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TURING DETERMINACY AND SUSLIN SETS

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Abstract. The relationship between the Axiom of Determinacy (AD) and the Axiom of Turing Determinacy has been open for over 50 years, and the attempts to understand that relationship have had a profound influence on Set Theory in a variety of ways. The prevailing conjecture is that these two determinacy hypotheses are actually equivalent, and the main theorem of this paper is that Turing Determinacy implies that every Suslin set is determined.

1. Introduction

The Axiom of Determinacy (AD) was introduced 60 years ago by Mycielski and Steinhaus [**6**] as an alternative to the Axiom of Choice. The modern view is that AD is an axiom for various inner models of the Universe of Sets, V, which contain all the real numbers (and all the ordinals). The simplest such inner model is $L(\mathbb{R})$ which is obtained by relativizing Gödel's construction of L to the reals, $\mathbb{R}$. As is standard in the theory of AD, we identify $\mathbb{R}$ with ω^ω, which with the product topology is homeomorphic to the Euclidean space of irrational numbers. Therefore strictly speaking

$$L(\mathbb{R}) = L(V_{\omega+1})$$

where $V_{\omega+1} = \mathcal{P}(V_\omega)$, which is the set of all $X \subseteq V_\omega$, and where V_ω is the union of all the finite transitive sets.

Thus in the development of the theory of AD, theorems can be viewed as theorems about inner models, such as $L(\mathbb{R})$ in which AD holds; or as theorems of ZF + AD + $\mathrm{DC}_\mathbb{R}$, where DC is the Axiom of Countable Dependent Choice, and $\mathrm{DC}_\mathbb{R}$ is DC restricted to partial orders on $\mathbb{R}$. The point is that if N is an inner model of V which contains the reals, then necessarily $N \vDash \mathrm{DC}_\mathbb{R}$.

Alternatively if

$$N \vDash \mathrm{ZF} + \mathrm{DC}$$

and $N \vDash$ "$V = L(\mathcal{P}(\mathbb{R}))$", then there is a partial order $\mathbb{P}$ such that

$$N^\mathbb{P} \vDash \mathrm{ZFC}$$

and such that $N^\mathbb{P}$ adds no new reals. Here $L(\mathcal{P}(\mathbb{R}))$, which is really $L(V_{\omega+2})$, is the generalization of Gödel's construction of L to the set of all sets of reals, $\mathcal{P}(\mathbb{R})$, and $N^\mathbb{P}$ denotes the Cohen extension of N given by $\mathbb{P}$.

Usually the context for the study of AD is ZF + $\mathrm{DC}_\mathbb{R}$ together with the assumption that $V = L(\mathcal{P}(\mathbb{R}))$, and this therefore explains the duality in perspectives (ignoring the distinction between DC and $\mathrm{DC}_\mathbb{R}$).

Suppose $x, y \in \mathbb{R}$. Then $x \leq_\mathbb{T} y$ if x is Turing reducible to y in the sense that x Turing computable with an oracle for y. If $x \leq_\mathbb{T} y$ and $y \leq_\mathbb{T} x$ then x and y are Turing equivalent, and this is an equivalence relation on $\mathbb{R}$.

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For each $x \in \mathbb{R}$, the Turing degree of x, denoted $[x]_{\mathbb{T}}$, is the corresponding equivalence class. The set of all Turing degrees, denoted $\mathcal{D}$, is a naturally a partial order where $[x]_{\mathbb{T}} \leq_{\mathbb{T}} [y]_{\mathbb{T}}$ if $x \leq_{\mathbb{T}} y$.

A Turing cone, is a set $X \subseteq \mathcal{D}$ which is of the form

$$X = \{d \in \mathcal{D} \mid d_0 \leq_{\mathbb{T}} d\}$$

for some Turing degree d_0, and the Turing cones generate a countably complete filter on (the subsets of) $\mathcal{D}$.

A striking early application of AD is the observation of Martin that assuming AD, if $X \subseteq \mathcal{D}$ is a set of Turing degrees, then either X or its complement $\mathcal{D} \backslash X$ must contain a Turing cone, see [**5**] for an overview of the development of the study of AD. Thus assuming AD, the simplest filter on $\mathcal{D}$ is an ultrafilter.

Turing Determinacy is the axiom which simply asserts that the conclusion of Martin's observation holds. Thus Turing Determinacy is an elegant and natural variation of AD, and the main open question is whether in ZF+$\mathrm{DC}_{\mathbb{R}}$, Turing Determinacy *implies* AD. Evidence that the answer is yes was provided by the following theorem from 1982. Unfortunately there has been little further progress on the general question of the equivalence of AD with Turing Determinacy since then.

Theorem 1.1. *The following are equivalent.*
(1) $L(\mathbb{R}) \vDash$ AD.
(2) $L(\mathbb{R}) \vDash$ Turing Determinacy. □

The proof of Theorem 1.1 has never been published, and there are now other proofs of closely related theorems, proved using the modern machinery which has been developed in the ensuing decades. This has been developed to the point where the actual theorem, Theorem 1.1, has now been proved by these methods [**1**]. Thus the original proof became in some sense more and more irrelevant to the general development of the subject since the subsequent alternative proofs arguably provided deeper insights.

Our main theorem is the following theorem and the key new feature is that there is *no* restriction on the complexity (such as requiring A belong to a canonical inner model generalizing $L(\mathbb{R})$) of the sets $A \subseteq \mathbb{R}$ to which the theorem applies. The notion that a set $A \subseteq \mathbb{R}$ be *Suslin* is a fundamental and central notion in the subject of AD, and it is a transfinite generalization of the notion of an analytic set, [**5**].

Theorem 1.2 (ZF + $\mathrm{DC}_{\mathbb{R}}$)**.** *Assume Turing Determinacy. Then every Suslin set is determined.* □

The key result of [**4**] is the following theorem, and the main theorem of [**4**] was obtained as an immediate corollary of this theorem. The proof of Theorem 1.1 used a much more technical variation of Theorem 1.3, also from [**4**], and this in part has blocked generalizing the original proof of Theorem 1.1, in the attempt to obtain further progress on the basic problem of Turing Determinacy versus AD.

Theorem 1.3 ([**4**])**.** *The following are equivalent.*
(1) $L(\mathbb{R}) \vDash$ AD.
(2) $L(\mathbb{R}) \vDash$ "Every Suslin set is determined". □

Thus Theorem 1.1 is an immediate corollary of Theorem 1.2, and this gives an entirely new proof of Theorem 1.1 which should be more amenable to generalization.

Remark 1.4. By [**2**], the assumption that every Suslin set is determined is relatively consistent with the existence of a non-principal ultrafilter on $\mathbb{N}$, and so the equivalence in the conclusion of Theorem 1.3 needs the restriction to a canonical inner model such as $L(\mathbb{R})$.□

2. Turing Determinacy and the Determinacy of Suslin Sets

We fix some conventions and notation. Suppose $\lambda \in \mathrm{Ord}$. A set T is a tree on $\omega \times \lambda$, if

$$T \subseteq \omega^{<\omega} \times \lambda^{<\omega}$$

and for each $(s, t) \in T$, $\mathrm{dom}(s) = \mathrm{dom}(t)$ and

$$(s|k, t|k) \in T$$

for all $k \in \mathrm{dom}(s)$. For each $s \in \omega^{<\omega}$, $T_s = \{t \in \lambda^{<\omega} \mid (s, t) \in T\}$.

The set of infinite branches of T is denoted $[T]$, and this is the set of all $(x, f) \in \omega^\omega \times \lambda^\omega$ such that

$$\{(x|k, f|k) \mid k < \omega\} \subseteq T.$$

The projection of T, this is denoted $p[T]$, is the set of $x \in \omega^\omega$ such that $(x, f) \in [T]$ for some $f \in \lambda^\omega$. For each $x \in \omega^\omega$,

$$T_x = \cup\{T_{x|k} \mid k < \omega\} \subseteq \lambda^{<\omega}.$$

Thus T_x is a tree on λ, and $x \in p[T]$ if and only if T_x has an infinite branch.

Finally for each $x \in p[T]$, f^T_x is the "left-most" infinite branch of T_x. This is the lexicographically least element $f \in \lambda^\omega$ such that $(x, f) \in [T]$.

A set $A \subseteq \omega^\omega$ is λ-Suslin if there is a tree T on $\omega \times \lambda$, in the above sense, such that

$$A = p[T].$$

Assuming the Axiom of Choice, every set $A \subseteq \mathbb{R}$ is λ-Suslin where $\lambda = |\mathbb{R}|$, but for example a set $A \subset \mathbb{R}$ is ω-Suslin if and only if A is an analytic set. Finally, a set $A \subseteq \omega^\omega$ is Suslin if A is λ-Suslin for some $\lambda \in \mathrm{Ord}$.

The first step in the original proof of Theorem 1.1 involved proving the following theorem, and the proof used, in key ways, that every Σ^1_2-set is the projection of a tree $T \in L$.

Note that the hypothesis of Theorem 2.1 must hold in V, if there exists an inner model

$$N \vDash \mathrm{ZF} + \text{Turing Determinacy}$$

such that $\mathbb{R} \subset N$ and such that $\mathrm{Ord} \subset N$. This shift in perspective in quite useful.

Theorem 2.1. *Suppose that the following hold where for each $x \in \mathbb{R}$, $\mathrm{Th}_2^{L_{\omega_1}[x]}$ is the Σ_2-theory of $L_{\omega_1}[x]$.*
(i) *ω_1 is strongly inaccessible in $L[x]$ for all $x \in \mathbb{R}$.*
(ii) *$\mathrm{Th}_2^{L_{\omega_1}[x]}$ is constant on a Turing cone.*
Then every Σ^1_2-set is determined. □

Here again there is now a completely different proof of Theorem 2.1 using modern methods, and in this case it is the core model theory of Jensen and Steel [**3**]. By [**3**], if there is no inner model with a Woodin cardinal then there is an inner model $K \subset V$ such that K is both Δ_2-definable in V and in every generic extension of V, and such that $\lambda^+ = (\lambda^+)^K$ for a proper class of limit cardinals λ.

We sketch how one can prove Theorem 2.1 by using the results of Jensen and Steel. Fix a real x_0 such that

$$\mathrm{Th}_2^{L_{\omega_1}[x]} = \mathrm{Th}_2^{L_{\omega_1}[x_0]}$$

for all $x \in \mathbb{R}$ such that $x_0 \in L[x]$ and assume toward a contradiction that

$$L_{\omega_1}[x_0] \vDash \text{"There is no inner model with a Woodin cardinal".}$$

Now apply the existence of K in

$$L_{\omega_1}[x_0] \vDash \mathrm{ZFC}$$

and consider the Σ_2-sentence which expresses:

"There exists a closed unbounded set $C \subset \omega_1$ of ordinals all of which have the same cofinality in K".

By the generic invariance of K and the fact that K computes λ^+ correctly for a proper class of λ, this Σ_2-sentence can be forced over $L_{\omega_1}[x_0]$ to be true and it can be forced over $L_{\omega_1}[x_0]$ to be false, (both by generically adding a real to $L_{\omega_1}[x_0]$), which contradicts the choice of x_0.

Here we also use from [**3**] that if there is no inner model with a Woodin cardinal then $K \cap V_\theta$ is uniformly definable in V_θ for all uncountable cardinals θ such that $\theta = |V_\theta|$ (with a definition which is independent of V).

Therefore

$$L_{\omega_1}[x_0] \vDash \text{"There is an inner model with a Woodin cardinal"}.$$

But then Σ^1_2-Determinacy holds in a generic extension of $L_{\omega_1}[x_0]$ and so it follows that Σ^1_2-Determinacy holds in V since Σ^1_2-Determinacy is upwards absolute, see the commnets on page 862.

This last step uses the following theorem from 1987, and a much stronger version of this theorem was later proved by Neeman [**7**].

Theorem 2.2. *Suppose δ is a Woodin cardinal and that $G \subset \mathrm{Coll}(\omega, \delta)$ is V-generic. Then*

$$V[G] \vDash \Sigma^1_2\text{-Determinacy}. \qquad \square$$

Our main theorem follows from Theorem 2.4 (stated and proved after the next lemma), which generalizes Theorem 2.1 to arbitrary trees, T, without making any additional assumptions about the trees. The original proof of Theorem 1.1, required that the tree be associated to a scaled pointclass (in the sense defined in [**5**]) and so needed the appropriate refinement of Theorem 1.3.

The point is that the orignal proof of Theorem 2.1 exploited that the inner models $L[x]$ have a rich internal structure and this is not true in general for the inner models $L[T][x]$.

However if the tree T is associated to a scaled pointclass, and assuming enough sets related to that pointclass are determined, then the inner models $L[T][x]$ again have a rich internal structure. This was the approach in the proof of Theorem 1.1.

We need a rather technical lemma and for this lemma we use the following notation. Suppose that there is a wellordering of $H(\omega_2)$ of length ω_2 which is definable from parameters in $H(\omega_2)$. Then for each set $Z \subset H(\omega_2)$, S_Z denotes the set given by the first ω_2-many ordinals $\eta > \omega_2$ such that

$$L_\eta\Big(H(\omega_2), Z\Big) \vDash \mathrm{ZFC}\backslash\mathrm{Replacement}.$$

Note that for each $\eta \in S_Z$, $S_Z \cap \eta$ is bounded in η and so

$$S_Z \cap \eta \in L_\eta\Big(H(\omega_2), Z\Big).$$

Lemma 2.3 (CH). *Suppose $\diamond(\omega_2)$ holds and is witnessed by a sequence which is definable from parameters in $H(\omega_2)$. Suppose*

$$\langle L_i : i < \omega\rangle$$

is an increasing sequence of subsets of ω_2 such that

$$|L_i| \le \omega_1$$

for all $i < \omega$. Then there exists a sequence $\langle B_i : i < \omega\rangle$ of subsets of ω_2 such that the following holds where for each $i < \omega$,

$$Z_i = \{(k, \alpha) \mid i \le k < \omega \text{ and } \alpha \in B_k\}.$$

(1) *For each $k < i < \omega$, the structure $(L_i, L_k, \in)$ is isomorphic to an initial segment of the structure $(S_{Z_i}, S_{Z_k}, \in)$.*

Proof. Define a subtree $T \subset 2^{<\omega_2}$ to be a *club-full* tree if the following hold for some closed unbounded set $C \subset \omega_2$.

(1.1) For each $\alpha < \omega_2$ and for all $s, t \in T$, if $\mathrm{dom}(s) = \alpha = \mathrm{dom}(t)$ and if

$$s \restriction (C \cap \alpha) = t \restriction (C \cap \alpha)$$

then $s = t$.

(1.2) Suppose $\pi : C \to \{0, 1\}$. Then for each $\alpha \in C$, there exists $s \in T$ such that $\mathrm{dom}(s) = \alpha$ and $\pi \restriction (C \cap \alpha) = s \restriction (C \cap \alpha)$.

Thus if T is a club-full tree then T is an $(<\omega_2)$-closed subtree of $2^{<\omega_2}$. Further the witness C is uniquely specified by T as the set of all $\alpha < \omega_2$ such that there exist $s, t \in T$ such that $\alpha \in \mathrm{dom}(s) \cap \mathrm{dom}(t)$, $s \restriction \alpha = t \restriction \alpha$, and such that $s(\alpha) \neq t(\alpha)$.

We first prove the following claim.

(2.1) Suppose $\langle T_\alpha : \alpha < \omega_1 \rangle$ is a sequence of club-full trees such that that $T_\beta \subseteq T_\alpha$ for all $\alpha < \beta < \omega_1$, and let

$$T = \bigcap_{\alpha < \omega_1} T_\alpha.$$

Then T is a club-full tree.

For each $\alpha < \omega_2$, let C_α be the closed unbounded subset of ω_2 which witnesses that T_α is a club-full tree. Thus $C_\beta \subseteq C_\alpha$ for all $\alpha < \beta < \omega_1$ since the witnesses are uniquely specified as indicated above. Let

$$C = \bigcap_{\alpha < \omega_1} C_\alpha.$$

We prove C witnesses T is a club-full tree. Fix

$$\pi : C \to \{0, 1\}.$$

We prove by induction on $\xi \in C$ that there exists a unique $s \in T$ such that

(3.1) $\mathrm{dom}(s) = \xi$,

(3.2) $s \restriction (C \cap \xi) = \pi \restriction (C \cap \xi)$.

This is an easy consequence of the definitions. For example, suppose ξ_0 is the least element of C and for each $\alpha < \omega_1$, let ξ_0^α be the least element of C_α. Thus for each $\alpha < \omega_1$, there exists uniquely $s_\alpha \in T_\alpha$ such that $\mathrm{dom}(s_\alpha) = \xi_0^\alpha$.

The key point is that for all $\alpha \leq \beta < \omega_1$, $\xi_0^\alpha \leq \xi_0^\beta$ and

$$s_\beta \restriction \xi_\alpha^0 = s_\alpha.$$

Clearly, one of the following must hold.

(4.1) $\xi_0^\alpha = \xi_0$ for all sufficiently large $\alpha < \omega_1$.

(4.2) For all $\alpha < \omega_1$, $\xi_0^\alpha < \xi_0$ and $\xi_0 = \sup\left\{\xi_0^\alpha \mid \alpha < \omega_1\right\}$.

Suppose (4.1) holds. Then there exists $\alpha < \omega_1$ such that $s_\alpha = s_\beta$ for all $\alpha < \beta < \omega_1$, and so $s_\alpha \in T$. Now suppose (4.2) holds and let

$$s = \cup \left\{s_0^\alpha \mid \alpha < \omega_1\right\}.$$

Thus $\mathrm{dom}(s) = \xi_0$ and for all $\eta < \xi_0$, $s \restriction \eta \in T$. Therefore $s \in T$ since T is $(<\omega_2)$-closed.

This gives existence, and the argument for uniqueness is similar. The induction steps are essentially the same when $\xi \in C$ is not a limit point of C, and if $\xi \in C$ is a limit point of C then the existence and uniqueness of s is immediate by the induction hypothesis since T is $(<\omega_2)$-closed. This proves (2.1)

Let Θ be the ordertype of $\cup\{L_i : i < \omega\}$ and for each $i < \omega$, let $\hat{L}_i \subset \Theta$ be the image of L_i under the isomorphism of $\cup\{L_i : i < \omega\}$ with Θ.

For each $\alpha < \omega_2$, let η_α be the α-th ordinal η such that

$$L_\eta(H(\omega_2)) \vDash \text{ZFC}\backslash\text{Replacement}.$$

For each club-full tree T, let $[T]$ be the set of cofinal branches of T, this is the set of all $b \in 2^{\omega_2}$ such that $b\restriction\alpha \in T$ for all $\alpha < \omega_2$. Since T is $(<\omega_2)$-closed every element $s \in T$ can be extended to a cofinal branch of T.

We note the following.

(5.1) Suppose T is a club-full tree, $C \subseteq \omega_2$ witnesses that T is a club-full tree, and that $D \subset C$ is closed and unbounded. Suppose

$$\pi : C\backslash D \to \{0, 1\}$$

and let T^* be the set $s \in T$ such that $s(\alpha) = \pi(\alpha)$ for all $\alpha \in \text{dom}(s)$ such that $\alpha \in C\backslash T$. Then T^* is a club-full tree with witness D.

We define by induction on $\alpha < \Theta$, a sequence $\langle T_i^\alpha : i < \omega\rangle$ of club-full trees such that the following hold for all $\alpha < \beta < \Theta$.

(6.1) $\langle T_i^\alpha : i < \omega\rangle \in L_{\eta_\alpha}(H(\omega_2))$

(6.2) Let i_α be the least i such that $\alpha \in \hat{L}_i$ and suppose $\langle b_k : k < \omega\rangle$ is a sequence such that $b_k \in [T_k^{\alpha+1}]$ for all $k < \omega$. Then the following hold where for each $k < \omega$, C_k^α is the witness that T_k^α is a club-full tree and $C_k^{\alpha+1}$ is the witness that $T_k^{\alpha+1}$ is a club-full tree.

a) $C_k^\alpha\backslash C_k^{\alpha+1}$ contains all the successor points of C_k^α.

b) Let A be the successor points of C_k^α. If $k < i_\alpha$, then the element of 2^{ω_2} defined by $b_k\restriction A$ and the order isomorphism of $(A, <)$ with $(\omega_2, <)$, codes a wellordering of ω_2, of length η_α.

c) $\langle b_k : k \geq i_\alpha\rangle$ is $L_{\eta_\alpha}(H(\omega_2))$-generic for the partial order given by the infinite product with full support

$$\prod_{i_\alpha \leq k < \omega} T_k^\alpha.$$

(6.3) $T_i^\beta \subseteq T_i^\alpha$ for all $i < \omega$.

Fix a sequence $\langle\sigma_\xi : \xi < \omega_2\rangle$ which witnesses $\diamond(\omega_2)$ and which is definable from parameters in $H(\omega_2)$, and let $<$ be the wellordering of $H(\omega_2)$ given by $\langle\sigma_\xi : \xi < \omega_2\rangle$.

Thus for each $\alpha < \Theta$, there is a wellordering of $L_{\eta_\alpha}(H(\omega_2))$ which is uniformly definable in $L_{\eta_\alpha}(H(\omega_2))$ from $\langle\sigma_\alpha : \alpha < \omega_2\rangle$.

We can now uniformly define by recursion $\langle T_i^\alpha : i < \omega\rangle \in L_{\eta_\alpha}(H(\omega_2))$ to be the least sequence which satisfies the requirements given

$$\left(\langle T_i^\xi : i < \omega\rangle : \xi < \alpha\right)$$

The only issue is at the successor steps $\alpha + 1$, and defining

$$\langle T_k^{\alpha+1} : i_\alpha \leq k < \omega\rangle.$$

But here one can use the $\diamond(\omega_2)$-sequence and a bijection

$$F : \omega_2 \to L_{\eta_\alpha}(H(\omega_2))$$

with $F \in L_{\eta_{\alpha+1}}(H(\omega_2))$, to guess both the sequence of branches and $\xi < \omega_2$, and then refine the trees $\langle T_k^\alpha : i_\alpha \leq k < \omega\rangle$ to ensure that if $F(\xi)$ is an open-dense set in the product partial order

$$\prod_{i_\alpha \leq k < \omega} T_k^\alpha$$

then the filter given by $\langle b_k : i_\alpha \le k < \omega\rangle$ meets this open-dense set.

We now give the details. Fix a surjection

$$F : \omega_2 \to L_{\eta_\alpha}(H(\omega_2))$$

with $F \in L_{\eta_{\alpha+1}}(H(\omega_2))$.

Working in $L_{\eta_{\alpha+1}}(H(\omega_2))$, we define a decreasing sequence

$$\langle C_\eta : \eta < \omega_2\rangle$$

of closed unbounded subsets of ω_2 and a sequence

$$\langle S^\eta_k : i_\alpha \le k < \omega, \eta < \omega_2\rangle$$

of club-full trees, by induction on $\eta < \omega$ such that the following hold.

(7.1) For all $\eta < \xi < \omega_2$, $C_\xi \cap \eta = C_\eta \cap \eta$.

(7.2) For all $\eta < \omega_2$ and for all $i_\alpha \le k < \omega$, $S^\eta_k \subseteq T^\alpha_k$ and C_η witnesses that S^η_k is club-full.

(7.3) For all $\eta < \xi < \omega_2$ and for all $i_\alpha \le k < \omega$, $S^\xi_k \subseteq S^\eta_k$ and

$$S^\xi_k \cap \{0,1\}^\eta = S^\eta_k \cap \{0,1\}^\eta .$$

(7.4) For all $\eta < \omega_2$, $\langle C_\xi : \xi < \eta\rangle \in L_{\eta_\alpha}(H(\omega_2))$.

(7.5) For all $\eta < \omega_2$, $\langle S^\xi_k : i_\alpha \le k < \omega, \xi < \eta\rangle \in L_{\eta_\alpha}(H(\omega_2))$.

Let C_0 be the limit points of the closed unbounded set $\cap\left\{C^\alpha_k \mid i_\alpha \le k < \omega\right\}$ and for each $i_\alpha \le k < \omega_1$, let S^0_k be the set of all $p \in T^\alpha_k$ such that $p(\xi) = 0$ for all $\xi \in C^\alpha_k \backslash C_0$.

We continue by induction on $\gamma < \omega_2$ to define

$$\langle C_\eta : \eta < \gamma\rangle$$

$$\langle S^\eta_k : i_\alpha \le k < \omega, \eta < \gamma\rangle$$

We can reduce to case that σ_γ guesses

$$(\langle b_k\restriction\gamma : i_\alpha \le k < \omega\rangle, \xi)$$

and that

(8.1) $\gamma \in \cap\left\{C_\eta \mid \eta < \gamma\right\}$,

(8.2) $b_k\restriction\gamma \in \cap\left\{S^\eta_k \mid \eta < \gamma\right\}$ for all $i_\alpha \le k < \omega$.

(8.3) $\xi < \gamma$ and $F(\xi)$ is open-dense in $\prod_{i_\alpha \le k<\omega} T^\alpha_k$.

Otherwise, we define

$$C_\gamma = \cap\left\{C_\eta \mid \eta < \gamma\right\}$$

and for each $i_\alpha \le k < \omega$, we define

$$S^\gamma_k = \cap\left\{S^\eta_k \mid \eta < \gamma\right\}.$$

Choose the <-least sequence

$$\langle (b^0_k, b^1_k) : i_\alpha \le k < \omega\rangle \in H(\omega_2)$$

such that the following hold.

(9.1) $b^0_k\restriction(\gamma+1) = (b_k\restriction\gamma)^\frown 0$ and $b^1_k\restriction(\gamma+1) = (b_k\restriction\gamma)^\frown 1$.

(9.2) For all $h : \{k < \omega \mid i_\alpha \le k\} \to \{0,1\}$,

$$\langle b^{h(k)}_k : i_\alpha \le k < \omega\rangle \in F(\xi).$$

There are only ω_1-many possibilities for h since CH holds and so $\langle (b^0_k, b^1_k) : i_\alpha \le k < \omega \rangle$ exists.

Now define

$$C_\gamma = \left(\cap \left\{ C_\eta \cap \gamma \mid \eta < \gamma \right\} \right) \cup \{\gamma\} \cup (C_0 \backslash (\theta + 1))$$

where $\theta = \sup \left\{ \mathrm{dom}(b^0_k), \mathrm{dom}(b^1_k) \mid i_\alpha \le k < \omega \right\}$.

For each $i_\alpha \le k < \omega$, define S^γ_k to be the set of

$$p \in \cap \left\{ S^\eta_k \mid \eta < \gamma \right\}$$

such that if $\mathrm{dom}(p) > \gamma$ then the following hold where

$$C = \cap \left\{ C_\eta \mid \eta < \gamma \right\}.$$

(10.1) If $p \restriction \gamma = b_k$ and $p(\gamma) = 0$ then $b^0_k \subseteq p$ and $p(\xi) = 0$ for all $\xi \in (C_0 \backslash C) \backslash \mathrm{dom}(b^0_k)$ such that $\xi \in \mathrm{dom}(p)$.

(10.2) If $p \restriction \gamma = b_k$ and $p(\gamma) = 1$ then $b^1_k \subseteq p$ and $p(\xi) = 0$ for all $\xi \in (C_0 \backslash C) \backslash \mathrm{dom}(b^1_k)$ such that $\xi \in \mathrm{dom}(p)$.

(10.3) If $p \restriction \gamma \neq b_k$ then $p(\xi) = 0$ for all $\xi \in (C_0 \backslash C) \cap \mathrm{dom}(p)$ such that $\xi > \gamma$.

Note that C_γ and $\langle S^\gamma_k : i_\alpha \le k < \omega \rangle$ are uniformly definable from the sequences

$$\langle C_\eta : \eta < \gamma \rangle$$

and

$$\langle S^\eta_k : i_\alpha \le k < \omega, \eta < \gamma \rangle$$

and an element of $H(\omega_2)$ (in the construction given above, $\langle (b^0_k, b^1_k) : i_\alpha \le k < \omega \rangle$ is that element).

Thus for all $\gamma < \omega_2$, the sequences

$$\langle C_\eta : \eta \le \gamma \rangle$$

and

$$\langle S^\eta_k : i_\alpha \le k < \omega, \eta \le \gamma \rangle$$

are definable in the structure

$$\left(H(\omega_2), C_0, \langle S^0_k : i_\alpha \le k < \omega \rangle \right)$$

from an element of $H(\omega_2)$. Therefore both sequences are elements of $L_{\eta_\alpha}(H(\omega_2))$ as required.

This proves the existence of

$$\left(\langle T^\alpha_i : i < \omega \rangle : \alpha < \Theta \rangle \right)$$

satisfying (6.1)–(6.3).

For each $i < \omega$ let

$$T_i = \bigcap_{\alpha < \Theta} T^\alpha_i.$$

Thus each $i < \omega$, T_i is a club-full tree. Therefore there exists a sequence

$$\langle b_i : i < \omega \rangle$$

such that for all $i < \omega$, $b_i \in 2^{\omega_2}$ and b_i is a cofinal branch of T_i. For each $i < \omega$, let

$$B_i = \{ \alpha < \omega_2 \mid b_i(\alpha) = 1 \}.$$

We verify $\langle B_i : i < \omega \rangle$ witnesses that the lemma holds.

For each $i < \omega$, let

$$Z_i = \{ (k, \alpha) \mid i \le k < \omega \text{ and } \alpha \in B_k \}.$$

We must show

(11.1) For each $k < i < \omega$, the structure $(L_i, L_k, \in)$ is isomorphic to an initial segment of the structure $(S_{Z_i}, S_{Z_k}, \in)$.

To prove (11.1), we simply have to show

(12.1) For each $k < \omega$, $\left\{\eta_\alpha \mid \alpha \in \hat{L}_k\right\}$ is an initial segment of S_{Z_k}.

We give the argument for $k = 0$. Note that for all $\xi > \omega_2$,

$$L_\xi(H(\omega_2), Z_0) = L_\xi(H(\omega_2), \langle b_k : k < \omega\rangle).$$

Fix $\alpha < \Theta$. If $\alpha \in \hat{L}_0$ then $\langle b_k : k < \omega\rangle$ is $L_{\eta_\alpha}(H(\omega_2))$-generic for the product partial order

$$\prod_{0 \le k < \omega} T_k^\alpha.$$

Therefore

$$L_{\eta_\alpha}(H(\omega_2), Z_0) \vDash \text{ZFC}\backslash\text{Replacement}$$

and so $\eta_\alpha \in S_{Z_0}$. Now suppose that $\alpha \notin \hat{L}_0$ (and so $i_\alpha > 0$). Then there is a wellordering of ω_2 of length η_α which is definable in

$$(H(\omega_2), b_0, A)$$

where A is the set of successor points of C_0^α and C_0^α is the closed unbounded subset of ω_2 which witness that $T_0^\alpha \in L_{\eta_\alpha}(H(\omega_2))$ is club-full, see (6.2)(b). Thus

$$L_{\eta_\alpha}(H(\omega_2), Z_0) \nvDash \text{ZFC}\backslash\text{Replacement},$$

and so $\eta_\alpha \notin S_{Z_0}$. The proof of (12.1) for $k \neq 0$ is identical. □

Theorem 2.4. *Suppose $\lambda \in \text{Ord}$, T is a tree on $\omega \times \lambda$, and that the following hold where for each $x \in \mathbb{R}$, $\text{Th}_2^{L[T][x]}(T, \omega_1^V)$ is the Σ_2-theory of $L[T][x]$ with parameter (T, ω_1^V).*

(i) *ω_1 is strongly inaccessible in $L[T][x]$ for all $x \in \mathbb{R}$.*

(ii) *$\text{Th}_2^{L[T][x]}(T, \omega_1^V)$ is constant on a Turing cone.*

Then $p[T]$ is determined.

Proof. We first assume in addition to the hypothesis of the theorem that the following hold.

(1.1) For some $x \in \mathbb{R}$, every $y \in \mathbb{R}$ is $L[T][x]$-generic for some partial order $\mathbb{P} \in L[T][x]$ such that $\mathbb{P}$ is countable in V.

(1.2) There exists an $L[T](\mathbb{R})$-generic filter

$$G \subset \text{Coll}(\omega_1, \mathbb{R})$$

such that $V = L[T](\mathbb{R})[G]$.

We prove that $p[T]$ is determined and:

(2.1) Either Player II has a winning strategy in the game $p[T]$, or Player I has a winning strategy in the game $p[T]$, which is Δ_2-definable from T.

For each $x \in \mathbb{R}$, let $\sigma_x = L[T][x] \cap \mathbb{R}$, and let $\text{Th}_2^V(T, \sigma_x, \omega_1)$ be the Σ_2-theory of V with parameter (T, σ_x, ω_1).

Thus $\text{Th}_2^V(T, \sigma_x, \omega_1)$ is constant on a Turing cone, and we fix $x_0 \in \mathbb{R}$ such that

(3.1) For all $x \in \mathbb{R}$, if $x_0 \le_{\mathrm{T}} x$ then

$$\text{Th}_2^V(T, \sigma_x, \omega_1) = \text{Th}_2^V(T, \sigma_{x_0}, \omega_1).$$

Thus for all $x \in \mathbb{R}$ such that $x_0 \le_{\mathrm{T}} x$, (1.1) holds for x.

We first prove the following.

(4.1) $L[T][x_0] \vDash \text{CH}$.

(4.2) $L[T][x_0] \cap H(\omega_1) \vDash \text{GCH}$.

Let $G_0 \subset \omega_1^{L[T][x_0]}$ be $L[T][x_0]$-generic where the partial order is by initial segments ordered by extension. Thus

$$L[T][x_0][G_0] \vDash \mathrm{CH}$$

and $\omega_1^{L[T][x_0]} = \omega_1^{L[T][x_0][G_0]}$.

Let

$$\langle \tau_\alpha : \alpha < \omega_1^{L[T][x_0][G_0]} \rangle \in L[T][x_0]$$

be a sequence of infinite subsets of ω such that for all $\alpha < \beta < \omega_1^{L[T][x_0][G_0]}$, $\tau_\alpha \cap \tau_\beta$ is finite. Using the partial order for Solovay's almost disjoint forcing, let $y \subset \omega$ be $L[T][x_0][G_0]$-generic such that

$$G_0 = \left\{ \alpha < \omega_1^{L[T][x_0][G_0]} \mid y \cap \tau_\alpha \text{ is finite} \right\}.$$

Thus

$$L[T][x_0][G_0][y] = L[T][x_0][y]$$

and

$$L[T][x_0][y] \vDash \mathrm{CH}.$$

But by the choice of x_0, $\mathrm{Th}_2^{L[T][x_0]}(T, \omega_1^V) = \mathrm{Th}_2^{L[T][x_0,y]}(T, \omega_1^V)$ and this proves (4.1).

Let $\gamma < \omega_1$ be a cardinal of $L[T][x_0]$ and now let $G_0 \subset \mathrm{Coll}(\omega, \gamma)$ be a filter which is $L[T][x_0]$-generic. Then there exists $y \in \mathbb{R}$ such that

$$L[T][x_0][G_0] = L[T][y]$$

and so by (4.1) and since $\mathrm{Th}_2^{L[T][x_0]}(T, \omega_1^V) = \mathrm{Th}_2^{L[T][y]}(T, \omega_1^V)$, necessarily

$$L[T][y] \vDash \mathrm{CH}.$$

This proves (4.2).

We next prove the following, and the proof is similar to the proof of (4.1)–(4.2).

(5.1) There is a wellordering of $H(\omega_2)^{L[T][x_0]}$ of length $\omega_2^{L[T][x_0]}$ which is definable in $H(\omega_2)^{L[T][x_0]}$ from parameters.

(5.2) $L[T][x_0] \vDash \diamond(\omega_2)$ and this is witnessed by a sequence

$$\langle Z_\alpha : \alpha < \omega_2^{L[T][x_0]} \rangle$$

which is definable in $H(\omega_2)^{L[T][x_0]}$ from parameters.

Of course (5.2) implies (5.1) but we shall achieve (5.2) by achieving a very strong form of (5.1).

We need a definition from the first edition of [**8**], see Definiton 8.15 in [**8**]. Suppose κ is an uncountable cardinal κ. Then *Strong Condensation* holds for $H(\kappa)$ if there is a bijection

$$\rho : \kappa \to H(\kappa)$$

such that for all countable

$$\mathcal{X} \prec (H(\kappa), \rho),$$

$\rho_{\mathcal{X}} \subset \rho$ where $\rho_{\mathcal{X}}$ is the image of $\rho \cap \mathcal{X}$ under the transitive collapse of $\mathcal{X}$.

By the main theorem of Wu [**10**], if CH holds then there is a partial order $\mathbb{P}$ such that if $G \subset \mathbb{P}$ is V-generic then the following hold.

(6.1) $\omega_1^V = \omega_1^{V[G]}$ and $\omega_2^V = \omega_2^{V[G]}$.

(6.2) $V[G] \vDash$ " Strong Condensation holds for $H(\omega_2)$".

Thus in V, and applying Wu's theorem in

$$L[T][x_0] \cap H(\omega_1) \vDash \text{ZFC} + \text{GCH},$$

there exists G_0 such that the following hold.

(7.1) $\omega_1^{L[T][x_0]} = \omega_1^{L[T][x_0][G_0]}$ and $\omega_2^{L[T][x_0]} = \omega_2^{L[T][x_0][G_0]}$.

(7.2) $L[T][x_0][G_0] \vDash$ " Strong Condensation holds for $H(\omega_2)$".

We claim:

(8.1) Suppose H is $L[T][x_0][G_0]$-generic for some partial order

$$\hat{\mathbb{P}} \in L[T][x_0][G_0] \cap H(\omega_1)$$

and that

$$(\omega_1)^{L[T][x_0][G_0]} = (\omega_1)^{L[T][x_0][G_0][H]}.$$

Then

$$H(\omega_2)^{L[T][x_0][G_0]}$$

is definable from parameters in $H(\omega_2)^{L[T][x_0][G_0][H]}$.

Fix a bijection

$$\rho : \omega_2^{L[T][x_0][G_0]} \to H(\omega_2)^{L[T][x_0][G_0]}$$

which witnesses in $L[T][x_0][G_0]$ that Strong Condensation holds for $H(\omega_2)$. Then for each

$$\alpha < \omega_2^{L[T][x_0][G_0]},$$

$\rho \restriction \alpha$ is uniformly definable from $\rho \restriction \omega_1^{L[T][x_0][G_0]}$ in $H(\omega_2)^{L[T][x_0][G_0][H]}$ as the only function with domain α which satisfies the condensation condition relative to $\rho \restriction \omega_1^{L[T][x_0][G_0]}$.

If

$$(\omega_2)^{L[T][x_0][G_0]} = (\omega_2)^{L[T][x_0][G_0][H]}$$

then this immediately implies (8.1), and if

$$(\omega_2)^{L[T][x_0][G_0]} < (\omega_2)^{L[T][x_0][G_0][H]}$$

then this again implies (8.1) by using the parameter

$$p = \left(\rho \restriction \omega_1^{L[T][x_0][G_0]}, (\omega_2)^{L[T][x_0][G_0]}\right).$$

This proves (8.1).

Fix a sequence $\langle \tau_\alpha : \alpha < \omega_2^{L[T][x_0][G_0]} \rangle$ such that

(9.1) $\langle \tau_\alpha : \alpha < \omega_2^{L[T][x_0][G_0]} \rangle$ is definable from parameters in $H(\omega_2)^{L[T][x_0][G_0]}$.

(9.2) For all $\alpha < \omega_2^{L[T][x_0][G_0]}$, $\tau_\alpha \subset \omega_1^{L[T][x_0][G_0]}$ and τ_α is cofinal in $\omega_1^{L[T][x_0][G_0]}$.

(9.3) For all $\alpha < \beta < \omega_2^{L[T][x_0][G_0]}$, $\tau_\alpha \cap \tau_\beta$ is bounded in $\omega_1^{L[T][x_0][G_0]}$.

Let $h_0 \subset \omega_2^{L[T][x_0][G_0]}$ be an $L[T][x_0][G_0]$-generic set where the partial order is by initial segments ordered by extension. Thus

$$L[T][x_0][G_0][h_0] \vDash \diamond(\omega_2).$$

By Solovay's almost disjoint forcing again, let

$$h_1 \subset \omega_1^{L[T][x_0][G_0]}$$

be $L[T][x_0][G_0]$-generic such that

$$h_0 = \left\{\alpha < \omega_2^{L[T][x_0][G_0]} \mid h_1 \cap \tau_\alpha \text{ is bounded in } \omega_1^{L[T][x_0][G_0]}\right\}.$$

Thus

$$L[T][x_0][G_0][h_0][h_1] = L[T][x_0][h_1]$$

and further h_0 is definable from parameters in $H(\omega_2)^{L[T][x_0][h_1]}$.

Therefore it follows that

$$L[T][x_0][h_1] \vDash \diamond(\omega_2)$$

and further, this is witnessed by a sequence which is definable from parameters in $H(\omega_2)^{L[T][x_0][h_1]}$.

Finally let

$$\langle \hat{\tau}_\alpha : \alpha < \omega_1^{L[T][x_0][G_0]} \rangle \in L[T][x_0][G_0]$$

be a sequence of infinite subsets of ω such that for all $\alpha < \beta < \omega_1^{L[T][x_0][G_0]}$, the set $\hat{\tau}_\alpha \cap \hat{\tau}_\beta$ is finite. By Solovay's almost disjoint forcing one last time, let $y \subset \omega$ be $L[T][x_0][G_0][h_1]$-generic such that

$$h_1 = \left\{ \alpha < \omega_1^{L[T][x_0][G_0]} \mid y \cap \hat{\tau}_\alpha \text{ is finite} \right\}.$$

Thus

$$L[T][x_0][G_0][h_1][y] = L[T][x_0][h_1][y] = L[T][x_0][y]$$

and since the partial order for which y is generic is ccc and has cardinality $\omega_1^{L[T][x_0][h_1]}$ in $L[T][x_0][h_1]$, necessarily

$$L[T][x_0][y] \vDash \diamond(\omega_2)$$

and further this is witnessed by a sequence which is definable from parameters in $H(\omega_2)^{L[T][x_0][y]}$.

Thus putting everything together the following hold in $L[T][x_0][y]$.

(10.1) There is a wellordering of $H(\omega_2)$ of length ω_2 which is definable from parameters in $H(\omega_2)$.

(10.2) $\diamond(\omega_2)$ holds and this is witnessed by a sequence which is definable from parameters in $H(\omega_2)$.

But by the choice of x_0,

$$\mathrm{Th}_2^{L[T][x_0]}(T, \omega_1^V) = \mathrm{Th}_2^{L[T][x_0,y]}(T, \omega_1^V)$$

and this proves (5.1)–(5.2).

Let θ be least such that $T \in V_\theta$ and such that $\theta = |V_\theta|$, and for each $x \in \mathbb{R}$, let

$$\mathrm{Th}_\omega^{V_\theta}(T, \sigma_x)$$

be the theory of V_θ with a constant for (T, σ_x).

Let $\mathcal{D}_0$ be the set of all $[x]_\mathbb{T} \in \mathcal{D}$ such that

$$\mathrm{Th}_\omega^{V_\theta}(T, \sigma_y) = \mathrm{Th}_\omega^{V_\theta}(T, \sigma_{x_0})$$

for all $[y]_\mathbb{T}$ such that $x \leq_\mathbb{T} y$. Thus $\mathcal{D}_0$ is Δ_2-definable in V from T and by the choice of x_0, $\mathcal{D}_0$ contains all $[x]_\mathbb{T}$ such that $x_0 \leq_\mathbb{T} x$.

For each $[x]_\mathbb{T} \in \mathcal{D}_0$, let S_x be the set given by the first $\omega_2^{L[T][x]}$ many ordinals $\xi > \omega_2^{L[T][x]}$ such that

$$L_\xi\left(H(\omega_2)^{L[T][x]}\right) \vDash \mathrm{ZFC}\backslash\mathrm{Replacement}.$$

Thus if $[y]_\mathbb{T} = [x]_\mathbb{T}$ then $S_x = S_y$, and for each $\xi \in S_x$, $S_x \cap \xi$ is bounded in ξ.

Define $Z \subseteq T$ to be T-*full* if for each $(s,t) \in Z$, there exists $w \in p[T]$ such that $s \subset w$ and such that

$$\left\{ (w|k, f_w^T|k) \mid k < \omega \right\} \subseteq Z,$$

recalling that f_w^T is the left most branch through T_w. Note that the T-full subsets of T are closed under arbitrary unions.

For each T-full set Z, let T^z be the transitive collapse of Z and let Θ_Z be the image of the ordinals under the transitive collapse of Z. Thus Θ_Z is just the ordinal which is isomorphic to ordertype of the set of ordinals occurring in Z and T_Z is the tree on $\omega \times \Theta_Z$ given by Z.

We claim:

(11.1) Suppose Z, W are each T-full and countable. Suppose

$$T^Z = T^W.$$

Then $Z = W$.

Let $\pi : T^Z \to T$ invert the transitive collapse of Z. Then since Z is T-full, π is pointwise minimal among all possible tree embeddings $\rho : T^Z \to T$ such that ρ is induced by an order preserving function

$$e : \Theta_Z \to \mathrm{Ord},$$

and the pointwise minimality is relative to the order $(s_0, t_0) \le (s_1, t_1)$ if $s_0 = s_1$ and $t_0 \le_{\mathrm{KB}} t_1$, where $\le_{\mathrm{KB}}$ be the Kleene-Brouwer order on $\mathrm{Ord}^{<\omega}$. This uniquely specifies Z from T_Z and so necessarily $Z = W$.

For each $[x]_{\mathbb{T}} \in \mathcal{D}_0$, let Σ^T_x be the union of all $Z \subset T$ such that the following hold.

(12.1) $Z \in L[T][x]$ and Z is countable in $L[T][x]$.

(12.2) Z is T-full.

Suppose $[x]_{\mathbb{T}} \in \mathcal{D}_0$. The following claims (13.1)–(13.2) are immediate from (1.1)–(1.2) and (11.1), and imply the third claim, (13.3).

(13.1) Suppose Z is T-full, Z is countable, and that $T^Z \in L[T][x]$. Then $Z \in L[T][x]$.

(13.2) Suppose that $Z \subseteq T$, $Z \in L[T][x]$, and Z is countable. Let Z^o be the maximum T-full set which is contained in Z. Then $Z^o \in L[T][x]$.

(13.3) Suppose $x \le_{\mathbb{T}} y$ and $L[T][y]$ is a forcing extension of $L[T][x]$ such that for all $\sigma \subset \mathrm{Ord}$, if σ is countable in $L[T][y]$ then there exists $\sigma^* \subset \mathrm{Ord}$ such that $\sigma^* \in L[T][x]$, σ^* is countable in $L[T][x]$, and such that $\sigma \subseteq \sigma^*$. Then

$$\Sigma^T_x = \Sigma^T_y.$$

For each $[x]_{\mathbb{T}} \in \mathcal{D}_0$, let $T[x]$ be the transitive collapse of Σ^T_x. Thus by (4.1), our simplifying assumptions (1.1)–(1.2), and the definition of $\mathcal{D}_0$: Σ^T_x is uniformly definable in V_θ from (σ_x, T). This implies

$$\Sigma^T_x \in L[T][x]$$

and so by (11.1) and since CH holds in $L[T][x]$, $T[x] \in H(\omega_2)^{L[T][x]}$.

For each $x, y \in \omega^\omega$ with $x \in \mathcal{D}_0$, and for each $0 < n < \omega$, let

$$T[x]_y \restriction n = \cup \left\{ T[x]_{y|m} \mid 0 < m \le n \right\}$$

and as above, let $\le_{\mathrm{KB}}$ be the Kleene-Brouwer order on $\mathrm{Ord}^{<\omega}$.

We now define the key auxiliary games $\mathcal{G}(x)$ for $p[T] \cap L[T][x]$ where $x \in \mathcal{D}_0$. In the i-th round, Player I plays $m_i \in \omega$ and Player II plays $(n_i, t_i) \in \omega \times \omega^\omega$.

The rules are as follows where $y \in \omega^\omega$ is the element such that $y(2k) = m_k$ and $y(2k+1) = n_k$, for all $k < \omega$, and Player II wins the run of the game if the rules are satisfied.

(14.1) For each $i < \omega$, $[t_i]_{\mathbb{T}} \in \mathcal{D}_0$ and $t_i \in L[T][x]$.

(14.2) For each $i < \omega$, $\omega_1^{L[T][t_i]} = \omega_1^{L[T][x]}$, $\omega_2^{L[T][t_i]} = \omega_2^{L[T][x]}$, and $T[t_i] = T[x]$.

(14.3) For each $0 < k < i$, $t_i \in L[T][t_k]$ and the structure $\left(T[t_0]_y \restriction i, T[t_0]_y \restriction k, \le_{\mathrm{KB}}\right)$ is isomorphic to an initial segment of the structure $(S_{t_i}, S_{t_k}, \in)$.

Note that for each $0 < i < \omega$, the last condition (14.3) depends only on previous plays. Thus this game is an open game for Player I and so it is determined. By (1.1)–(1.2), and since $[x]_{\mathbb{T}} \in \mathcal{D}_0$, $\mathcal{G}(x)$ is Δ_2-definable in $L[T][x]$ from (T, ω_1^V).

If Player II has a winning strategy in $\mathcal{G}(x)$ for some $[x]_{\mathbb{T}} \in \mathcal{D}_0$, then trivially Player II has a winning strategy in $L[T][x]$ for the game given by $p[T] \cap L[T][x]$ since

$$p[T] \cap L[T][x] = (p[S])^{L[T][x]}$$

where $S = T[x]$. Therefore by absoluteness, that winning strategy for Player II must be a winning strategy in V for the game $p[T]$. Thus we can reduce to the case that Player I has a winning strategy in $\mathcal{G}(x)$ for all $[x]_{\mathbb{T}} \in \mathcal{D}_0$.

We now come to the key technical claim in the proof. Suppose $x \in \mathcal{D}_0$ and that

$$\langle L_i : 0 < i < \omega \rangle \in L[T][x]$$

is an increasing sequence of subsets of $\omega_2^{L[T][x]}$ such that

$$|L_i|^{L[T][x]} \leq \omega_1^{L[T][x]}$$

for all $0 < i < \omega$. The claim is that there exists $z \in \mathbb{R}$ and a sequence $\langle z_i : 0 < i < \omega \rangle$ such that the following hold.

(15.1) $x \leq_{\mathbb{T}} z$.

(15.2) $\omega_1^{L[T][x]} = \omega_1^{L[T][z]}$, $\omega_2^{L[T][x]} = \omega_2^{L[T][z]}$, and $T[x] = T[z]$.

(15.3) $\langle z_i : 0 < i < \omega \rangle \in L[T][y]$ and $x \leq_{\mathbb{T}} z_i \leq_{\mathbb{T}} z_k \leq_{\mathbb{T}} z$, for all $0 < k < i < \omega$.

(15.4) For each $0 < k < i$, the structure $(L_i, L_k, \in)$ is isomorphic to an initial segment of the structure $(S_{z_i}, S_{z_k}, \in)$.

By Lemma 2.3, and setting $L_0 = L_1$, there exists a sequence $\langle B_i : i < \omega \rangle \in L[T][x]$ of subsets of $\omega_2^{L[T][x]}$ such that the following holds where for each $i < \omega$,

$$Z_i = \{(k, \alpha) \mid i \leq k < \omega \text{ and } \alpha \in B_k\}$$

and where S_{Z_i} is set given by the first $\omega_2^{L[T][x]}$ many ordinals $\xi > \omega_2^{L[T][x]}$ such that

$$L_\xi\Big(H(\omega_2)^{L[T][x]}, Z_i\Big) \vDash \text{ZFC}\backslash\text{Replacement}.$$

(16.1) For each $0 < k < i$, the structure $(L_i, L_k, \in)$ is isomorphic to an initial segment of the structure $(S_{Z_i}, S_{Z_k}, \in)$.

Fix a sequence $\langle \sigma_\alpha : \alpha < \omega_2^{L[T][x]} \rangle$ of cofinal subsets of $\omega_1^{L[T][x]}$ such that

(17.1) $\langle \sigma_\alpha : \alpha < \omega_2 \rangle$ is definable from parameters in $H(\omega_2)^{L[T][x]}$.

(17.2) $\sigma_\alpha \cap \sigma_\beta$ is bounded in $\omega_1^{L[T][x]}$ for all $\alpha < \beta < \omega_2^{L[T][x]}$.

Force over $L[T][x]$ with the countable product with full support of almost disjoint forcing to get a sequence $\langle \hat{B}_i : i < \omega \rangle$ of subsets of $\omega_1^{L[T][x]}$ such that for all $i < \omega$

$$B_i = \left\{\alpha < \omega_2^{L[T][x]} \mid \hat{B}_i \cap \sigma_\alpha \text{ is bounded in } \omega_1^{L[T][x]}\right\}.$$

Thus defining for each $i < \omega$,

$$\hat{Z}_i = \left\{(k, \alpha) \mid i \leq k < \omega \text{ and } \alpha \in \hat{B}_k\right\},$$

it follows that for each $i < \omega$, S_{Z_i} is the set given by the first $\omega_2^{L[T][x]}$ many ordinals $\xi > \omega_2^{L[T][x]}$ such that

$$L_\xi\Big(H(\omega_2)^{L[T][x]}, \hat{Z}_i\Big) \vDash \text{ZFC}\backslash\text{Replacement}.$$

Now we code down to subsets of ω. Fix a sequence $\langle \hat{\sigma}_\alpha : \alpha < \omega_1^{L[T][x]} \rangle$ of infinite subsets of ω such that

(18.1) $\langle \hat{\sigma}_\alpha : \alpha < \omega_2 \rangle \in L[T][x]$.
(18.2) $\hat{\sigma}_\alpha \cap \hat{\sigma}_\beta$ is finite for all $\alpha < \beta < \omega_1^{L[T][x]}$.

Now force over $L[T][x][\langle \hat{B}_i : i < \omega \rangle]$ with the countable product with finite support of almost disjoint forcing to get a sequence $\langle b_i : i < \omega \rangle$ of subsets of ω such that for all $i < \omega$

$$\hat{B}_i = \left\{ \alpha < \omega_1^{L[T][x]} \mid b_i \cap \hat{\sigma}_\alpha \text{ is finite} \right\}.$$

Thus defining for each $i < \omega$,

$$z_i = \{(k, m) \mid i \leq k < \omega \text{ and } m \in b_k\} \times \{n < \omega \mid n \in x\}$$

and viewing $z_i \in \mathbb{R}$, it follows that for each $i < \omega$, S_{Z_i} is the set given by the first $\omega_2^{L[T][x]}$ many ordinals $\xi > \omega_2^{L[T][x]}$ such that

$$L_\xi\Big(H(\omega_2)^{L[T][x]}, z_i\Big) \vDash \text{ZFC}\backslash\text{Replacement};$$

in other words, for each $i < \omega$, $S_{z_i} = S_{Z_i}$.

Thus setting $z = z_0$, z and $\langle z_i : 0 < i < \omega \rangle$ are as required provided

$$T[x] = T[z].$$

But $L[T][x]$ is closed under ω-sequences in $L[T][x][\langle \hat{B}_i : i < \omega \rangle]$ and $L[T][z]$ is a ccc forcing extension of $L[T][x][\langle \hat{B}_i : i < \omega \rangle]$. Therefore since $[x]_{\mathbb{T}} \in \mathcal{D}_0$,

$$T[x] = T[z]$$

by (13.3). This proves the key technical claim.

For each $s \in \omega^{<\omega}$, and for each $[x]_{\mathbb{T}} \in \mathcal{D}_0$, if dom$(s)$ is even let $\mathcal{G}(x)_s$ be the game where the condition (14.3) is replaced by the condition

- For each $0 < k < i$, the structure $\Big(T[t_0]_z \upharpoonright (n + i), T[t_0]_z \upharpoonright (n + k), \leq_{\text{KB}}\Big)$ is isomorphic to an initial segment of the structure $(S_{t_i}, S_{t_k}, \in)$;

where $z = s^\frown y$ and $n = \text{dom}(s)$.

Thus if $s = \emptyset$ then $\mathcal{G}(x)_s = \mathcal{G}(x)$. Note the following and for (19.1) we appeal to our simplifying assumptions (1.1)-(1.2).

(19.1) For each $x \in \mathcal{D}_0$ and for each $s \in \omega^{<\omega}$, if dom(s) is even then $\mathcal{G}(x)_s$ is uniformly Δ_2-definable in $L[T][x]$ from (T, κ, s) where $\kappa = \omega_1^V$.
(19.2) Suppose p is a legal (rules are satisfied) position in $\mathcal{G}(x)$ of even nonzero length (so it is Player I's turn to move and Player I has not already won) and s is the integer part of p. Suppose that t_i is the last real played by Player II. Then the following are equivalent.
   a) p is a winning position for Player I in $\mathcal{G}(x)$.
   b) Player I has a winning strategy in $\mathcal{G}(t_i)_s$.

We use (19.2) to define a strategy F^∞ for Player I. For each $s \in \omega^{<\omega}$ such that dom(s) is even, and for each $x \in \mathbb{R}$ with $x_0 \leq_{\mathbb{T}} x$, if Player I has a winning strategy in in game $\mathcal{G}(x)_s$, let F_s^x be the canonical winning strategy (where Player I plays the least $k < \omega$ which decreases the rank of the resulting position in $\mathcal{G}(x)_s$ if Player I has not already won); and if Player II has a winning strategy in the game $\mathcal{G}(x)_s$, let F_s^x be the constant function with value 0.

Notice that for all $s \in \omega^{<\omega}$ and for all $x \in \mathbb{R}$, if dom(s) is even and $x_0 \leq_{\mathbb{T}} x$, then by (3.1), F_s^x is independent of the choice of x and in particular,

(20.1) $F_s^x(s) = F_s^{x_0}(s)$.

This defines a strategy F^∞ for Player I, where

$$F^\infty(s) = F_s^{x_0}(s)$$

for all $s \in \omega^{<\omega}$ such that dom(s) is even.

If F^∞ is a winning strategy for Player I in the game given by $p[T]$ then we are done, and so we can reduce to the case that for some $y \in \mathbb{R}$,

(21.1) $y \notin p[T]$,
(21.2) For all $k < \omega$, $y(2k) = F^\infty(y \restriction 2k)$.

Fix $x_1 \in \mathbb{R}$ such that $x_0 \leq_{\mathbb{T}} x_1$ and such that $y \in L[T][x_1]$. Thus

$$y \notin p[T[x_1]].$$

We now use the key technical claim (see page 858) to construct $z \in \mathbb{R}$ with $x_0 \leq_{\mathbb{T}} z$ and a play against $F^z_\emptyset$ which defeats $F^z_\emptyset$.

Since $y \notin p[T[x_1]]$, the tree $(T[x_1])_y$ has no infinite branch and so in $L[T][x_1]$, the linear order on $(T[x_1])_y$ given by the Kleene-Brouwer order is a wellordering with length θ_y such that $\theta_y < \omega_2^{L[T][x_1]}$. For each $0 < n < \omega$, let $L_n \subseteq \theta_y$ be the image of $(T[x_1])_y \restriction n$ under the isomorphism

$$\pi : \left((T[x_1])_y, \leq_{\mathrm{KB}}\right) \cong (\theta_y, \in)$$

By that claim, there exists $z \in \mathbb{R}$ and a sequence $\langle z_i : 0 < i < \omega\rangle$ such that the following hold where we set $z_0 = z$.

(22.1) $x_1 \leq_{\mathbb{T}} z$.
(22.2) $\omega_1^{L[T][x_1]} = \omega_1^{L[T][z]}$, $\omega_2^{L[T][x_1]} = \omega_2^{L[T][z]}$, and $T[x_1] = T[z]$.
(22.3) $\langle z_i : i < \omega\rangle \in L[T][z]$ and $x_1 \leq_{\mathbb{T}} z_i \leq_{\mathbb{T}} z_k \leq_{\mathbb{T}} z$, for all $k < i < \omega$.
(22.4) For each $0 < k < i$, the structure $(L_i, L_k, \in)$ is isomorphic to an initial segment of the structure $(S_{z_i}, S_{z_k}, \in)$.

But then playing $\langle (y(2i+1), z_i) : i < \omega\rangle$ against the canonical winning strategy for Player I in in the game $\mathcal{G}(z)$ defeats that strategy.

The point is that by induction on $i < \omega$, it follows from (19.2), (20.1), and the definition of F^x_s, (and using in particular that $x_1 \leq_{\mathbb{T}} z_i \leq_{\mathbb{T}} z_k \leq_{\mathbb{T}} z$, for all $k < i < \omega$), that for each $i < \omega$, that strategy must play $y(2i)$.

This is a contradiction, and this proves that F^∞ is a winning strategy for Player I in the game $p[T]$. Finally, F^∞ is uniformly definable in $V_{\theta+1}$ from $(T, [x]_{\mathbb{T}})$ for any $x \in \mathbb{R}$ such that $x_0 \leq_{\mathbb{T}} x$, and so F^∞ is Δ_2-definable from T.

This finishes the proof of (2.1), and the proof of the theorem under the simplifying assumptions (1.1)–(1.2).

We now use this special case and prove the theorem without assuming (1.1)–(1.2). Fix $x_0 \in \mathbb{R}$ such that for all $x \in \mathbb{R}$, if $x_0 \leq_{\mathbb{T}} x$ then

$$\mathrm{Th}_2^{L[T][x]}(T, \omega_1^V) = \mathrm{Th}_2^{L[T][x_0]}(T, \omega_1^V)$$

Let $G_0 \subset \mathrm{Coll}(\omega, {<}\omega_1)$ be V-generic and let $G \subset \omega_1$ be a $L[T][x_0][G_0]$-generic subset of ω_1 where the partial order is the partial order of bounded subsets of ω_1 in $L[T][x_0][G_0]$ ordered by extension.

Let $\mathbb{R}_{G_0} = \mathbb{R}^{L[T][x_0][G_0]}$. Thus $L[T][x_0](\mathbb{R}_{G_0})$ is a symmetric forcing extension of $L[T][x_0]$ for $\mathrm{Coll}(\omega, {<}\kappa)$ where $\kappa = \omega_1^V$ and so κ is strongly inaccessible in $L[T][x_0]$. Further G is $L[T][x_0](\mathbb{R}_{G_0})$-generic and

$$L[T][x_0][G] = L[T][x_0](\mathbb{R}_{G_0})[G].$$

Thus $L[T][x_0][G]$ is a homogenous forcing extension of $L[T][x_0]$ for a partial order which is Δ_2-definable in $L[T][x_0]$ with parameter ω_1^V.

Therefore by the choice of x_0, the definability of forcing, and since ω_1 is strongly inaccessible in $L[T][x_0]$, it follows that the hypothesis of the theorem holds in $L[T][x_0][G]$.

But (1.1)–(1.2) hold in $L[T][x_0][G]$ and so in $L[T][x_0][G]$, $p[T]$ is determined. Further by (2.1) either there is a a winning strategy for Player II, or there is a winning strategy for Player I which is Δ_2-definable in $L[T][x_0][G]$ with parameter T.

We first suppose that there is a winning strategy for Player II. Then in V, there exists $\alpha < \omega_1$ and an $L[T][x_0]$-generic filter $g \subset \mathrm{Coll}(\omega, {<}\alpha)$ such that in $L[T][x_0][g]$ there is a winning strategy F for Player II in the game $p[T] \cap L[T][x_0][g]$. But then by absoluteness, F must be a winning strategy in V for Player II in the game $p[T]$.

Finally suppose that in $L[T][x_0][G]$, there is a winning strategy F for Player I such that F is Δ_2-definable in $L[T][x_0][G]$ with parameter T. This implies that the strategy F must belong to $L[T][x_0]$ and further that F is Δ_2-definable in $L[T][x_0]$ with parameter (T, κ) where

$$\kappa = \omega_1^{L[T][x_0][G]} = \omega_1^V,$$

But in V and by the choice of x_0, for all $x \in \mathbb{R}$, if $x_0 \leq_{\mathbb{T}} x$ then

$$\mathrm{Th}_2^{L[T][x]}(T, \omega_1^V) = \mathrm{Th}_2^{L[T][x_0]}(T, \omega_1^V).$$

Therefore it follows that in V, for all $x \in \mathbb{R}$, if $x_0 \leq_{\mathbb{T}} x$ then in $L[T][x]$, F is a winning strategy for Player I in the game $p[T] \cap L[T][x]$. This trivially implies that in V, F is a winning strategy for Player I in the game $p[T]$. □

3. The Next Questions

Given the results of this paper, the following questions are the natural incremental questions in the general program of the analysis of Turing Determinacy and its relationship to AD.

The notion that a set $A \subseteq \mathbb{R}$ be an ${}^\infty$Borel set is a transfinite generalization of the notion of a borel set. Assuming Turing Determinacy, many sets $A \subseteq \mathbb{R}$ can be verified to be ${}^\infty$Borel sets, and further one obtains an elegant characterization of the ${}^\infty$Borel sets. Theorem 3.1 was proved as a precursor to Theorem 1.1, and arguably provided strong evidence that at least assuming $V = L(\mathbb{R})$, Turing Determinacy is equivalent to AD.

The reason is that in ZF, if a set $A \subseteq \mathbb{R}$ is ${}^\infty$Borel and there is no uncountable sequence of distinct reals, then A is Lebesgue measurable, A has the property of Baire, and more genrally A has all the usual regularity properties. Thus as a corollary of Theorem 3.1, assuming Turing Determinacy, every set in $L(\mathbb{R})$ is Lebesgue measurable etc., verifying that in $L(\mathbb{R})$, Turing Determinacy suffices to prove many of the consequences of AD.

Theorem 3.1 (ZF+$\mathrm{DC}_{\mathbb{R}}$)**.** *Assume Turing Determinacy and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*
(1) *A is an ${}^\infty$Borel set.*
(2) *There exists a set $S \subset$ Ord such that $A \in L(S, \mathbb{R})$.* □

By Theorem 3.1, a positive answer to either of the following questions would immediately yield a very strong generalization of the equivalence of Turing Determinacy and AD in $L(\mathbb{R})$.

Question 3.2 (ZF + $\mathrm{DC}_{\mathbb{R}}$)**.** Assume Turing Determinacy. Must every ${}^\infty$Borel set be determined?

The second question seems likely to be a more general question (which raises yet another interesting question) and the formulation involves the notion of an ${}^\infty$Borel code S for an ${}^\infty$Borel set A.

This is a transfinite generalization of the notion of a borel code for a borel set, and ${}^\infty$Borel codes can be viewed, depending on the presentation, as trees on $\omega \times \lambda$ for some $\lambda \in$ Ord, by generalization the definition of $p[T]$, or simply as sets $S \subset$ Ord which code transfinite words in the free Boolean algebra on ω-many generators (to be interpreted by the basic open subsets of ω^ω given by $\omega^{<\omega}$).

Whichever way one chooses to define the notion of an ${}^{\infty}$Borel code, it really does not affect the answer to the following question.

Question 3.3 (ZFC)**.** Suppose that S is an ${}^{\infty}$Borel code for $A \subseteq \mathbb{R}$ and for each $x \in \mathbb{R}$ and for each $\theta \in \mathrm{Ord}$, let $\mathrm{Th}_2^{L[S][x]}(S, \theta)$ be the Σ_2-theory of $L[S][x]$ with parameter (S, θ). Suppose that the following hold.
(i) ω_1 is strongly inaccessible in $L[S][x]$ for all $x \in \mathbb{R}$.
(ii) For each $\theta \in \mathrm{Ord}$, $\mathrm{Th}_2^{L[S][x]}(S, \theta)$ is constant on a Turing cone.
Must $A \cap L[S][x]$ be determined in $L[S][x]$ on a Turing cone?

Finally, coming full circle, another basic question concerns Σ^1_2-Turing Determinacy itself, but in the context of Second Order Number Theory, which is far too weak a theory in which to implement any of the arguments here, or from 1982, for obtaining Σ^1_2-Determinacy.

This problem is mostly resolved in [**9**] which is a sequel to this paper, and the main theorem is that in Second Order Number Theory, if one adds a Cohen real then Δ^1_2-Turing Determinacy holds in $N[c]$ if and only if Σ^1_2-Determinacy holds in $N[c]$.

The proof requires first proving Martin's theorem [**5**] on the equivalence of Δ^1_2-Determinacy with Σ^1_2-Determinacy, but again working in just Second Order Number Theory. The point here is that Martin's proof does not obviously work in just Second Order Number Theory since it needs a version of the countable Axiom of Choice.

Note that Martin's theorem implies (in ZFC) the upward absoluteness of Σ^1_2-Determinacy, a fact we used in the proof sketch on page 848.

This is by Shoenfield's theorem on Π^1_2-absoluteness which shows that a winning strategy for Player I in a Π^1_2-game, is a winning strategy for Player I in that Π^1_2-game in all outer models.

Thus since Shoenfield's theorem holds in Second Order Number Theory, if N is a model of Second Order Number Theory and if Δ^1_2-Turing Determinacy holds in N, then Δ^1_2-Turing Determinacy holds in $N[c]$ where c is an N-generic Cohen real. But then by the main theorem of [**9**], Σ^1_2-Determinacy holds in $N[c]$.

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