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INVARIANT AND NATURAL PROJECTIONS INTO THE DUAL OF BANACH ALGEBRAS

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Abstract. We study the classes of invariant and natural projections in the dual of a Banach algebra A. These type of projections are relevant by their connections with the existence problem of bounded approximate identities in closed ideals of Banach algebras. It is known that any invariant projection is a natural projection. In this article we consider the issue of when a natural projection is an invariant projection.

1. Invariant and Natural Projections

Let A be a complex Banach algebra. It is known that its dual space A^* is a Banach A-bimodule. Besides A^{**} admits two products $\square, \diamond$, that turn it into eventually different Banach algebras $(A^{**}, \square)$ and $(A^{**}, \diamond)$ which contain a subalgebra isometrically isomorphic to A [**1**], [**2**].

The uniform Banach algebra $(\mathrm{B}(A^*), \circ)$ of bounded linear operators on A^* becomes a Banach A-bimodule if for $x \in A$, $T \in B(A^*)$ and $x' \in A^*$ we write xT, Tx in $B(A^*)$ as $(xT)(x') = xT(x')$ and $(Tx)(x') = T(xx')$ respectively.

We shall write ${}_A\mathrm{End}(A^*)$, $\mathrm{End}_A(A^*)$ and $\mathrm{End}(A^*)$ to the classes of left, right and bilateral bounded A-module endomorphisms of A^*.

Let $\mathrm{P}(A^*)$ be the set of projections (or idempotents) of $\mathrm{B}(A^*)$. We shall write

$$\begin{aligned} {}_A\mathrm{IP}(A^*) &\triangleq \mathrm{P}(A^*) \cap_A \mathrm{End}(A^*) \\ \mathrm{IP}_A(A^*) &\triangleq \mathrm{P}(A^*) \cap \mathrm{End}_A(A^*), \\ \mathrm{IP}(A^*) &\triangleq_A \mathrm{IP}(A^*) \cap \mathrm{IP}_A(A^*) \end{aligned}$$

to the set of *left, right* and *two sided invariant projections*, respectively. Besides, if A is abelian, let

$$\mathrm{NP}(A^*) = \{P \in \mathrm{P}(A^*) : P(h) \in \{0_{A^*}, h\} \text{ if } h \in \sigma(A)\}$$

be the set of *natural projections*, where $\sigma(A)$ denotes the carrier space of A (cf. [**7**], p. 110). In particular, we shall write $\mathrm{NP}(A^*) = \mathrm{P}(A^*)$ if $\sigma(A) = \emptyset$.

A Banach subspace S of A^* is called *left* (or *right*) *invariantly complemented* if it is the range of a left (or right) invariant projection. Besides S is called *invariantly complemented* or *naturally complemented* if it is the range of an invariant or a natural projection respectively.

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1.1. Why research invariant and natural projections? Invariant projections play a significant role concerning to the existence of bounded approximate identities in closed ideals of Banach algebras (cf. [**4**], Theorem 1 and [**3**], Proposition 6.4). On the other hand, natural projections are related to the notion of weak bounded approximate identities ([**5**], p. 4164), i.e. given a closed ideal I of a Banach algebra A, a weak bounded approximate identity of I consists of a net $\{u_i\} \subseteq I$ so that $\langle u_i, \mathfrak{h}\rangle \to 1$ for each $\mathfrak{h} \in \sigma(I)$.

1.2. Every invariant projection is a natural one. (cf. [**5**], Lemma 3.2) Let A be an abelian complex semisimple Banach algebra endowed with a bounded approximate identity. Given $P \in \mathrm{P}(A^*)$, P is a natural projection if and only if for each $h \in \sigma(A)$ and $x \in A$, $P(xh) = xP(h)$. Consequently $\mathrm{IP}(A^*) \subseteq \mathrm{NP}(A^*)$.[1]

Now, if $x \in A$ and $h \in \sigma(A)$ it is readily seen that $xh = \langle x, h\rangle h$. Hence if $P \in \mathrm{NP}(A^*)$, $P(xx') = xP(x')$ if $x \in A$ and $x' \in \mathrm{span}(\sigma(A))^-$. Thus we have that $\mathrm{span}(\sigma(A)) \subseteq A^*A = AA^*$.

Further, these spaces are equal if besides $\kappa_A(A)$ is an ideal of A^{**}, where $\kappa_A : A \hookrightarrow A^{**}$ is the usual isometric immersion of A into A^{**} ([**5**], p. 4164).

Remark 1.1. There may exist natural non-invariant projections. For, let us suppose that $A^*A \neq A^*$, for instance if $(A^{**}, \Box)$ (or $(A^{**}, \Diamond)$) is non-unitary (cf. [**6**], Proposition 2.2). As A has a bounded approximate identity by Cohen's factorization theorem A^*A becomes closed. If $x_0' \in A^* - A^*A$ by the Hahn-Banach theorem there exists $x_0'' \in (A^*A)^\perp$ so that $\langle x_0', x_0''\rangle = 1$. Let $P : A^* \to A^*$ so that $P(x') = \langle x', x_0''\rangle x_0'$ if $x' \in A^*$. Clearly $P \in \mathrm{B}(A^*)$ and $P^2 = P$. Now, P is a natural projection because $P(\sigma(A)) = \{0_{A^*}\}$. However, if P were invariant $\langle xx', x_0''\rangle x_0' = \langle x', x_0''\rangle xx_0'$ for all $x \in A$ and $x' \in A^*$. But $\langle xx', x_0''\rangle = 0$, i.e. $xx_0' = 0_{A^*}$ for all $x \in A$ or $\langle x', x_0''\rangle = 0$ for all $x' \in A^*$, both give contradiction.

1.3. Our matter and main results. We consider the issue of when a natural projection is an invariant projection. Our main result is Theorem 2.1 and the consequential Corollaries 2.2 and 2.3.

2. The Structure of Invariant Projections

In what follows, if a Banach algebra A has a left, right or a 2-sided bounded approximate identity we shall simply write $A \in LBAI$, $A \in RBAI$ or $A \in BAI$ respectively. The first and second conditions are equivalent to the existence of a left or a right unit in $(A^{**}, \Diamond)$ and $(A^{**}, \Box)$ respectively, and the third one to the existence of a *mixed unit* $E \in A^{**}$, i.e.

$$x''\Box E = E\Diamond x'' = x'' \text{ for all } x'' \in A^{**}.$$

Theorem 2.1. *Let A be a complex Banach algebra.*

(1) *If $A \in RBAI$ there exists a conjugate linear operator $\Psi : B(A^*) \to A^{**}$ so that $\Psi \mid End_A(A^*)$ defines a Banach algebra monomorphism between $(End_A(A^*), \circ)$ and $(A^{**}, \Box)$.*

[1] As the reviewer pointed to us, every invariant projection is a normal projection in the more general setting of abelian Banach algebras through the following argument:
Given $P \in \mathrm{IP}(A^*)$, $P \in \mathrm{NP}(A^*)$ if and only if $xP(h) = \langle x, h\rangle P(h)$ if $x \in A$ and $h \in \sigma(A)$. The necessity follows immediately. On the other hand, given $h \in \sigma(A)$ and $x, y \in A$ the condition implies that $\langle xy, P(h)\rangle = \langle x, h\rangle\langle y, P(h)\rangle$. As A is commutative $\langle x, h\rangle\langle y, P(h)\rangle = \langle y, h\rangle\langle x, P(h)\rangle$ and there exists $c_h \in \mathbb{C}$ so that $P(h) = c_h h$. Finally the assertion follows because $P^2 = P$.

(2) *If $A \in LBAI$ there exists a conjugate linear operator $\Phi : B(A^*) \to A^{**}$ so that $\Phi \mid_A End(A^*)$ defines a Banach algebra monomorphism between $({}_A End(A^*), \circ)$ and $(A^{**}, \diamond^{op})$.*

Proof. (1) Let $A \in RBAI$ and let F be a right unit in $(A^{**}, \Box)$. We define $\Psi : \mathrm{B}(A^*) \to A^{**}$ so that $\Psi(T) = T^*(F)$ whenever $T \in \mathrm{B}(A^*)$. Plainly, Ψ is a conjugate linear functional and it is bounded because $\|\Psi\| \le \|F\|$.
Let $S, T \in \mathrm{End}_A(A^*)$, $x' \in A^*$, $x \in A$. Then

$$\begin{aligned} \langle x, T^*(F)x' \rangle &= \langle x'x, T^*(F) \rangle \\ &= \langle T(x'x), F \rangle \\ &= \langle T(x')x, F \rangle \\ &= \langle T(x'), \kappa_A(x) \Box F \rangle \\ &= \langle x, T(x') \rangle, \end{aligned}$$

hence we have $T^*(F)x' = T(x')$. Thus

$$\begin{aligned} \langle x', \Psi(S \circ T) \rangle &= \langle x', (S \circ T)^*(F) \rangle \\ &= \langle S(T(x')), F \rangle \\ &= \langle T(x'), S^*(F) \rangle \\ &= \langle T^*(F)x', S^*(F) \rangle \\ &= \langle x', S^*(F) \Box T^*(F) \rangle \\ &= \langle x', \Psi(S) \Box \Psi(T) \rangle \end{aligned}$$

and $\Psi \mid \mathrm{End}_A(A^*)$ is multiplicative. Further, let $\mathrm{L}_\Box : A^{**} \to \mathrm{B}(A^*)$ so that $\mathrm{L}_\Box(x'')(x') = x''x'$ if $x' \in A^*$ and $x'' \in A^{**}$. Then

$$(\mathrm{L}_\Box \circ \Psi)(T)(x') = \Psi(T)x' = T^*(F)x' = T(x').$$

Hence

$$\mathrm{L}_\Box \circ \Psi \mid \mathrm{End}_A(A^*) = \mathrm{Id}_{\mathrm{End}_A(A^*)} \tag{2.1}$$

and the assertion follows.

(2) Analogously, if $A \in LBAI$ we choose a left unit E in $(A^{**}, \diamond)$. The map $\Phi : \mathrm{B}(A^*) \to A^{**}$ so that $\Phi(T) = T^*(E)$ if $T \in \mathrm{B}(A^*)$ defines a conjugate linear operator whose restriction to ${}_A\mathrm{End}(A^*)$ is a Banach algebra homomorphism between $({}_A\mathrm{End}(A^*), \circ)$ and $(A^{**}, \diamond^{\mathrm{op}})$. Now, let $\mathrm{R}_\diamond : A^{**} \to \mathrm{B}(A^*)$, $\mathrm{R}_\diamond(x'')(x') = x'x''$ if $x' \in A^*, x'' \in A^{**}$. Then

$$\mathrm{R}_\diamond \circ \Phi \mid_A \mathrm{End}(A^*) = \mathrm{Id}_{{}_A\mathrm{End}(A^*)},$$

i.e. $\Phi \mid_A \mathrm{End}(A^*)$ becomes injective.

$\Box$

Corollary 2.2. *Let A be a complex Banach algebra and $P \in P(A^*)$.*

(1) *If $A \in RBAI$ and $P \in IP_A(A^*)$ there is an idempotent $y''_P \in (A^{**}, \Box)$ and $P(x') = y''_P x'$ if $x' \in A^*$.*

(2) *If $A \in LBAI$ and $P \in_A IP(A^*)$ there is an idempotent $x''_P \in (A^{**}, \diamond)$ and $P(x') = x'x''_P$ if $x' \in A^*$.*

(3) *If $A \in BAI$ and $P \in IP(A^{**})$ there is $z''_P \in A^{**}$ so that $z''_P = z''_P \Box z''_P = z''_P \diamond z''_P$ and $P(x') = z''_P x' = x' z''_P$ if $x' \in A^*$.*

Proof. (1) With the notation of Theorem 2.1(1), let $y''_P = \Psi(P)$ for a given $P \in \mathrm{IP}_A(A^*)$. Since P is an idempotent and $\Psi \mid \mathrm{End}_A(A^*)$ is multiplicative y''_P becomes an idempotent in $(A^{**}, \Box)$. The conclusion follows by (2.1).

(2) The proof is analogous in the context of Theorem 2.1(2).

(3) It follows as the conjunction of the previous points.

□

Corollary 2.3. *Let A be an abelian Banach algebra endowed with bounded approximate identity. If $P \in NP(A^*)$, then $P \in IP(A^*)$ if and only if there is $\Phi \in A^{**}$ so that $\Phi \Box \Phi = \Phi \Diamond \Phi = \Phi$ and $P(x') = \Phi x' = x' \Phi$ if $x' \in A^*$.*

Remark 2.4. The idempotent y''_P in Corollary 2.2 may not be unique.

For instance, if $y''_P x' = z'' x'$ for every $x' \in A^*$ and some idempotent $z'' \in (A^{**}, \Box)$ then $z'' - y''_P \in (A^* A)^{\perp}$. Since in this case $A \in RBAI$ by Cohen's factorization theorem $(A^* A)^{\perp} = (0_{A^{**}})$ if and only if $A^* = A^* A$. Thus y''_P is uniquely determined if and only if $A^* A = A^*$.

Analogously, the idempotent x''_P in Corollary 2.2 is uniquely determined if and only if $AA^* = A^*$.

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