding only finitely many handles then we call it a* finite handlebody.

In this definition we take $\mathbb{B}^\ell = \{x \in \mathbb{R}^\ell \ / \ |x| \leq 1\}$, the unit ball in $\mathbb{R}^\ell$, so $\mathbb{S}^{\ell-1}$ is the boundary sphere of $\mathbb{B}^\ell$. Of course when constructing a handlebody, of necessity the first handle to be added must be a 0-handle as $\partial\varnothing = \varnothing$ and $\mathbb{S}^{k-1} \neq \varnothing$ when $k > 0$. A handlebody is a manifold with boundary.

The following two theorems characterise the existence of handlebody structures on topological manifolds.

Theorem 4. [**3**, Theorem 9.2] *A metrisable manifold fails to have a handlebody decomposition if and only if it is an unsmoothable 4-manifold.*

Theorem 5. [**3**, Theorem 8.2] *Every connected, non-compact, metrisable 4-manifold is smoothable.*

A basic result needed in our proof is the following proposition.

Proposition 6. [**4**, Proposition 3.17] *Suppose that M is a handlebody and $K \subset M$ is compact. Then there is a finite handlebody $W \subset M$ which is a neighbourhood of K.*

We also require the following weak version of the collaring theorem of Brown.

Theorem 7. [**2**] *Let W be a finite handlebody. Then there is an embedding $e : \partial W \times [a, b] \to W$ $(a < b)$ such that $e(x, b) = x$ for each $x \in \partial W$.*

The embedding e is called a *collar* of the boundary ∂W. Using the notation of Theorem 7, we will call the set $e(\partial W \times \{c\})$ a *level* of the collar and the subset $W \setminus e(\partial W \times (c, b])$ will be said to be *inside the level* $e(\partial W \times \{c\})$. A set inside a level of a handlebody is a compact subset of W; moreover the boundary of this set is a manifold of one lower dimension and, when $c > a$, the set inside the level $e(\partial W \times \{c\})$ is homeomorphic to W so is itself a finite handlebody.

In this section we construct a neighbourhood basis of a compactum in a manifold where the neighbourhoods making up the basis are all handlebodies. We then use this construction to prove Adjamagbo's conjecture.

Proposition 8. *Let M^m be a connected topological manifold and $V_0 \subset M$ a non-empty, relatively compact, open subset of M such that the frontier of V_0 in M is an $(m-1)$-manifold. Suppose further in the case where M is closed that $\overline{V_0} \neq M$. Then for each real number $r \in (0, 1)$ there is a finite handlebody W_r such that for each $r \in [0, 1)$, the collection $\{\mathrm{Int}(W_s) \ / \ s > r\}$ is a neighbourhood basis of W_r, where W_0 is the closure of V_0.*

Proof. Suppose given M^m and $V_0 \subset M$ as in Proposition 8. In the case where M is closed pick a point $p \in M \setminus \overline{V_0}$. It follows that M (in the case where M is open) or $M \setminus \{p\}$ (in the case where M is closed) may be embedded properly in some euclidean space, see [**4**, Theorem 2.1(1⇔36)] for example. Let d be the metric on M or $M \setminus \{p\}$ as appropriate inherited from the euclidean space under some fixed proper embedding.

For each natural number n let

$$U_n = \left\{ x \in M \ / \ d(x, \overline{V_0}) < \frac{1}{n} \right\}.$$

Then $\overline{U_{n+1}} \subset U_n$ for each n and the collection $\{U_n \ / \ n = 1, 2, \dots\}$ is a neighbourhood basis for $\overline{V_0}$.

For each n use Proposition 6 to find a finite handlebody $X_n \subset U_n$ containing $\overline{U_{n+1}}$ in its interior. By Theorem 7 we may find a collar $e_n : \partial X_n \times \left[\frac{1}{n+2}, \frac{1}{n}\right] \to X_n$ such that $e_n\left(x, \frac{1}{n}\right) = x$ for each $x \in \partial X_n$. Further, compactness of the disjoint sets $\overline{U_{n+1}}$ and ∂X_n allows us to assume that the image of e_n is disjoint from $\overline{U_{n+1}}$.

For each $r \in (0, 1)$ choose W_r as follows. There is a unique natural number n such that $\frac{1}{n+1} \leq r < \frac{1}{n}$. Let W_r be the set inside the level $e_n(\partial X_n \times \{r\})$ of the handlebody X_n. As noted above, W_r is a handlebody. Moreover, for each $r \in [0, 1)$, the collection $\{\mathrm{Int}(W_s) \ / \ s > r\}$ is a neighbourhood basis of W_r. □

We are now ready to prove Adjamagbo's conjecture in the topological category.

Theorem 9 (Topological Category)**.** *Let M be a topological manifold and $V_0 \subset M$ be a relatively compact non-empty open subset of M such that the boundary of V_0 is a manifold. Then there exists a family $\{V_r \ / \ r \in [0, \infty)\}$ of relatively compact open subsets of M the boundaries of which are submanifolds such that M is the union of the elements of the family, and that for any $r \in [0, \infty)$ the family $\{V_s \ / \ s > r\}$ is a fundamental system of neighbourhoods of the closure of V_r. If $\overline{V_0}$ is connected then so is each V_r.*

Proof. The case where $\overline{V_0} = M$ is trivial so we assume that $\overline{V_0} \neq M$.

Applying Proposition 8 to the case $\overline{V_0} \neq M$, for each $r \in (0, 1)$ set $V_r = \mathrm{Int}(W_r)$. Then each V_r is open and relatively compact with boundary a topological manifold, and for each $r < 1$ the family $\{V_s \ / \ s > r\}$ is a neighbourhood basis of the closure of V_r. The proof is complete in the case where M is closed if we set $V_r = M$ for each $r \geq 1$.

Suppose M is open. In this case follow the procedure in the proof of Proposition 8 but now replace the sets U_n by sets $U'_n = \left\{x \in M \ / \ d(x, \overline{V_0}) < n\right\}$ and the handlebodies X_n by finite handlebodies X'_n whose boundaries lie in $U'_{n+1} \setminus \overline{U'_n}$. For each natural number n and each $r \in [n, n+1)$ we then construct the open sets V_r to be the interiors of sets inside appropriate levels of the handlebody X'_n.

To ensure that each V_r is connected when $\overline{V_0}$ is, when we construct the handlebodies X_n, we discard any supernumerary components of the handlebodies. □

Remark 10. *Just as in the smooth and (as will be evident) piecewise linear cases, the sets V_r are all regular-open, ie $V_r = \mathrm{Int}\overline{V_r}$, for all $r > 0$.*

4. Adjamagbo's Conjecture in the Piecewise Linear Category

Finally we consider the piecewise linear case. Because the proof is similar to the topological case we just point out the differences. The main difference is that we use regular neighbourhoods, see [**7**, Section 12] for example, instead of handlebodies.

Proposition 11. *Let M^m be a connected piecewise linear manifold and $V_0 \subset M$ a non-empty, relatively compact, open subset of M such that the frontier of V_0 in M is an $(m-1)$-submanifold. Then for each real number $r \in (0,1)$ there is a piecewise linear manifold W_r such that for each $r \in [0,1)$, the collection $\{\mathrm{Int}(W_s) \ / \ s > r\}$ is a neighbourhood basis of W_r, where W_0 is the closure of V_0.*

Proof. If $\overline{V_0} = M$ then we may set $W_r = M$ for all $r \in (0,1)$. So suppose that $\overline{V_0} \neq M$.

As in the proof of Proposition 8, embed M or $M \setminus \{p\}$ in some euclidean space and for each natural number n let

$$U_n = \left\{ x \in M \ / \ d(x, \overline{V_0}) < \frac{1}{n} \right\}.$$

Compactness of $\overline{V_0}$ means that $\overline{U_{n+1}}$ is also compact so is contained in a finite union of simplices of M with the union of these simplices lying in U_n. Let $X_n \subset U_n$ be a regular neighbourhood of this union of simplices, hence a piecewise linear manifold. Then the sets W_r may be obtained just as in the proof of Proposition 8 by using a piecewise linear collar on ∂X_n. □

Theorem 12 (Piecewise Linear Category)**.** *Let M be a piecewise linear manifold and $V_0 \subset M$ a relatively compact non-empty open subset of M such that the boundary of V_0 is a submanifold. Then there exists an increasing family $\langle V_r \rangle_{r \in [0,\infty)}$ of relatively compact open subsets of M the boundaries of which are submanifolds such that M is the union of the elements of the family, and that for any $r \in [0,\infty)$, the family $\langle V_s \rangle_{s>r}$ is a fundamental system of neighbourhoods of the closure of V_r. If $\overline{V_0}$ is connected then so is each V_r.*

Proof. The proof is essentially the same as the proof of Theorem 9 but we make use of Proposition 11 rather than Proposition 8. □

5. Continuity

In this section we address the continuity of the choice of V_r in Theorems 2, 9 and 12 as r varies. Since, as observed in Remark 10, the sets V_r are regular-open and often the study of continuity involving multifunctions is restricted to the choice of closed sets, we will look at how $\overline{V_r}$ varies with r, noting that $\overline{V_r}$ is compact for all r.

For multifunctions there are two main concepts of continuity: upper semi-continuity and lower semi-continuity, both introduced by Michael in [**5**].

Definition 13. *Fix two topological spaces X and Y. A* multifunction *from X to Y is a function assigning to each point of X a subset of Y. We will use the notation $f : X \twoheadrightarrow Y$ to denote such a function. We will also restrict our multifunctions to the case where $f(x)$ is a closed subset of Y for each $x \in X$.*

Suppose that $f : X \twoheadrightarrow Y$ is a multifunction.

(i) *f is* upper semi-continuous *at $x \in X$ if for each open subset $V \subset Y$ for which $f(x) \subset V$ there is a neighbourhood U of x such that $f(\xi) \subset V$ for each $\xi \in U$. When f is upper semi-continuous at x for each $x \in X$ then f is* upper semi-continuous*.*

(ii) *f is* lower semi-continuous at $x \in X$ *if for each open subset $V \subset Y$ for which $f(x) \cap V \neq \varnothing$ there is a neighbourhood U of x such that $f(\xi) \cap V \neq \varnothing$ for each $\xi \in U$. When f is lower semi-continuous at x for each $x \in X$ then f is* lower semi-continuous.

We define the multifunction $V : [0, \infty) \twoheadrightarrow M$ by $V(r) = \overline{V_r}$ as constructed in Theorems 2, 9 and 12.

Proposition 14. *The function V is upper semi-continuous.*

Proof. Fix $r \in [0, \infty)$ and an open set $V \subset M$ such that $\overline{V_r} = V(r) \subset V$. Since $\{V_s \ / \ s > r\}$ is a fundamental neighbourhood system for $V(r)$ it follows that there is $s > r$ such that $V_s \subset V$. Let $U = [0, s)$. Then for each $t \in U$ we have $V(t) \subset V_s \subset V$. □

On the other hand in the non-compact case the function V is not lower semi-continuous at any of the points n and $\frac{1}{n}$ for n a positive integer. Indeed, suppose that r is either n or $\frac{1}{n}$ for n a positive integer. Then from the construction we have that $\overline{\bigcup_{s<r} V_s} \subsetneqq V_r$ so the open set $M \setminus \left(\overline{\bigcup_{s<r} V_s}\right)$ is an open subset of M meeting $\overline{V_r}$ but no $\overline{V_s}$ for $s < r$. Similar comments apply to most cases where M is compact.

Here is a description of an attempt to overcome this. Comments here will relate to the proofs of Proposition 8 and Theorem 9 but they may be adapted to the other two categories. In a sense what we want to do is to construct a fundamental system of neighbourhoods of $M \setminus V_1$ indexed by an interval of the form $(a, 1)$, and replacing the neighbourhoods V_r for, say, $\frac{1}{2} < r < 1$ by the complements of the closures of these neighbourhoods. Somehow we need to continue this process to fill in the gaps.

We can make the process described in the previous paragraph more precise as follows. Replace the sets U_n in the proof of Proposition 8 by the set $\{x \in M \ / \ d(x, M \setminus V_1\}$ and then the handlebodies X_n by handlebodies using these new open sets. In this way we end up with neighbourhoods V_r for, say, $\frac{1}{2} < r < 1$ that ensure lower semi-continuity at $r = 1$ but will have lost upper-semicontinuity. However if at each of the jumps in the sets V_r we remove a small interval as previously described and then insert neighbourhoods as just described to ensure lower semi-continuity or as in Proposition 8 to restore upper semi-continuity then we may hope to obtain a multifunction $V : [0, \infty) \twoheadrightarrow M$ that is both lower and upper semi-continuous. Unfortunately this procedure is doomed to fail as there can be only countable many mutually disjoint closed intervals used in the process so cannot cover the interval $[0, 1]$ as otherwise their end points will form a countable, closed, perfect set, which is impossible in a complete metric space.

This raises the question.

Question 15. *Given a relatively compact non-empty open subset V_0 of a manifold M^m such that ∂V_0 is a manifold, is it possible to find a family $\{V_r \ / \ r \in [0, \infty)\}$ such that each of the following conditions holds?*

- *each set V_r is relatively compact and open;*
- *the boundary of each set V_r is an $(m-1)$-submanifold of M;*
- $M = \bigcup_{r \geq 0} V_r$*;*
- *for each $r \in [0, \infty)$ the family $\{V_s \ / \ s > r\}$ is a fundamental system of neighbourhoods of $\overline{V_r}$;*

- *the multifunction* $V : [0,\infty) \twoheadrightarrow M$ *defined by* $V(r) = \overline{V_r}$ *is both lower and upper semi-continuous.*

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