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THE CLOSURE-COMPLEMENT-FRONTIER PROBLEM IN SATURATED POLYTOPOLOGICAL SPACES

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Abstract. Let X be a space equipped with n topologies $\tau_1, ..., \tau_n$ which are pairwise comparable and saturated, and for each $1 \leq i \leq n$ let k_i and f_i be the associated topological closure and frontier operators, respectively. Inspired by the closure-complement theorem of Kuratowski, we prove that the monoid of set operators $\mathcal{KF}_n$ generated by $\{k_i, f_i : 1 \leq i \leq n\} \cup \{c\}$ (where c denotes the set complement operator) has cardinality no more than $2p(n)$ where $p(n) = \frac{5}{24}n^4 + \frac{37}{12}n^3 + \frac{79}{24}n^2 + \frac{101}{12}n + 2$. The bound is sharp in the following sense: for each n there exists a saturated polytopological space $(X, \tau_1, ..., \tau_n)$ and a subset $A \subseteq X$ such that repeated application of the operators k_i, f_i, c to A will yield exactly $2p(n)$ distinct sets. In particular, following the tradition for Kuratowski-type problems, we exhibit an explicit initial set in $\mathbb{R}$, equipped with the usual and Sorgenfrey topologies, which yields $2p(2) = 120$ distinct sets under the action of the monoid $\mathcal{KF}_2$.

1. Introduction

In his 1922 thesis [**8**], Kuratowski posed and solved the following problem: given a topological space (X, τ), what is the largest number of distinct subsets that can be obtained by starting from an initial set $A \subseteq X$, and applying the topological closure and complement operators, in any order, as often as desired? The answer is 14. This result, now widely known as Kuratowski's *closure-complement theorem*, is both thought-provoking and amusing, and has inspired a substantial number of authors to study generalizations, variants, and elaborations of the original closure-complement problem. We recommend consulting the admirable survey of Gardner and Jackson [**6**], or visiting Bowron's website *Kuratowski's Closure-Complement Cornucopia* [**3**] for an indexed list of all relevant literature.

Shallit and Willard [**10**] considered a natural extension of Kuratowski's problem. If we equip a space X with not one but two distinct topologies τ_1 and τ_2, how many distinct subsets may be obtained by starting with an initial set, and applying each of the two associated closure operators k_1, k_2, and the set complement operator c, in any order, as often as desired? The authors construct an example of a bitopological space (X, τ_1, τ_2) where it is possible to obtain infinitely many subsets from a certain initial set. Consequently, the monoid $\mathcal{K}_2$ of set operators generated by $\{k_1, k_2, c\}$ may have infinitely many elements in general. In their example, the topologies τ_1 and τ_2 are incomparable, which suggests that the monoid may yet be finite in case $\tau_1 \supseteq \tau_2$.

In [**1**], Banakh, Chervak, Martynyuk, Pylypovych, Ravsky, and Simkiv verify this last possibility, and generalize the closure-complement theorem to polytopological

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spaces, i.e. sets X equipped with families of topologies $\mathcal{T}$ in which the topologies are linearly ordered by inclusion. If the family is a finite set $\mathcal{T} = \{\tau_1, ..., \tau_n\}$, they give an explicit formula for the maximal cardinality of the monoid $\mathcal{K}_n$ generated by $\{k_j : 1 \leq j \leq n\} \cup \{c\}$. This maximal cardinality is of course 14 when $n = 1$, and grows exponentially as $n \to \infty$.

The authors of [**1**] also consider the special case where the topologies involved are *saturated*, i.e., for any $1 \leq j, \ell \leq n$, if a nonempty set U is τ_j-open, then U has nonempty τ_ℓ-interior. In the saturated case, the cardinality bound on the monoid is given by $\#\mathcal{K}_n \leq 12n + 2$. The most natural example is the case of the real line $\mathbb{R}$ equipped with $\tau_2 =$ the usual topology and $\tau_1 =$ the Sorgenfrey topology. Then one may obtain no more than $12 \cdot 2 + 2 = 26$ distinct sets by applying k_1, k_2, c to any particular initial set, and indeed this upper bound is obtainable in $(\mathbb{R}, \tau_1, \tau_2)$, as demonstrated explicitly in [**1**].

In [**5**], Gaida and Eremenko solved a *closure-complement-frontier problem* by showing that in any topological space (X, τ), the monoid $\mathcal{KF}_1$ generated by $\{k, f, c\}$ (where f is the frontier operator, or topological boundary operator) has cardinality ≤ 34; moreover there are examples of spaces in which it is possible to obtain 34 distinct subsets by applying the operators to a single initial set. This problem also appeared as Problem E3144 in *American Mathematical Monthly* [**2**]. The purpose of this paper is to study the extension of Gaida and Eremenko's problem to the setting of saturated polytopological spaces as in [**1**].

To state our result, we consider a polytopological space $(X, \tau_1, ..., \tau_n)$, and we denote by $\mathcal{KF}_n = \mathcal{KF}_n(X, \tau_1, ..., \tau_n)$ the monoid of set operators generated by $\{k_j, f_j : 1 \leq j \leq n\} \cup \{c\}$. We also let $\mathcal{KF}_n^0 = \mathcal{KF}_n^0(X, \tau_1, ..., \tau_n)$ denote the monoid generated by $\{k_j, i_j, f_j : 1 \leq j \leq n\}$, where i_j is the interior operator associated to τ_j. Since $i_j = ck_jc$, we have that $\mathcal{KF}_n^0 \subseteq \mathcal{KF}_n$, and in fact, in Section 2 we observe that

$$\mathcal{KF}_n = \mathcal{KF}_n^0 \cup c\mathcal{KF}_n^0$$

so that $\mathcal{KF}_n^0$ comprises the submonoid of *even operators* of $\mathcal{KF}_n$, and

$$\#\mathcal{KF}_n = 2 \cdot \#\mathcal{KF}_n^0.$$

Our main theorem follows.

Theorem 1.1. *Let $(X, \tau_1, ..., \tau_n)$ be a saturated polytopological space. Then $\#\mathcal{KF}_n^0 \leq p(n)$ and $\#\mathcal{KF}_n = 2 \cdot \#\mathcal{KF}_n^0 \leq 2p(n)$, where*

$$p(n) = \tfrac{5}{24}n^4 + \tfrac{37}{12}n^3 + \tfrac{79}{24}n^2 + \tfrac{101}{12}n + 2.$$

Thus for $n = 1$ we recover Gaida-Eremenko's result with $p(n) = 17$ and $2p(n) = 34$. The next few upper bounds are $p(2) = 60$, $p(3) = 157$, $p(4) = 339$, and $p(5) = 642$.

We also demonstrate that the bound $p(n)$ is sharp.

Theorem 1.2. *For every $n \geq 1$, there exists a saturated polytopological space $(X, \tau_1, ..., \tau_n)$ in which $\#\mathcal{KF}_n^0 = p(n)$ and $\#\mathcal{KF}_n = 2p(n)$. In fact, there is an initial set $A \subseteq X$ such that $\#\{oA : o \in \mathcal{KF}_n\} = 2p(n)$.*

The explicit examples we give are natural and easy to understand (disjoint unions of copies of $\mathbb{R}$ equipped with combinations of the Sorgenfrey and Euclidean topologies), but not finite. By the results of [**9**] (see [**6**] Theorem 4.1 and surrounding remarks), we deduce abstractly that there must exist a finite polytopological space $(X, \tau_1, ..., \tau_n)$ on which $\#\mathcal{KF}_n^0 = p(n)$, but we do not know how many points are necessary.

Question 1.3. What is the minimal cardinality of a polytopological space $(X, \tau_1, ..., \tau_n)$ for which $\#\mathcal{KF}_n^0 = p(n)$ exactly? What is the minimal cardinality of a space in which one can find an initial set A with $\#\{oA : o \in \mathcal{KF}_n\} = 2p(n)$?

It would be interesting to know the answer even for $n = 2$. It is known that the minimal number of points needed for a space to contain a Kuratowski 14-set is 7; see [**7**]. During the preparation of this article, Bowron has communicated to us that if $n = 1$, then the minimal number of points needed for $\#\mathcal{KF}_1^0 = 17$ is four, while the minimal number of points needed to contain a 34-set is 8.

Another interesting question that remains open is to solve the closure-complement-frontier problem for polytopological spaces which are not necessarily saturated.

Question 1.4. Let $(X, \tau_1, ..., \tau_n)$ be a polytopological space which is not necessarily saturated. What is the maximal cardinality of the monoid $\mathcal{KF}_n$ generated by $\{k_j, f_j : 1 \leq j \leq n\} \cup \{c\}$?

Finally, it would be interesting to study some of the variants described in Section 4 of [**6**] in the larger context of polytopological spaces. For example, it was shown independently by Gardner and Jackson [**6**] and by Sherman [**11**] that in any topological space (X, τ), the greatest number of sets one may obtain from an initial set $A \subseteq X$ by applying the set operators $\{k, i, \cup, \cap\}$ is 35.

Question 1.5. Let $(X, \tau_1, ..., \tau_n)$ be a (saturated?) polytopological space. What is the largest number of sets one may obtain from an initial set $A \subseteq X$ by applying the set operators k_j, i_j $(1 \leq j \leq n)$, $\cup$, and $\cap$ in any order, as often as desired?

2. Preliminaries and Notation

Recall from the introduction that a *polytopological space* is a set X equipped with a family of topologies $\mathcal{T}$ which is linearly ordered by the inclusion relation. In this paper we will work only with finite families $\mathcal{T} = \{\tau_1, ..., \tau_n\}$ and assume $\tau_1 \supseteq ... \supseteq \tau_n$. In this case we refer to $(X, \tau_1, ..., \tau_n)$ as an *n-topological space.*

For each topology τ_j, we permanently associate the *closure operator* k_j, the *interior operator* i_j, and the *frontier operator* f_j. We use c to denote the *set complement operator.* The operators k_j and i_j are idempotent, so $k_j k_j = k_j$ and $i_j i_j = i_j$, and the operator c is an involution, so $cc = \mathrm{Id}$, where Id denotes the *identity operator.* For each set $A \subseteq X$ we have $f_j A = k_j A \cap k_j cA$; we summarize this symbolically by writing

$$f_j = k_j \wedge k_j c = k_j \wedge c i_j.$$

From the identity above, we see that

$$\boxed{f_j c = f_j.}$$

We permanently denote by $\mathcal{KF}_n = \mathcal{KF}_n(X, \tau_1, ..., \tau_n)$ the smallest monoid of set operators which contains k_j, f_j $(1 \leq j \leq n)$ and c. We also denote by $\mathcal{KF}_n^0 = \mathcal{KF}_n^0(X, \tau_1, ..., \tau_n)$ the smallest monoid of set operators which contains k_j, i_j, and f_j $(1 \leq j \leq n)$. By DeMorgan's laws, we have $ck_jc = i_j$ and thus it is immediate that $\mathcal{KF}_n^0 \subseteq \mathcal{KF}_n$.

Since we are requiring that $\mathcal{KF}_n^0$ be a monoid, it contains the identity operator Id. It also contains the *zero operator* 0, i.e. the set operator for which $0A = \emptyset$, for every $A \subseteq X$. This follows from the work of Gaida and Eremenko [**5**], who observed that

$$i_1 f_1 k_1 = 0.$$

We also define the *one operator* by the rule $1 = c0$, so $1A = X$ for every $A \subseteq X$ and $1 \in \mathcal{KF}_n$.

Proposition 2.1. *The sets $\mathcal{KF}_n^0$ and $c\mathcal{KF}_n^0$ are disjoint and $\mathcal{KF}_n$ is equal to their union.*

Proof. By examining the generators k_j, i_j, f_j of $\mathcal{KF}_n^0$, it is clear that $o\emptyset = \emptyset$ and $co\emptyset = X$ for any operator $o \in \mathcal{KF}_n^0$. Therefore, $\mathcal{KF}_n^0$ and $c\mathcal{KF}_n^0$ are disjoint.

To see that $\mathcal{KF}_n \subseteq \mathcal{KF}_n^0 \cup c\mathcal{KF}_n^0$, we can argue by induction on word length of elements of $\mathcal{KF}_n$. Let $\mathcal{W}_m \subseteq \mathcal{KF}_n$ be the set of operators which can be written as a word of length $\leq m$ in the generators k_j, f_j, c. Assume that $\mathcal{W}_m \subseteq \mathcal{KF}_n^0 \cup c\mathcal{KF}_n^0$ (which is certainly true if $m = 1$). Then $\mathcal{W}_{m+1}$ is the union of sets of the form $k_j\mathcal{W}_m$, $f_j\mathcal{W}_m$, and $c\mathcal{W}_m$. But by invoking DeMorgan's laws and the identity $f_jc = f_j$, the inductive hypothesis implies the following inclusions:

$$\begin{aligned} k_j\mathcal{W}_m &\subseteq k_j\mathcal{KF}_n^0 \cup k_jc\mathcal{KF}_n^0 \\ &= k_j\mathcal{KF}_n^0 \cup ci_j\mathcal{KF}_n^0 = \mathcal{KF}_n^0 \cup c\mathcal{KF}_n^0; \end{aligned}$$

$$\begin{aligned} f_j\mathcal{W}_m &\subseteq f_j\mathcal{KF}_n^0 \cup f_jc\mathcal{KF}_n^0 \\ &= f_j\mathcal{KF}_n^0 \cup f_j\mathcal{KF}_n^0 = \mathcal{KF}_n^0; \end{aligned}$$

$$c\mathcal{W}_m \subseteq c\mathcal{KF}_n^0 \cup cc\mathcal{KF}_n^0 = \mathcal{KF}_n^0 \cup c\mathcal{KF}_n^0;$$

which concludes the inductive step and the proof. □

By the previous proposition, we are now justified in referring to the elements of $\mathcal{KF}_n^0$ as the *even operators*, and those in $c\mathcal{KF}_n^0$ as the *odd operators*. By direct algebraic manipulation, it is easy to see that any operator in $\mathcal{KF}_n$ may be rewritten as a word in which the generator c appears either zero times (the even case) or exactly one time (the odd case). For example $k_1ci_1cck_1cf_1k_1c = k_1i_1f_1i_1$.

Corollary 2.2. $\#\mathcal{KF}_n = 2 \cdot \#\mathcal{KF}_n^0$.

In the special case $n = 1$, the results of Gaida-Eremenko [**5**] imply that $\mathcal{KF}_1^0$ consists of no more than 17 distinct even operators, which may be listed explicitly as below:

$$\mathcal{KF}_1^0 = \{\mathrm{Id}, k_1, i_1, k_1i_1, i_1k_1, i_1k_1i_1, k_1i_1k_1, f_1, f_1f_1, f_1k_1, f_1i_1, i_1f_1, \\ k_1i_1f_1, 0, f_1k_1i_1, f_1i_1k_1, f_1i_1f_1\}.$$

Adding c to the left of each operator above yields the odd operators, for a total of $\#\mathcal{KF}_n \leq 34$. The operators are indeed distinct when, for instance, $X = \mathbb{R}$ and τ_1 is the usual topology on the reals, and in this case we get $\#\mathcal{KF}_n = 34$.

We are ready to state some elementary algebraic identities in $\mathcal{KF}_n^0$, which are easily proven. The first one is prominent in the solution to Kuratowski's original closure-complement problem.

Lemma 2.3. *In any n-topological space $(X, \tau_1, ..., \tau_n)$,*

(1) (Kuratowski) for each $1 \leq x \leq n$,

$$\boxed{k_xi_xk_xi_x = k_xi_x} \text{ and } \boxed{i_xk_xi_xk_x = i_xk_x;}$$

(2) for each $1 \leq x, y \leq n$,

$$\boxed{k_xk_y = k_{\max(x,y)}} \text{ and } \boxed{i_xi_y = i_{\max(x,y)};}$$

(3) for each $1 \leq x, y \leq n$,

$$\boxed{\text{if } x \leq y \text{ then } k_xf_y = f_y.}$$

Recall that an n-topological space $(X, \tau_1, ..., \tau_n)$ is *saturated* if whenever $1 \leq x, y \leq n$ and U is a nonempty τ_x-open set, then $i_yU \neq \emptyset$. For the remainder of the paper, we assume that our space $(X, \tau_1, ..., \tau_n)$ is saturated. The most basic and important identity, which we use extensively, is proven in [**1**]:

Lemma 2.4 (Banakh, Chervak, Martynyuk, Pylypovych, Ravsky, Simkiv)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y \leq n$, $k_xi_y = k_xi_x$ and $i_xk_y = i_xk_x$.*

This identity means that, assuming saturation, the second index in a word of the form k_xi_y or i_xk_y is irrelevant in determining the action of the operator. For this reason, we find it convenient to adopt a *star notation*, and simply write

$$\text{for each } 1 \leq x, y \leq n, \boxed{k_xi_y = k_xi_*} \text{ and } \boxed{i_xk_y = i_xk_*.}$$

We employ this notation in the following lemma.

Lemma 2.5 (IF Lemma)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y \leq n$,*

$$\boxed{i_xf_y = i_xf_*.}$$

Proof. Since interiors distribute over intersections, by Lemma 2.4 we have $i_xf_y = i_xk_y \wedge i_xk_yc = i_xk_* \wedge i_xk_*c = i_xf_*$. □

For other types of words, as below, it turns out that the value of y is irrelevant if $y \leq x$, but may matter if $y > x$.

Lemma 2.6 (FK Lemma)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y \leq n$,*

$$\boxed{f_x k_y = f_x k_{\max(x,y)}.}$$

Proof. If $y \geq x$ then the statement is trivial. Otherwise $y < x$, and we compute using Lemmas 2.3 and 2.4 that $f_x k_y = k_x k_y \wedge c i_x k_y = k_x \wedge c i_x k_* = f_x k_x = f_x k_{\max(x,y)}$. □

For many of our algebraic lemmas involving k_x or i_x, we may use DeMorgan's law to instantly deduce a "dual" corollary.

Lemma 2.7 (FI Lemma)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y \leq n$,*

$$\boxed{f_x i_y = f_x i_{\max(x,y)}.}$$

Proof. By duality: $f_x i_y = f_x c k_y c = f_x k_y c = f_x k_{\max(x,y)} c = f_x c i_{\max(x,y)} = f_x i_{\max(x,y)}$. □

Lemma 2.8 (FKF Lemma)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. Then for each $1 \leq x, y, z \leq n$,*

$$\boxed{\text{if } y \leq \max(x,z), \text{ then } f_x k_y f_z = f_x f_z.}$$

Proof. If $y \leq z$ then $k_y f_z = f_z$ by Lemma 2.3. Otherwise $y \leq x$, in which case we compute

$$\begin{aligned} f_x k_y f_z &= k_x k_y f_z \wedge c i_x k_y f_z \\ &= k_x f_z \wedge c i_x k_* f_z \\ &= k_x f_z \wedge c i_x f_z = f_x f_z. \end{aligned}$$

□

Lemma 2.9 (FIKI/FKIK/FKIF Lemma)**.** *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y \leq n$,*

- $\boxed{\text{if } y \leq x, \text{ then } f_x i_y k_* i_* = f_x k_x i_*.}$
- $\boxed{\text{if } y \leq x, \text{ then } f_x k_y i_* k_* = f_x i_x k_*.}$
- $\boxed{\text{if } y \leq x, \text{ then } f_x k_y i_* f_* = f_x i_x f_*.}$

Proof. For the first item, by Lemmas 2.3 and 2.4, compute

$$\begin{aligned} f_x i_y k_* i_* &= k_x i_* k_* i_* \wedge k_x c i_y k_* i_* \\ &= k_x i_* \wedge c i_x i_y k_* i_* \\ &= k_x k_x i_* \wedge c i_x k_x i_* \\ &= f_x k_x i_*. \end{aligned}$$

The second item follows from the first by duality. The third follows from the second, by observing that $f_x k_y i_* f_* = f_x k_y i_* k_* f_* = f_x i_x k_* f_* = f_x i_x f_*$. □

The next lemma is a generalization of Gaida-Eremenko's observation, together with its dual statement.

Lemma 2.10 (IFK/IFI Lemma). *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space.*

- *For any $1 \leq x, y, z \leq n$, $\boxed{i_x f_y k_z = 0.}$*
- *For any $1 \leq x, y, z \leq n$, $\boxed{i_x f_y i_z = 0.}$*

Proof. It suffices to prove that $i_n f_y k_z = 0$, for if there existed a set $A \subseteq X$ with $i_x f_y k_z A \neq \emptyset$, then by saturation, we would have $i_n f_y k_z A = i_n i_x f_y k_z A \neq \emptyset$, which would contradict $i_n f_y k_z = 0$.

We can use Lemma 2.5 to rewrite $i_n f_y k_z = i_n f_* k_z = i_n f_n k_z$. Then use Lemma 2.6 to write $i_n f_y k_z = i_n f_n k_n = 0$. □

Lemma 2.11 (FFK/FFI/FFF Lemma). *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. For each $1 \leq x, y, z \leq n$, the following hold.*

- $\boxed{f_x f_y k_z = k_x f_y k_z.}$
- $\boxed{\text{If } x \leq y, \text{ then } f_x f_y k_z = f_y k_z.}$
- $\boxed{f_x f_y i_z = k_x f_y i_z.}$
- $\boxed{\text{If } x \leq y, \text{ then } f_x f_y i_z = f_y i_z.}$
- $\boxed{f_x f_y f_z = k_x f_y f_z.}$
- $\boxed{\text{If } x \leq y, \text{ then } f_x f_y f_z = f_y f_z.}$

Proof. It suffices to prove the first statement, as the second follows immediately; the third and fourth follow from duality; and the fifth and sixth follow from the observation that $f_x f_y f_z = f_x f_y k_z f_z$.

Using Lemma 2.10, we compute

$$\begin{aligned} f_x f_y k_z &= k_x f_y k_z \wedge k_x c f_y k_z \\ &= k_x f_y k_z \wedge c i_x f_y k_z \\ &= k_x f_y k_z \wedge c0 \\ &= k_x f_y k_z \wedge 1 = k_x f_y k_z. \end{aligned}$$

□

Lemma 2.12 (FKFK/FKFI Lemma). *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space.*

- *For any $1 \leq x, y, z, w \leq n$, $\boxed{f_x k_y f_z k_w = k_{\max(x,y)} f_z k_w.}$*
- *For any $1 \leq x, y, z, w \leq n$, $\boxed{f_x k_y f_z i_w = k_{\max(x,y)} f_z i_w.}$*

Proof. Using Lemma 2.10 again,

$$\begin{aligned} f_x k_y f_z k_w &= k_x k_y f_z k_w \wedge c i_x k_y f_z k_w \\ &= k_{\max(x,y)} f_z k_w \wedge c i_x k_* f_z k_w \\ &= k_{\max(x,y)} f_z k_w \wedge c i_x f_z k_w \\ &= k_{\max(x,y)} f_z k_w \wedge c0 \\ &= k_{\max(x,y)} f_z k_w \wedge 1 = k_{\max(x,y)} f_z k_w . \end{aligned}$$

□

3. The Case of Two Topologies

In this section we look closely at the special case where $n = 2$, and solve the closure-complement-frontier problem for a saturated 2-topological space. The prototypical example is $(\mathbb{R}, \tau_s, \tau_u)$ where $\tau_s =$ the Sorgenfrey topology (in which basic open neighborhoods have the form $[a, b) = \{x \in \mathbb{R} : a \le x < b\}$) and $\tau_u =$ the usual Euclidean topology.

It is instructive to use Lemmas 2.3 through 2.12 to write out the distinct elements of $\mathcal{KF}_2^0$ explicitly. There turn out to be at most 60 of them. This is an enjoyable computation and we postpone the details until the more general case of Section 4, where n is arbitrary. The reader may verify the truth of the following proposition by observing that applying any of the generators k_x, i_x, or f_x $(x = 1, 2)$ to the left of any of the 60 words listed below will always simply produce another word on the list, and thus the entire monoid $\mathcal{KF}_2^0$ is accounted for.

Proposition 3.1. *The monoid $\mathcal{KF}_2^0$ consists of at most 60 elements, which are listed in the table below. Consequently, the monoid $\mathcal{KF}_2$ consists of at most 120 elements.*

Word Length	*Operators*	*Count*
0	Id	*1*
1	$i_1, i_2,$ $k_1, k_2,$ f_1, f_2	*6*
2	$k_1 i_*, k_2 i_*,$ $i_1 k_*, i_2 k_*,$ $f_1 i_1, f_1 i_2, f_2 i_2,$ $i_1 f_*, i_2 f_*,$ $f_1 k_1, f_1 k_2, f_2 k_2,$ $k_2 f_1,$ $f_1 f_1, f_1 f_2, f_2 f_1, f_2 f_2$	*17*
3	$i_1 k_* i_*, i_2 k_* i_*,$ $k_1 i_* k_*, k_2 i_* k_*,$ $f_1 k_1 i_*, f_1 k_2 i_*, f_2 k_2 i_*,$ $f_1 i_1 k_*, f_1 i_2 k_*, f_2 i_2 k_*,$ $0,$ $k_2 f_1 i_1, k_2 f_1 i_2,$ $k_1 i_* f_*, k_2 i_* f_*,$ $k_2 f_1 k_1, k_2 f_1 k_2,$ $f_1 k_2 f_1,$ $k_2 f_1 f_1, k_2 f_1 f_2,$ $f_1 i_1 f_*, f_1 i_2 f_*, f_2 i_2 f_*$	*23*
4	$f_1 i_2 k_* i_*,$ $f_1 k_2 i_* k_*,$ $k_2 f_1 k_1 i_*, k_2 f_1 k_2 i_*,$ $k_2 f_1 i_1 k_*, k_2 f_1 i_2 k_*,$ $f_1 k_2 i_* f_*,$ $k_2 f_1 i_1 f_*, k_2 f_1 i_2 f_*,$ $k_2 f_1 k_2 f_1$	*10*
5	$k_2 f_1 k_2 i_* k_*,$ $k_2 f_1 i_2 k_* i_*,$ $k_2 f_1 k_2 i_* f_*$	*3*

It is also straightforward to check, on a case-by-case basis, that the 60 operators in $\mathcal{KF}_2^0$ are distinct, in the sense that for any ω_1, ω_2 as in the table above with $\omega_1 \neq \omega_2$, there exists a subset A^{ω_1,ω_2} of some 2-topological space (X, τ_1, τ_2) for which $\omega_1 A^{\omega_1,\omega_2} \neq \omega_2 A^{\omega_1,\omega_2}$.

Combining this observation with the simple lemma below, we obtain the stronger fact that there exists a 2-topological space with an initial subset A which distinguishes all of the operators in $\mathcal{KF}_2$ simultaneously.

Lemma 3.2. *Suppose that for any distinct pair of operators $\omega_1, \omega_2 \in \mathcal{KF}_n^0$, there exists a saturated n-topological space X^{ω_1,ω_2} and a subset $A^{\omega_1,\omega_2} \subseteq X^{\omega_1,\omega_2}$ in which $\omega_1 A^{\omega_1,\omega_2} \neq \omega_2 A^{\omega_1,\omega_2}$. Then there exist a saturated n-topological space X and a subset $A \subseteq X$ such that $\omega_1 A \neq \omega_2 A$, for each pair of distinct operators $\omega_1, \omega_2 \in \mathcal{KF}_n^0$.*

Proof. If the assumption is true, then we can construct the n-topological disjoint union $X = \bigcup_{\substack{\omega_1,\omega_2 \in \mathcal{KF}_n^0 \\ \omega_1 \neq \omega_2}} X^{\omega_1,\omega_2}$ and form the initial set $A = \bigcup_{\omega_1 \neq \omega_2} A^{\omega_1,\omega_2}$. Then for any operators $\omega_1 \neq \omega_2$ in $\mathcal{KF}_n^0$, we have $(\omega_1 A)\Delta(\omega_2 A) \supseteq (\omega_1 A^{\omega_1,\omega_2})\Delta(\omega_2 A^{\omega_1,\omega_2}) \neq \emptyset$ (where Δ denotes the symmetric difference), and therefore $\omega_1 A \neq \omega_2 A$. □

Despite the preceding, we would like to follow the tradition of the closure-complement theorem by exhibiting an explicit initial set $A \subseteq \mathbb{R}$ which simultaneously distinguishes the operators in $\mathcal{KF}_2$.

Example 3.3 (An Initial Set For $\mathcal{KF}_2$ in the Usual/Sorgenfrey Line)**.** We consider the 2-topological space $(\mathbb{R}, \tau_1, \tau_2)$ where $\tau_1 = \tau_s$ is the Sorgenfrey topology and $\tau_2 = \tau_u$ is the usual Euclidean topology. We define

$$S^0 = \bigcup_{k=0}^{\infty} \left(\frac{1}{3^{2k+1}}, \frac{1}{3^{2k}} \right), S^1 = \bigcup_{k=0}^{\infty} \left[\frac{1}{3^{2k+1}}, \frac{1}{3^{2k}} \right),$$
$$S^2 = \bigcup_{k=0}^{\infty} \left[\frac{1}{3^{2k+1}}, \frac{1}{3^{2k}} \right], S^* = \bigcup_{k=0}^{\infty} \left(\frac{1}{3^{2k+1}}, \frac{1}{3^{2k}} \right],$$
$$T^0 = \bigcup_{k=1}^{\infty} \left(\frac{1}{3^{2k}}, \frac{2}{3^{2k}} \right), T^1 = \bigcup_{k=1}^{\infty} \left[\frac{1}{3^{2k}}, \frac{2}{3^{2k}} \right),$$
$$T^2 = \bigcup_{k=1}^{\infty} \left[\frac{1}{3^{2k}}, \frac{2}{3^{2k}} \right], T^* = \bigcup_{k=1}^{\infty} \left(\frac{1}{3^{2k}}, \frac{2}{3^{2k}} \right],$$

and we take the following initial set:

$$A = \left(S^0 \cap \mathbb{Q}\right) \cup T^1 \cup \left(\left(2 - S^0\right) \cap \mathbb{Q}\right) \cup \left(2 - T^0\right) \cup ((2,3) \cap \mathbb{Q}) \cup \{4\} \cup (5,6) \cup (6,7)$$
$$\cup \bigcup_{n=0}^{\infty} \left(\left(\frac{1}{2^{n+2}} S^2 + 8 - \frac{1}{2^n} \right) \cap \mathbb{Q} \right) \cup \bigcup_{n=0}^{\infty} \left(\frac{1}{2^{n+2}} S^2 + 10 - \frac{1}{2^n} \right) \cup (10, 11).$$

It is possible to verify by hand that applying the 60 operators of the monoid $\mathcal{KF}_2^0$ to A yields 60 distinct sets. The results of such a computation appear in a previous draft of this paper (posted August 3, 2019) accessible via `arXiV.org`. Bowron, in private communication, has also provided us with an elegant and brief computer-assisted verification. Rather than presenting such a verification here, we will turn to a stronger result, by first considering the natural partial order on the monoid $\mathcal{KF}_n^0$.

The partial order is defined as follows: for every $o_1, o_2 \in \mathcal{KF}_n^0$,

$$o_1 \leq o_2 \qquad \text{if and only if} \qquad o_1 A \subseteq o_2 A \text{ for every } A \subseteq X.$$

The partial orderings on $\mathcal{K}_1^0$, $\mathcal{KF}_1^0$ (see Figure 1), and other related monoids have been diagrammed by various authors; see especially [**6**] and [**4**]. It is clear that $\mathcal{KF}_n$ has a minimal element 0 and a maximal element k_n, and that $0 \leq i_n \leq \ldots \leq i_1 \leq \text{Id} \leq k_1 \leq \ldots \leq k_n$. It is also clear that for any set operator o we have $i_j o \leq o \leq k_j o$.

By the definition, for any operators o_1, o_2, o_3, if $o_1 \leq o_2$ then $o_1 o_3 \leq o_2 o_3$, so order is preserved by multiplication on the right. The operators i_j and k_j ($1 \leq j \leq n$) are also left order-preserving in the sense that if $o_1 \leq o_2$, then $i_j o_1 \leq i_j o_2$ and $k_j o_1 \leq k_j o_2$. On the other hand, f_j is not left order-preserving in general.

Example 3.4 (Exhibiting the Partial Order on $\mathcal{KF}_2^0$)**.** We will now show there exists a set A in a 2-topological space with the property that $o_1 \leq o_2$ if and only if $o_1 A \subseteq o_2 A$, for each $o_1, o_2 \in \mathcal{KF}_2^0$. In particular, the 60 operators of $\mathcal{KF}_2^0$ applied to A yield 60 distinct sets.

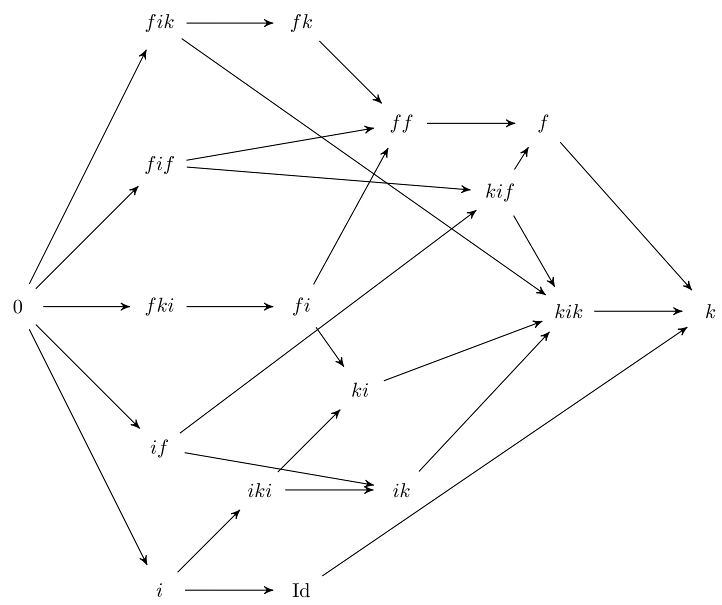


FIGURE 1. The partial ordering on the 17 operators of $\mathcal{KF}_1^0$, which was computed by Gaida and Eremenko but did not appear in the printed version of their article [**5**]; see also [**4**]. Subscripts are omitted from the notation since only one topology is involved.

We first present a list of apparently non-obvious inequalities in the partially ordered set $\mathcal{KF}_2^0$.

Proposition 3.5. *The following relations hold in any saturated* 2*-topological space* (X, τ_1, τ_2)*:*
(a) $f_1 i_1 \leq f_1 i_2$ *and* $f_1 k_1 \leq f_1 k_2$*;*
(b) $f_1 k_1 i_* \leq f_1 i_2 k_* i_*$ *and* $f_1 i_1 k_* \leq f_1 k_2 i_* k_*$*;*
(c) $f_1 f_1 \leq f_1 k_2 f_1$*;*
(d) $f_1 k_2 f_1 \leq f_1 f_2$*;*
(e) $f_1 k_2 i_* k_* \leq f_1 k_2$ *and* $f_1 i_2 k_* i_* \leq f_1 i_2$*;*
(f) $f_1 k_2 \leq f_1 f_2$.

Proof. For (a), we have $f_1 i_1 = k_1 i_* \wedge c i_1 = k_1 i_* \wedge k_1 c \leq k_1 i_* \wedge k_2 c = k_1 i_* \wedge c i_2 = f_1 i_2$, and the second statement follows in a dual way, because we can multiply the first inequality on the right by c, and get $f_1 k_1 = f_1 i_1 c \leq f_1 i_2 c = f_1 k_2$.

For (b), we have

$$f_1 k_1 i_* = k_1 k_1 i_* \wedge k_1 c k_1 i_* = k_1 i_* \wedge k_1 i_* k_* c.$$
$$f_1 i_2 k_* i_* = k_1 i_* k_* i_* \wedge k_1 c i_2 k_* i_* = k_1 i_* \wedge k_2 i_* k_* c.$$

The second statement follows dually.

For (c), we compute $f_1 f_1 = f_1 \wedge c i_1 f_1$ and $f_1 k_2 f_1 = k_2 f_1 \wedge c i_1 k_* f_1 = k_2 f_1 \wedge c i_1 f_1$, so the inequality $f_1 f_1 \leq f_1 k_2 f_1$ follows from $f_1 \leq k_2 f_1$.

For (d), compute $f_1 f_1 = f_1 \wedge c i_1 f_1$ and $f_1 k_2 f_1 = k_2 f_1 \wedge c i_1 k_* f_1 = k_2 f_1 \wedge c i_1 f_1$, so the inequality $f_1 f_1 \leq f_1 k_2 f_1$ follows from $f_1 \leq k_2 f_1$.

For (e), compute $f_1 k_2 i_* k_* = k_1 k_2 i_* k_* \wedge k_1 c k_2 i_* k_* = k_2 i_* k_* \wedge k_1 i_* c \leq k_2 \wedge k_1 c k_2 = f_1 k_2$.

Lastly, for (f), note that $k_1 i_* \leq k_1 \leq k_2$. Hence $f_1 k_2 = k_1 k_2 \wedge k_1 c k_2 = k_2 \wedge k_1 i_* c = k_2 \wedge (k_2 c \wedge k_1 i_* c) \leq [(k_2 \wedge k_2 c) \wedge k_1 i_* c] \vee [(k_2 \wedge k_2 c) \wedge k_1 i_*] = (k_2 \wedge k_2 c) \wedge (k_1 i_* c \vee k_1 i_*) = f_2 \wedge (k_1 c k_* \vee k_1 c k_* c) = f_2 \wedge k_1 (c k_2 \vee c k_2 c) = k_1 f_2 \wedge k_1 c (k_2 \wedge k_2 c) = f_1 f_2$. □

Using the inequalities in the proposition, together with the facts that closure and interior are left order-preserving, and all operators are right order-preserving, we obtain the diagram of the partially ordered set $\mathcal{KF}_2^0$ depicted in Figure 2.

To show that no further inequalities hold in general, we define a partition $P = \{P_0, \ldots, P_{12}\}$ of $\mathbb{R}^+$ such that for each inequality $o_1 \leq o_2$ ($o_1, o_2 \in \mathcal{KF}_2^0$) not implied by Figure 2, there exist integers $0 \leq \alpha_1 < \cdots < \alpha_n \leq 12$ $(1 \leq n \leq 12)$ satisfying $o_1(P_{\alpha_1} \cup \cdots \cup P_{\alpha_n}) \not\subseteq o_2(P_{\alpha_1} \cup \cdots \cup P_{\alpha_n})$ in $(\mathbb{R}^+, \tau_1, \tau_2)$ where $\tau_1 = \tau_s$ is the Sorgenfrey topology and $\tau_2 = \tau_u$ is the usual Euclidean topology.

The partition $\{\pi_1, \ldots, \pi_8\}$ of $(0, 1]$ is defined as follows:

$$\pi_1 = \bigcup_{n=1}^{\infty} \left\{\tfrac{1}{3^{2n}}\right\} \qquad \pi_3 = \bigcup_{n=1}^{\infty} \left\{\tfrac{2}{3^{2n}}\right\} \qquad \pi_5 = \bigcup_{n=1}^{\infty} \left\{\tfrac{1}{3^{2n-1}}\right\}$$
$$\pi_2 = \bigcup_{n=1}^{\infty} \left(\tfrac{1}{3^{2n}}, \tfrac{2}{3^{2n}}\right) \qquad \pi_4 = \bigcup_{n=1}^{\infty} \left(\tfrac{2}{3^{2n}}, \tfrac{1}{3^{2n-1}}\right) \qquad \pi_6 = \bigcup_{n=1}^{\infty} \left(\tfrac{1}{3^{2n-1}}, \tfrac{1}{3^{2n-2}}\right) \cap \mathbb{Q}$$
$$\pi_7 = \bigcup_{n=1}^{\infty} \left(\tfrac{1}{3^{2n-1}}, \tfrac{1}{3^{2n-2}}\right) \setminus \mathbb{Q}$$
$$\pi_8 = \{1\}.$$

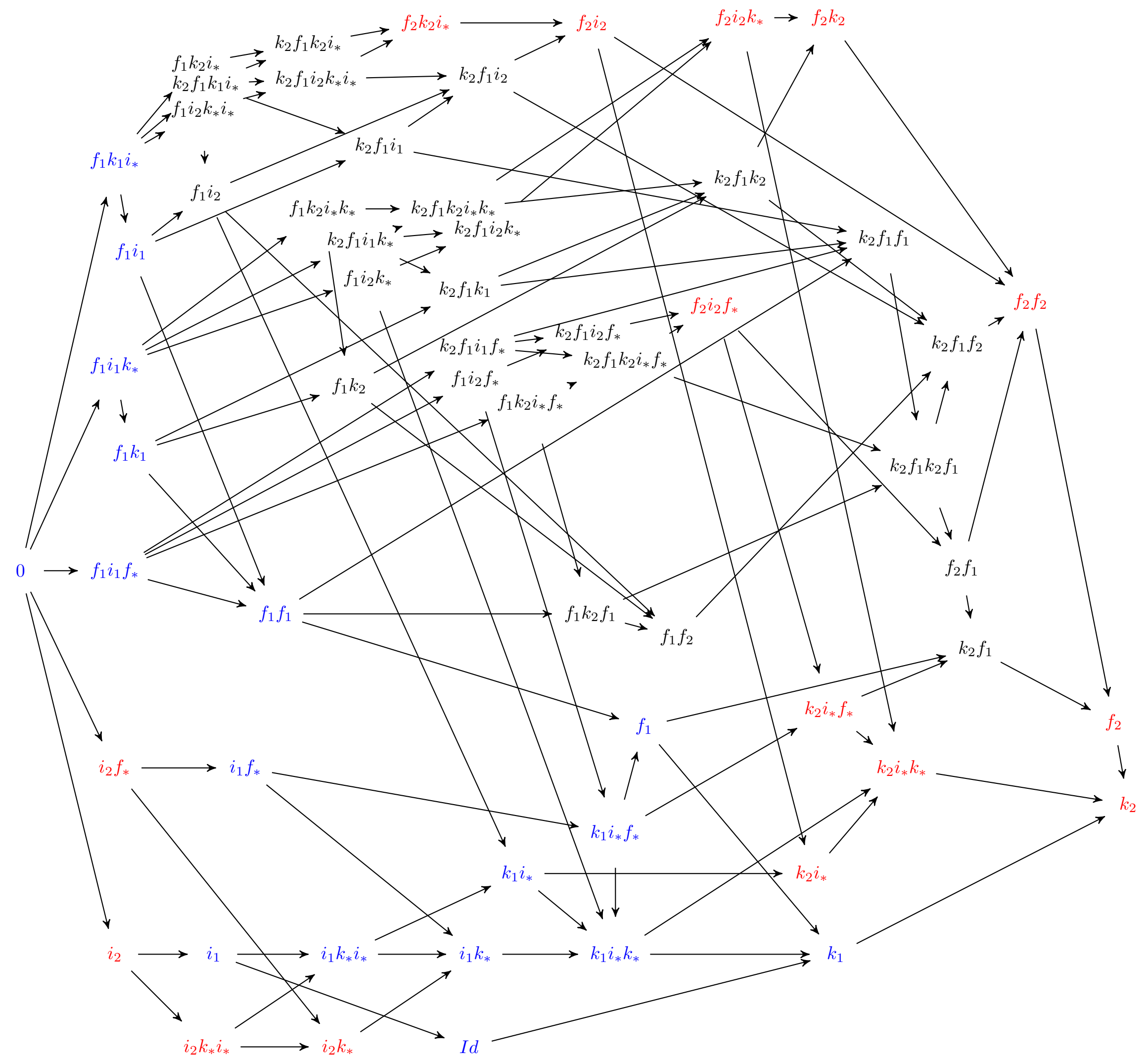


FIGURE 2. The partial ordering on $\mathcal{KF}_2^0$. The blue operators are operators that can be built using exclusively the τ_1 topology. The red operators are operators built using the topology τ_2 that cannot also be built using τ_1. The black operators are those built using a combination of both topologies.

For $1 \le j \le 8$, let $P_j = \bigcup_{n=0}^{\infty} \left(1 - \frac{1}{2^n} + \frac{1}{2^{n+2}}\pi_j\right)$. Thus

$$P_1 \cup \cdots \cup P_8 = (0, \tfrac{1}{4}] \cup (\tfrac{1}{2}, \tfrac{5}{8}] \cup (\tfrac{3}{4}, \tfrac{13}{16}] \cup \cdots .$$

To complete the definition of P, set

$$P_0 = \left\{1 - \tfrac{1}{2^n} : n = 1, 2, \ldots\right\} \qquad P_{10} = \{1\} \qquad P_{11} = (1, \infty) \cap \mathbb{Q}$$
$$P_9 = (\tfrac{1}{4}, \tfrac{1}{2}) \cup (\tfrac{5}{8}, \tfrac{3}{4}) \cup (\tfrac{13}{16}, \tfrac{7}{8}) \cup \cdots \qquad P_{12} = (1, \infty) \setminus \mathbb{Q}.$$

Then each of the following equations holds in $(\mathbb{R}^+, \tau_1, \tau_2)$:

$$\begin{aligned}
k_1P_0 &= P_0 & k_2P_0 &= P_0 \cup P_{10}\\
k_1P_1 &= P_0 \cup P_1 & k_2P_1 &= P_0 \cup P_1 \cup P_{10}\\
k_1P_2 &= P_0 \cup P_1 \cup P_2 & k_2P_2 &= P_0 \cup P_1 \cup P_2 \cup P_3 \cup P_{10}\\
k_1P_3 &= P_0 \cup P_3 & k_2P_3 &= P_0 \cup P_3 \cup P_{10}\\
k_1P_4 &= P_0 \cup P_3 \cup P_4 & k_2P_4 &= P_0 \cup P_3 \cup P_4 \cup P_5 \cup P_{10}\\
k_1P_5 &= P_0 \cup P_5 & k_2P_5 &= P_0 \cup P_5 \cup P_{10}\\
k_1P_6 &= P_0 \cup P_5 \cup P_6 \cup P_7 & k_2P_6 &= P_0 \cup P_1 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_{10}\\
k_1P_7 &= P_0 \cup P_5 \cup P_6 \cup P_7 & k_2P_7 &= P_0 \cup P_1 \cup P_5 \cup P_6 \cup P_7 \cup P_8 \cup P_{10}\\
k_1P_8 &= P_8 & k_2P_8 &= P_8 \cup P_{10}\\
k_1P_9 &= P_8 \cup P_9 & k_2P_9 &= P_0 \cup P_8 \cup P_9 \cup P_{10}\\
k_1P_{10} &= P_{10} & k_2P_{10} &= P_{10}\\
k_1P_{11} &= P_{10} \cup P_{11} \cup P_{12} & k_2P_{11} &= P_{10} \cup P_{11} \cup P_{12}\\
k_1P_{12} &= P_{10} \cup P_{11} \cup P_{12} & k_2P_{12} &= P_{10} \cup P_{11} \cup P_{12}.
\end{aligned}$$

Using these equations, all inclusions not implied by Figure 2 may be eliminated computationally. Bowron has written the following C program and Python script which verify the eliminations:

`https://github.com/mathematrucker/polytopological-spaces/blob/master/figure_2.c`
`https://github.com/mathematrucker/polytopological-spaces/blob/master/figure_2.py`

Following Lemma 3.2, we may take the disjoint union of all possible sets of the form $P_{\alpha_1} \cup \cdots \cup P_{\alpha_n}$ to obtain an initial set A with the property that if $o_1, o_2 \in \mathcal{KF}_2^0$ and $o_1 \not\le o_2$, then $o_1A \not\subseteq o_2A$. Consequently, $o_1 \le o_2$ if and only if $o_1A \subseteq o_2A$, for all $o_1, o_2 \in \mathcal{KF}_2^0$.

4. The General Case

We are ready to solve the closure-complement-frontier problem in the general setting of a saturated n-topological space where n is arbitrary. The surprising fact which underlies our computation is that every reduced word in $\mathcal{KF}_n^0$ has length ≤ 5, and in fact has the same form as one of the reduced words which we already computed in Section 3 for $\mathcal{KF}_2^0$.

In order to prove this observation we define the following subsets of $\mathcal{KF}_n^0$:

$$K = \{k_j : 1 \leq j \leq n\}$$
$$I = \{i_j : 1 \leq j \leq n\}$$
$$F = \{f_j : 1 \leq j \leq n\}$$

We also allow the formation of product sets in $\mathcal{KF}_n^0$ in the usual way, so we may write, for example, $KFI = \{kfi : k \in K, i \in I, f \in F\}$. So if $n = 2$, we could explicitly write

$$KFI = \{f_1i_1, f_1i_2, k_2f_1i_1, k_2f_1i_2, f_2i_*\}.$$

We will now adopt a notational convention which will not lead to ambiguity in the context of this paper, and which will help us clearly delineate word types in $\mathcal{KF}_n^0$. Suppose E is a set which is the n-times product of the sets K, I, and F (in any order). Then we denote by $(E)_r$ the set of all *reduced* words $\omega \in E$, i.e. those which do not admit any representation as a word of length $< n$. So, under this convention, if $n = 2$ we would have

$$(KFI)_r = \{k_2f_1i_1, k_2f_1i_2\}.$$

We are now ready to prove our main Theorem 1.1, which is a consequence of the more detailed theorem below.

Theorem 4.1. *Let $(X, \tau_1, ..., \tau_n)$ be a saturated n-topological space. Then $\mathcal{KF}_n^0$ is contained in the union of the sets in the left-hand column of the table below. The number of distinct elements in each such set is at most as listed in the right-hand column.*

Word-Type	*Number of Words*
$\{\text{Id}\}$	1
$IFK = \{0\}$	1
I	n
K	n
IK	n
KI	n
IKI	n
KIK	n
F	n
IF	n
FF	n^2
FI	$n + \binom{n}{2}$
FK	$n + \binom{n}{2}$
FIF	$n + \binom{n}{2}$
KIF	n
FIK	$n + \binom{n}{2}$

Word-Type	*Number of Words*
FKI	$n + \binom{n}{2}$
$(KF)_r$	$\binom{n}{2}$
$(KFK)_r$	$2\binom{n+1}{3}$
$(KFI)_r$	$2\binom{n+1}{3}$
$(KFF)_r$	$\binom{n}{2} \cdot n$
$(FKF)_r$	$\binom{n}{2} + 2\binom{n}{3}$
$(FIKI)_r$	$\binom{n}{2}$
$(FKIK)_r$	$\binom{n}{2}$
$(FKIF)_r$	$\binom{n}{2}$
$(KFIK)_r$	$2\binom{n+1}{3}$
$(KFKI)_r$	$2\binom{n+1}{3}$
$(KFIF)_r$	$2\binom{n+1}{3}$
$(KFKF)_r$	$\binom{n}{2} + 5\binom{n}{3} + 5\binom{n}{4}$
$(KFIKI)_r$	$\binom{n}{2} + 2\binom{n}{3}$
$(KFKIK)_r$	$\binom{n}{2} + 2\binom{n}{3}$
$(KFKIF)_r$	$\binom{n}{2} + 2\binom{n}{3}$

Consequently, the number of elements of $\mathcal{KF}_n^0$ *is at most*

$$\begin{aligned} p(n) &= 5\binom{n}{4} + 10\binom{n+1}{3} + 13\binom{n}{3} + (n+14)\binom{n}{2} + n^2 + 14n + 2 \\ &= \frac{5}{24}n^4 + \frac{37}{12}n^3 + \frac{79}{24}n^2 + \frac{101}{12}n + 2 \end{aligned}$$

and the number of elements of $\mathcal{KF}_n$ *is at most* $2p(n)$.

Proof. Let $\mathcal{X}$ be the union of all of the sets in the table above, so we want to prove $\mathcal{KF}_n^0 \subseteq \mathcal{X}$. For this, it suffices to check that **(A)** for each set E listed in the table above, and for each $1 \leq x \leq n$, we have $k_xE, i_xE, f_xE \subseteq \mathcal{X}$. Our second goal **(B)** is to establish the listed upper bound for the cardinality of each set.

We can begin the verification by making these observations:

- Every 0- and 1-letter word type in $\mathcal{KF}_n^0$ (i.e. the elements of $\{\mathrm{Id}\}$, K, I, and F) is accounted for in the table.
- There are $3^2 = 9$ possible 2-letter word types. By Lemma 2.3, we have $II = I$ and $KK = K$, and the other seven possible types are accounted for on the table. So all elements of $\mathcal{KF}_n^0$ which admit a word representation of length ≤ 2 are contained in $\mathcal{X}$.
- There are $3^3 = 27$ possible 3-letter word types. Ten of these reduce to 2-letter words using $II = I$ and $KK = K$, which by the previous bullet point, are already accounted for in the table. At most seventeen types remain, and among these, we know that $IFK = IFI = 0$ by Lemma 2.10, while $FFK = KFK$, $FFI = KFI$, and $FFF = KFF$ by Lemma 2.11. Also $IKF = IF$ by Lemmas 2.4 and 2.3, and since $F \subseteq KF$, we have $IFF \subseteq IFKF \subseteq \{0\}F \subseteq \{0\}$. This leaves eleven other possible 3-letter word types, each of which is listed in the table. Therefore, all elements of $\mathcal{KF}_n^0$ which admit a word representation of length ≤ 3 are already contained in a subset listed in the table.

By the last bullet point above, we see that whenever E consists of $\leq$ 2-letter words, then indeed we have $k_xE, i_xE, f_xE \subseteq \mathcal{X}$ for each $1 \leq x \leq n$, which establishes **(A)** for the sets $\{\mathrm{Id}\}$, K, I, IK, KI, F, IF FF, FI, FK, and $(KF)_r$. **(A)** is also immediate for the set $\{0\}$.

The cardinality bounds **(B)** are immediate for the sets $\{\mathrm{Id}\}$, $\{0\}$, K, I, IK, KI, F, and FF. By Lemma 2.5 the set IF consists of words of the form i_xf_* ($1 \leq x \leq n$), of which there are n many. The set $(KF)_r$ consists of elements of the form k_xf_y which do not reduce to 1-letter representations; by Lemma 2.3, it is necessary that $x > y$. There are $\binom{n}{2}$ many pairs (x, y) with $x > y$, so $\#(KF)_r \leq \binom{n}{2}$. Lastly, by Lemma 2.6, the set FK consists of words of the form f_xk_y where $1 \leq x \leq y \leq n$; there are $n + \binom{n}{2}$ many such pairs (x, y), and thus FK consists of no more than $n + \binom{n}{2}$ elements. A similar argument yields the same number for FI.

So to finish the proof, it remains only to check **(A)** and **(B)** for those sets E which consist of words of length ≥ 3.

<u>The sets IKI and KIK.</u> By Lemma 2.4, every element of IKI has the form $i_yk_*i_*$ for some $1 \leq y \leq n$, and thus $\#IKI \leq n$, establishing **(B)**. Note that for any

$1 \leq x \leq n$, we have $k_x i_y k_* i_* = k_x i_* k_* i_* = k_x i_*$ by Lemma 2.3, so $k_x IKI \subseteq KI \subseteq \mathcal{X}$. Also $i_x i_y k_* i_* = i_{\max(x,y)} k_* i_*$ by Lemma 2.3, so $i_x IKI \subseteq IKI \subseteq \mathcal{X}$. The word $f_x i_y k_* i_*$ either reduces to a ≤ 3-letter word, in which case it is a member of $\mathcal{X}$ by our previous remarks; or it does not reduce, in which case $f_x i_y k_* i_* \in (FIKI)_r \subseteq \mathcal{X}$. This establishes **(A)**, and the arguments are similar for KIK.

The set FIF. By Lemma 2.5, every element of FIF has the form $f_y i_z f_*$, so $FIF \subseteq FIf_1$. Therefore **(B)** $\#FIF \leq \#FI \leq n + \binom{n}{2}$. For $1 \leq x \leq n$, we have either $k_x f_y i_z f_* \in (KFIF)_r$ or $k_x f_y i_z f_*$ reduces to a shorter word; in either case we obtain $k_x f_y i_z f_* \in \mathcal{X}$ and hence $k_x FIF \subseteq \mathcal{X}$. By Lemma 2.10 we see $i_x f_y i_z f_* = 0 f_* = 0 \in \mathcal{X}$, and by Lemma 2.11 we see $f_x f_y i_z f_* = k_x f_y i_z f_* \in \mathcal{X}$, establishing **(A)**.

The set KIF. By Lemmas 2.4 and 2.5, every element of KIF has the form $k_y i_* f_*$, where $1 \leq y \leq n$, so **(B)** holds. For any $1 \leq x \leq n$, $k_x k_y i_* f_* = k_{\max(x,y)} i_* f_* \in KIF \subseteq \mathcal{X}$, and $i_x k_y i_* f_* = i_x k_* i_* k_* f_* = i_x k_* f_* \in IKF \subseteq \mathcal{X}$. The word $f_x k_y i_* f_*$ either reduces to a ≤ 3-letter word or else lies in $(FKIF)_r$; in either case it lies in $\mathcal{X}$, establishing **(A)**.

The sets FIK and FKI. Elements of FIK have the form $f_y i_z k_*$, so $FIK = FIk_1$, and **(B)** $\#FIK \leq \#FI \leq n + \binom{n}{2}$. For **(A)**, note that for any x, the word $k_x f_y i_z k_* = f_x f_y i_z k_*$ either reduces to a ≤ 3 letter word or else lies in $(KFIK)_r$, so it lies in $\mathcal{X}$, while $i_x FIK = \{0\}K = \{0\} \subseteq \mathcal{X}$ as well. The arguments are similar for FKI.

The sets $(KFK)_r$ and $(KFI)_r$. Elements of $(KFK)_r$ have the form $k_y f_z k_w$. To establish **(A)**, we note that for $1 \leq x \leq n$, we have $k_x k_y f_z k_w = k_{\max(x,y)} f_z k_w$ by Lemma 2.3, $i_x k_y f_z k_w = i_x k_* f_z k_w = i_x f_z k_w$ by Lemma 2.4, and $f_x k_y f_z k_w = k_{\max(x,y)} f_z k_w$ by Lemma 2.12. Then all three words admit representations of length ≤ 3, and therefore lie in $\mathcal{X}$.

For **(B)**, since $k_y f_z k_w$ cannot be written with ≤ 2 letters, by Lemma 2.3 it is necessary that $y > z$. Also, by Lemma 2.6, we may assume that $w \geq z$. The number of triples (y, z, w) with $y > z$ and $z \leq w$ may be found by the following reasoning: either $z = w$ or $z \neq w$. If $z = w$, we find $\binom{n}{2}$ many triples (y, z, z) with $y > z$. If $z \neq w$, either $w = y$ or $w \neq y$. If $w = y$ we again obtain $\binom{n}{2}$ many triples (y, z, y). If $w \neq y$, then there are $\binom{n}{3}$ many sets of distinct numbers $\{y, z, w\}$ where z is minimal; these each yield two choices of ordered triples (y, z, w) or (w, z, y). So the cardinality of $(KFK)_r$ is no more than $\binom{n}{2} + \binom{n}{2} + 2\binom{n}{3} = 2\binom{n+1}{3}$. The arguments are similar for $(KFI)_r$.

The set $(KFF)_r$. Elements of $(KFF)_r$ have the form $k_y f_z f_w$, which can be rewritten as $k_y f_z k_w f_w$; thus the arguments to establish **(A)** are exactly analogous to those given for the case of $(KFK)_r$. For **(B)**, we note that since $k_y f_z f_w$ cannot be written as a word of length ≤ 2, it must be the case that $k_y f_z \in (KF)_r$. Therefore $\#(KFF)_r \leq \#(KF)_r \cdot \#F = \binom{n}{2} \cdot n$.

The set $(FKF)_r$. Elements of $(FKF)_r$ have the form $f_y k_z f_w$. To establish **(A)**, note that for $1 \leq x \leq n$, we have $k_x f_y k_z f_w = f_x f_y k_z f_w$ by Lemma 2.11, and this word either admits a word representation of length ≤ 3 and therefore lies in $\mathcal{X}$, or else it lies in $(KFKF)_r \subseteq \mathcal{X}$. Also $i_x f_y k_z f_w = 0 f_w = 0 \in \mathcal{X}$.

For **(B)**, since $f_y k_z f_w$ cannot be written with ≤ 2 letters, by Lemma 2.8 it is necessary that $z > y$ and $z > w$. We have either $y = w$ or $y \neq w$. If $y = w$ we are looking for triples of the form (y, z, y) with $z > y$, of which there $\binom{n}{2}$ many. If $y \neq w$, we find $\binom{n}{3}$ many sets $\{y, z, w\}$ of distinct numbers where z is maximal; each of these yields two choices of ordered triples (y, z, w) or (w, z, y). So the cardinality of $(FKF)_r$ is no more than $\binom{n}{2} + 2\binom{n}{3}$.

At this point, we pause to observe the following: combining all the arguments in the previous parts, we have shown that if $o \in \mathcal{KF}_n^0$ admits any representation as a word of length ≤ 3, then for every $1 \leq x \leq n$, we have $k_x o, i_x o, f_x o \in \mathcal{X}$. All words of length ≤ 4 have this form, so put in other words, we have now shown:

- All elements of $\mathcal{KF}_n^0$ which admit a word representation of length ≤ 4 are already contained in a subset listed in the table.

The sets $(FIKI)_r$, $(FKIK)_r$, and $(FKIF)_r$. Elements of $(FIKI)_r$ have the form $f_y i_z k_* i_*$, where $y < z$ by Lemma 2.9. There are $\binom{n}{2}$ many such pairs (y, z), so **(B)** $\#(FIKI)_r \leq \binom{n}{2}$. For **(A)**, note that for any x, the word $k_x f_y i_z k_* i_* = f_x f_y i_z k_* i_*$ either reduces to a ≤ 4 letter word or else lies in $(KFIKI)_r$, so it lies in $\mathcal{X}$; while $i_x FIKI = \{0\} KI = \{0\} \subseteq \mathcal{X}$ as well. The arguments are similar for $(FKIK)_r$ and $(FKIF)_r$.

The sets $(KFIK)_r$, $(KFKI)_r$, and $(KFIF)_r$. All elements of $(KFIK)_r$ have the form $k_y f_z i_w k_*$ where $y > z$ and $z \leq w$, which implies $(KFIK)_r \subseteq (KFI)_r k_1$ and therefore $\#(KFIK)_r \leq \#(KFI)_r \leq 2\binom{n+1}{3}$, establishing **(B)**. For $1 \leq x \leq n$, we have $k_x k_y f_z i_w k_* = k_{\max(x,y)} f_z i_w k_*$ by Lemma 2.3, and $i_x k_y f_z i_w k_* = i_x k_* f_z i_w k_* = i_x f_z i_w k_*$ by Lemma 2.4, and $f_x k_y f_z i_w k_* = k_{\max(x,y)} f_z i_w k_*$ by Lemma 2.12. Each of these words has a representation of length ≤ 4, and therefore lies in $\mathcal{X}$, establishing **(A)**. The arguments are similar for $(KFKI)_r$ and $(KFIF)_r$.

The set $(KFKF)_r$. For $1 \leq x \leq n$, we have $k_x KFKF \subseteq KFKF \subseteq \mathcal{X}$ by Lemma 2.3, $i_x KFKF \subseteq IFKF \subseteq \mathcal{X}$ by Lemma 2.4, and $f_x KFKF \subseteq KFKF \subseteq \mathcal{X}$ by Lemma 2.12, so **(A)** holds.

To establish **(B)**, we observe that every element of $(KFKF)_r$ has the form $k_x f_y k_z f_w$, and because this cannot be shortened to a word of length ≤ 3, we must have $x > y$ by Lemma 2.3, and $y < z$, $z > w$ by Lemma 2.8. So we are looking for ordered quadruples (x, y, z, w) which alternate in magnitude with $x > y$, $y < z$, $z > w$. There are $\binom{n}{2}$ many such quadruples if $x = z$ and $y = w$; there are $2\binom{n}{3}$ many if $x = z$ but $y \neq w$; and there are $2\binom{n}{3}$ many if $y = w$ but $x \neq z$. If $x = w$, then necessarily $y < x$ and $z > x$, which yields an additional $\binom{n}{3}$ possible quadruples. If all of x, y, z, w are distinct, then either x or z is maximal. If x is maximal then the choice of minimality for y or w determines the quadruple, yielding $2\binom{n}{4}$ quadruples. If z is maximal then either y or w is minimal; if w is minimal the quadruple is determined, whereas if y is minimal then there are 2 ways to assign x and w. This gives another $(1+2)\binom{n}{4}$ quadruples where z is maximal. Thus we compute a bound of $\#(KFKF)_r \leq \binom{n}{2}+(2+2+1)\binom{n}{3}+(2+1+2)\binom{n}{4}$, as in the table.

At this point, our computations up to this point have shown:

- All elements of $\mathcal{KF}_n^0$ which admit a word representation of length ≤ 5 are already contained in a subset listed in the table.

The sets $(KFIKI)_r$, $(KFKIK)_r$, and $(KFKIF)_r$. Every element of $(KFIKI)_r$ has the form $k_y f_z i_w k_* i_*$, and for any $1 \leq x \leq n$, we have $k_x k_y f_z i_w k_* i_* = f_x k_y f_z i_w k_* i_* = k_{\max(x,y)} f_z i_w k_* i_*$ by Lemmas 2.3 and 2.12, while $i_x k_y f_z i_w k_* i_* = i_x k_* f_z i_w k_* i_* = i_x f_z i_w k_* i_*$ by Lemma 2.4. In all three cases we find representations of length ≤ 5, so $k_x(KFIKI)_r$, $i_x(KFIKI)_r$, $f_x(KFIKI)_r \subseteq \mathcal{X}$ and we have proven **(A)**.

For **(B)**, we note that since $k_y f_z i_w k_* i_*$ does not reduce to a word of length ≤ 4, we must have $y > z$ by Lemma 2.3, and by Lemma 2.9 we have $w > z$. Thus we are looking for triples (y, z, w) with $y > z$ and $z < w$. By arguments analogous to those in the case of $(FKF)_r$, we compute that $\#(KFIKI)_r \leq \binom{n}{2} + 2\binom{n}{3}$. The arguments for $(KFKIK)_r$ and $(KFKIF)_r$ are similar.

This completes the proof. □

Example 4.2 (Separating KFKF Words)**.** In [**1**], the authors show that $\#\mathcal{K}_n \leq 12n+2$ for a saturated n-topological space, so we expect the size of the Kuratowski monoid to grow linearly with n. Our corresponding formula $p(n)$ in Theorem 1.1 implies quartic growth for the Kuratowski-Gaida-Eremenko monoid $\mathcal{KF}_n$. As is evident from the proof, the sole reason for this is that the set of reduced words $(KFKF)_r = \{k_x f_y k_z f_w : x > y, y < z, z > w, 1 \leq x, y, z, w \leq n\}$ is expected to contain $\binom{n}{2} + 5\binom{n}{3} + 5\binom{n}{4}$ elements.

It is interesting to see a natural example of a saturated 4-topological space in which the elements of $(KFKF)_r$ are distinct. Consider $(\mathbb{R}^3, \tau_1, \tau_2, \tau_3, \tau_4)$, where $\tau_1 = \tau_s \times \tau_s \times \tau_s$, $\tau_2 = \tau_s \times \tau_s \times \tau_u$, $\tau_3 = \tau_s \times \tau_u \times \tau_u$, and $\tau_4 = \tau_u \times \tau_u \times \tau_u$. Define $B = ((1,2) \times (0,2) \times (0,2)) \cup ((0,2) \times (1,2) \times (0,2)) \cup ((0,2) \times (0,2) \times (1,2))$, and let $\{C_n : n \in \mathbb{N}\}$ be a countably infinite collection of pairwise disjoint τ_4-closed sub-cubes of $(0,1) \times (0,1) \times (0,1)$ with the property that if $C = \bigcup_{n\in\mathbb{N}} C_n$, then the set of τ_4-derived points of C is exactly $C' = k_4 C \backslash C = (\{1\} \times [0,1] \times [0,1]) \cup ([0,1] \times \{1\} \times [0,1]) \cup ([0,1] \times [0,1] \times \{1\})$. We denote $B_{\mathbb{Q}} = B \cap (\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q})$, and we take for our initial set $A = B_{\mathbb{Q}} \cup \left(\bigcup_{n\in\mathbb{N}} i_4 C\right)$.

We also consider the particular open cube $i_4 C_0 \subseteq A$, say $i_4 C_0 = (x_0, x_1) \times (y_0, y_1) \times (z_0, z_1)$, and we label the following sets:

$$\phi = (x_0, x_1) \times (y_0, y_1) \times \{z_1\} \quad = \text{the upper face of } C_0;$$
$$\psi = (x_0, x_1) \times \{y_1\} \times (z_0, z_1) \quad = \text{the forward face of } C_0;$$
$$q = (0,1) \times \{1\} \times (0,1) \quad = \text{the inner rear face of } B;$$
$$r = \{1\} \times (0,1) \times (0,1) \quad = \text{the inner left face of } B;$$
$$Q = (0,2) \times \{2\} \times (0,2) \quad = \text{the forward face of } B;$$

$$R = \{2\} \times (0,2) \times (0,2) \qquad = \text{the right face of } B;$$
$$U = [((1,2) \times (0,2)) \cup ((0,2) \times (1,2))] \times \{0\} \qquad = \text{the outer lower face of } B;$$
$$V = ((1,2) \times \{0\} \times (0,2)) \cup ((0,2) \times \{0\} \times (1,2)) \qquad = \text{the outer rear face of } B.$$

Then by direct computation, one may verify the following properties about the sets $k_x f_y k_z f_w A$, which differentiate all possible ordered quadruples (x, y, z, w) satisfying $x > y, y < z, z > w$:

(1) (a) If $w = 1$ then ϕ, ψ are disjoint from $k_x f_y k_z f_w A$.
(b) If $w = 2$ then $\phi \subseteq k_x f_y k_z f_w A$ but $\psi \cap k_x f_y k_z f_w A = \emptyset$.
(c) If $w = 3$ then $\phi, \psi \subseteq k_x f_y k_z f_w A$.

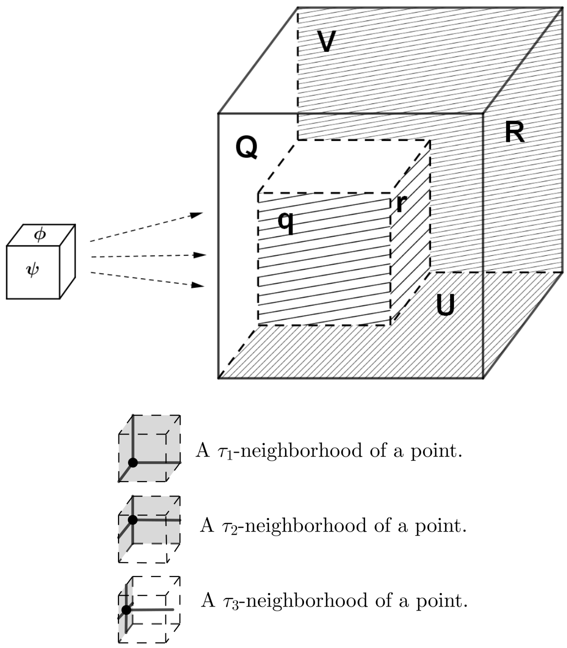
(2) (a) If $z = 2$ then Q, R are disjoint from $k_x f_y k_z f_w A$.
(b) If $z = 3$ then $Q \subseteq k_x f_y k_z f_w A$ but $R \cap k_x f_y k_z f_w A = \emptyset$.
(c) If $z = 4$ then $Q, R \subseteq k_x f_y k_z f_w A$.

(3) (a) If $y = 1$ then U, V are disjoint from $k_x f_y k_z f_w A$.
(b) If $y = 2$ then $U \subseteq k_x f_y k_z f_w A$ but $V \cap k_x f_y k_z f_w = \emptyset$.
(c) If $y = 3$ then $U, V \subseteq k_x f_y k_z f_w A$.

(4) (a) If $x = 2$ then q, r are disjoint from $k_x f_y k_z f_w A$.
(b) If $x = 3$ then $q \subseteq k_x f_y k_z f_w A$ but $r \cap k_x f_y k_z f_w A = \emptyset$.
(c) If $x = 4$ then $q, r \subseteq k_x f_y k_z f_w A$.

From the above, distinct quadruples (x, y, z, w) yield distinct sets $k_x f_y k_z f_w A$, and therefore

FIGURE 3. From left to right: the set C_0 and its faces; the set B and its faces; typical basic open neighborhoods in τ_1, τ_2, τ_3.



$$\#(KFKF)_r[A] = \#\{k_x f_y k_z f_w A : 1 \le x, y, z \le n, x > y, y < z, z > w\} = \binom{4}{2} + 5\binom{4}{3} + 5\binom{4}{2} = 31.$$

5. Separating Kuratowski-Gaida-Eremenko Words

The goal of this section is to prove that our upper bound $p(n)$ is sharp for every n. Guided by the results of the previous section, we introduce the following definition: a word in the generators $\{k_x, i_x, f_x : 1 \le x \le n\}$ (formally, an element of the free semigroup on $3n$ letters) will be called a **Kuratowski-Gaida-Eremenko word**, or KGE-word, if it has one of the following forms:

- Id or $0 = i_* f_* k_*$,
- k_x, i_x, $i_x k_*$, $k_x i_*$, $i_x k_* i_*$, $k_x i_* k_*$, f_x, $i_x f_*$, or $k_x i_* f_*$,
- $f_x f_y$,
- $k_x f_y$ where $x > y$,
- $f_x i_y k_* i_*$, $f_x k_y i_* k_*$, or $f_x k_y i_* f_*$ where $x < y$,
- $f_x i_y$, $f_x k_y$, $f_x i_y f_*$, $f_x i_y k_*$, or $f_x k_y i_*$ where $x \le y$,
- $k_x f_y f_z$ where $x > y$,
- $f_x k_y f_z$ where $x < y$ and $y < z$,
- $k_x f_y k_z$, $k_x f_y i_z$, $k_x f_y i_z k_*$, $k_x f_y k_z i_*$, or $k_x f_y i_z f_*$ where $x > y$ and $y \le z$,
- $k_x f_y i_z k_* i_*$, $k_x f_y k_z i_* k_*$, or $k_x f_y k_z i_* f_*$ where $x > y$ and $y < z$,
- $k_x f_y k_z f_w$ where $x > y$, $y < z$, and $z > w$.

We understand the $*$-notation as imposing an equivalence relation on the KGE-words: for example, although strictly speaking $i_1 f_1 k_1$ and $i_1 f_1 k_2$ are distinct words in the free semigroup, we regard them here as merely two representations of the same KGE-word 0; on the other hand $f_1 f_1$ and $f_1 f_2$ are distinct KGE-words. With this understanding in place, the number of KGE-words is $p(n)$. For convenience, we allow words and operators to be used interchangeably when the precise meaning is clear. So each KGE-word corresponds to at most one element of $\mathcal{KF}_n$, whereas *a priori* an element of $\mathcal{KF}_n$ may be represented by more than one KGE-word.

We note that in any monoid $\mathcal{KF}_n^0$, by Lemmas 2.3 through 2.12, we have the following set inclusions:

$$\begin{aligned}
KF &\supseteq (KF)_r \cup F;\\
KFK &\supseteq (KFK)_r \cup FK;\\
KFI &\supseteq (KFI)_r \cup FI;\\
KFIF &\supseteq (KFIF)_r \cup FIF;\\
KFKIF &\supseteq (KFKIF)_r \cup (FKIF)_r;\\
KFIK &\supseteq (KFIK)_r \cup FIK;\\
KFKIK &\supseteq (KFKIK)_r \cup (FKIK)_r;\\
KFKI &\supseteq (KFKI)_r \cup FKI;\\
KFIKI &\supseteq (KFIKI)_r \cup (FIKI)_r;\\
KFKF &\supseteq (KFKF)_r \cup (KFF)_r \cup (FKF)_r \cup FF.
\end{aligned}$$

Therefore the following holds.

Proposition 5.1. *Each KGE-word belongs to at least one of the following sets in* $\mathcal{KF}_n^0$*:*

- $\{\mathrm{Id}\}$ *or* $\{0\}$*;*
- K*,* I*,* IK*,* KI*,* KIK*,* IKI*,* IF*,* KIF*;*
- KF*,* KFK*,* KFI*;*
- $KFKF$*; or*
- $KFIF \cup KFKIF$*,* $KFIK \cup KFKIK$*,* $KFKI \cup KFIKI$*.*

For the reader's convenience, we note that the 17 sets above correspond to the 17 distinct even operators which comprise the monoid $\mathcal{KF}_1^0$ from [**5**].

Theorem 1.2. *For every* $n \geq 1$*, there exists a saturated polytopological space* $(X, \tau_1, ..., \tau_n)$ *in which* $\#\mathcal{KF}_n^0 = p(n)$ *and* $\#\mathcal{KF}_n = 2p(n)$*. In fact, there is an initial set* $A \subseteq X$ *such that* $\#\{oA : o \in \mathcal{KF}_n\} = 2p(n)$*.*

Proof. Applying Lemma 3.2, it suffices to demonstrate the following: For any pair of distinct KGE-words $\omega_1, \omega_2 \in \mathcal{KF}_n$, there exists a saturated n-topological space X^{ω_1,ω_2} and a subset $A^{\omega_1,\omega_2} \subseteq X^{\omega_1,\omega_2}$ in which $\omega_1 A^{\omega_1,\omega_2} \neq \omega_2 A^{\omega_1,\omega_2}$. We verify the claim for $\omega_1 \neq \omega_2$ by using the cases delineated in Proposition 5.1.

Case 1: $\omega_1 \in E_1$ and $\omega_2 \in E_2$, where E_1 and E_2 are distinct subsets from Proposition 5.1. Then we may take for our separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_n = \tau_u$, and take for our initial set A the example exhibited by Gaida-Eremenko in [**5**]. In this case, because all topologies are equal, the monoid $\mathcal{KF}_n^0$ is actually equal to $\mathcal{KF}_1^0$ and we get the following reductions: $KFIF \cup KFKIF = FIF$, $KFIK \cup KFKIK = FIK$, $KFKI \cup KFIKI = FKI$, $KFKF = FF$, $KF = F$, $KFK = FK$, $KFI = FI$. But elements ω_1, ω_2 taken from distinct word types will produce different sets $\omega_1 A \neq \omega_2 A$, as demonstrated by Gaida and Eremenko.

Case 2: $\omega_1, \omega_2 \in E$ where $E = K,\ I,\ IK,\ KI,\ KIK,\ IKI,\ IF$, or KIF. Assume that, for example, that $\omega_1, \omega_2 \in KIK$. We have $\omega_1 = k_{x_1} i_* k_*$ and $\omega_2 = k_{x_2} i_* k_*$ where $1 \leq x_1, x_2 \leq n$, and since $\omega_1 \neq \omega_2$, we have $x_1 \neq x_2$. Assume without loss of generality that $x_1 < x_2$, and take for a separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{x_1} = \tau_s$ and $\tau_{x_1+1} = ... = \tau_n = \tau_u$. Take the initial set A from Example 3.3. Then $\omega_1 A = k_1 i_* k_* A \neq k_n i_* k_* A = \omega_2 A$. The proofs for the other sets $E = K, I, ...$ etc. are similar because words in these sets E depend on only one index, and we leave them to the reader.

Case 3: $\omega_1, \omega_2 \in KF$. If $\omega_1, \omega_2 \in KF$, then we have $\omega_1 = k_{x_1} f_{y_1}$ and $\omega_2 = k_{x_2} f_{y_2}$ where $1 \leq x_1, y_1, x_2, y_2 \leq n$, $x_1 \geq y_1$, and $x_2 \geq y_2$. Assuming $(x_1, y_1) \neq (x_2, y_2)$, we have either $x_1 \neq x_2$ or $y_1 \neq y_2$.

Sub-Case (a): Suppose $y_1 \neq y_2$; without loss of generality assume $y_1 < y_2$. Then take for a separating space $(\mathbb{R}, \tau_1, .., \tau_n)$ where $\tau_1 = ... = \tau_{y_1} = \tau_s$ and $\tau_{y_1+1} = ... = \tau_n = \tau_u$, and take for an initial set A as in Example 3.3. Then we have $\omega_1 = k_{x_1} f_{y_1} = k_{x_1} f_1$, which is equal to either f_1 or $k_n f_1$ depending on the value of x_1. On the other hand since $x_2 \geq y_2 > y_1$, we have $\omega_2 = k_{x_2} f_{y_2} = k_n f_n = f_n$. Since $f_1 A \neq k_n f_1 A \neq f_n A$, we conclude $\omega_1 A \neq \omega_2 A$ as desired.

Sub-Case (b): Suppose $y_1 = y_2$ but $x_1 \neq x_2$; without loss of generality assume $x_1 < x_2$. Take for a separating space $(\mathbb{R}, \tau_1, .., \tau_n)$ where $\tau_1 = ... = \tau_{x_1} = \tau_s$ and $\tau_{x_1+1} = ... = \tau_n = \tau_u$, and take the usual initial set A as in Example 3.3. Then since $y_1 \leq x_1$, we have $\omega_1 = k_{x_1} f_{y_1} = k_1 f_1 = f_1$, while since $y_2 = y_1 \leq x_1$, we have $\omega_2 = k_{x_2} f_{y_2} = k_n f_1$. So $\omega_1 A \neq \omega_2 A$ as in Example 3.3.

Case 4: $\omega_1, \omega_2 \in KFK$. The idea of this proof is the same as in Case 3. Suppose that $\omega_1, \omega_2 \in KFK$, then we have $\omega_1 = k_{x_1} f_{y_1} k_{z_1}$ and $\omega_2 = k_{x_2} f_{y_2} k_{z_2}$ where $1 \leq x_1, y_1, z_1, x_2, y_2, z_2 \leq n$, $x_1 \geq y_1$, $y_1 \leq z_1$, $x_2 \geq y_2$, $y_2 \leq z_2$. We have $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$, and therefore $x_1 \neq x_2$, $y_1 \neq y_2$ or $z_1 \neq z_2$.

Sub-Case (a): Suppose $z_1 \neq z_2$, so without loss of generality $z_1 < z_2$. Take for a separating space $(\mathbb{R}, \tau_1, .., \tau_n)$ where $\tau_1 = ... = \tau_{z_1} = \tau_s$ and $\tau_{z_1+1} = ... = \tau_n = \tau_u$, and take for an initial set A as in Example 3.3. Then since $y_1 \leq z_1$, we have $\omega_1 = k_{x_1} f_{y_1} k_{z_1} = k_{x_1} f_1 k_1$, which is equal to either $f_1 k_1$ or $k_n f_1 k_1$ depending on the value of x_1. On the other hand $\omega_2 = k_{x_2} f_{y_2} k_n$, so ω_2 is equal to either $k_1 f_1 k_n = f_1 k_n$, $k_n f_1 k_n$, or $k_n f_n k_n = f_n k_n$, depending on the values of x_2, y_2. These five distinct possibilities yield five distinct sets when applied to A, so we conclude $\omega_1 A \neq \omega_2 A$ as desired.

Sub-Case (b): Suppose $z_1 = z_2$ but $y_1 < y_2$. Take for a separating space $(\mathbb{R}, \tau_1, .., \tau_n)$ where $\tau_1 = ... = \tau_{y_1} = \tau_s$ and $\tau_{y_1+1} = ... = \tau_n = \tau_u$, and take the usual initial set A as in Example 3.3. Then, considering all possible values of x_1, z_1, we compute that $\omega_1 = k_{x_1} f_1 k_{z_1} \in \{f_1 k_1, f_1 k_n, k_n f_1 k_1, k_n f_1 k_n\}$. On the other hand since $x_2, z_2 \geq y_2 > y_1$, we have $\omega_2 = k_n f_n k_n = f_n k_n$. So $\omega_1 A \neq \omega_2 A$.

Sub-Case (c): Suppose $z_1 = z_2$ and $y_1 = y_2$ but $x_1 < x_2$. Take for a separating space $(\mathbb{R}, \tau_1, .., \tau_n)$ where $\tau_1 = ... = \tau_{x_1} = \tau_s$ and $\tau_{x_1+1} = ... = \tau_n = \tau_u$, and take the usual initial set A as in Example 3.3. We compute $\omega_1 = k_{x_1} f_{y_1} k_{z_1} = k_1 f_1 k_{z_1} \in \{f_1 k_1, f_1 k_n\}$, and $\omega_2 = k_{x_2} f_{y_2} k_{z_2} = k_n f_{y_2} k_{z_2} \in \{k_n f_1 k_1, k_n f_1 k_n, f_n k_n\}$, so $\omega_1 A \neq \omega_2 A$.

Case 5: $\omega_1, \omega_2 \in KFI$. In this case take the same separating space as in Case 4, but for an initial set take cA where A is the initial set from Case 4. We are done if $\omega_1 cA \neq \omega_2 cA$, and this follows from Case 4 because both $\omega_1 c$ and $\omega_2 c$ are elements of KFK. (To verify this, write $\omega_1 = k_{x_1} f_{y_1} i_{z_1}$ where $1 \leq x_1, y_1, z_1 \leq n$, $x_1 \geq y_1$, and $y_1 \leq z_1$. Then $\omega_1 c = k_{x_1} f_{y_1} c k_{z_1} = k_{x_1} f_{y_1} k_{z_1} \in KFK$, and similarly for ω_2.)

Case 6: $\omega_1, \omega_2 \in KFKF$. We proceed similarly to Cases 3 and 4. We have $\omega_1 = k_{x_1} f_{y_1} k_{z_1} f_{w_1}$ and $\omega_2 = k_{x_2} f_{y_2} k_{z_2} f_{w_2}$ where $1 \leq x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2 \leq n$, $x_1 \geq y_1$, $y_1 \leq z_1$, $z_1 \geq w_1$, $x_2 \geq y_2$, $y_2 \leq z_2$, and $z_2 \geq w_2$. We also know $(x_1, y_1, z_1, w_1) \neq (x_2, y_2, z_2, w_2)$, which gives us four sub-cases.

Sub-Case (a): Suppose $w_1 \neq w_2$, so without loss of generality $w_1 < w_2$. We consider $(\mathbb{R}, \tau_1, ..., \tau_n)$ with $\tau_1 = ... = \tau_{w_1} = \tau_s$ and $\tau_{w_1+1} = ... = \tau_n = \tau_u$. Considering all possible values of $x_1, y_1, z_1, x_2, y_2, z_2$, we compute that

$$\omega_1 = k_{x_1} f_{y_1} k_{z_1} f_1 \in \{f_1 f_1, f_n f_1, f_1 k_n f_1, k_n f_1 f_1, k_n f_1 k_n f_1\}$$
$$\omega_2 = k_{x_2} f_{y_2} k_{z_2} f_n \in \{f_1 f_n, f_n f_n, k_n f_1 f_n\}$$

from which we conclude $\omega_1 A \neq \omega_2 A$, where A is the initial set from Example 3.3.

Sub-Case (b): Suppose $w_1 = w_2$ but $z_1 < z_2$, and consider $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{z_1} = \tau_s$ and $\tau_{z_1+1} = ... = \tau_n = \tau_u$. Since $w_1, y_1 \leq z_1$, we get $\omega_1 = k_{x_1} f_1 k_1 f_1 \in \{f_1 f_1, k_n f_1 f_1\}$ whereas $\omega_2 = k_{x_2} f_{y_2} k_n f_1 \in \{f_1 k_n f_1, k_n f_1 k_n f_1, f_n f_1\}$, so $\omega_1 A \neq \omega_2 A$ where A is as in Example 3.3.

Sub-Case (c): Suppose now $w_1 = w_2$, $z_1 = z_2$ but $y_1 < y_2$, and consider $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{y_1} = \tau_s$ and $\tau_{y_1+1} = ... = \tau_n = \tau_u$. Since $z_1 = z_2 \geq y_2$, we get $\omega_1 = k_{x_1} f_1 k_n f_{w_1} \in \{f_1 k_n f_1, k_n f_1 k_n f_1, f_1 f_n, k_n f_1 f_n\}$, whereas since $x_2, z_2 \geq y_2$, we have $\omega_2 = k_n f_n k_n f_{w_2} \in \{f_n f_1, f_n f_n\}$, so $\omega_1 A \neq \omega_2 A$ where A is as in Example 3.3.

Sub-Case (d): Suppose $w_1 = w_2$, $z_1 = z_2$, $y_1 = y_2$ but $x_1 < x_2$, and consider $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{x_1} = \tau_s$ and $\tau_{x_1+1} = ... = \tau_n = \tau_u$. Since $y_1 \leq x_1$, we get $\omega_1 = k_1 f_1 k_{z_1} f_{w_1} \in \{f_1 f_1, f_1 f_n, f_1 k_n f_1\}$ whereas $\omega_2 = k_n f_{y_2} k_{z_2} f_{w_2} \in \{k_n f_1 f_1, k_n f_1 f_n, k_n f_1 k_n f_1, f_n f_1\}$, so $\omega_1 A \neq \omega_2 A$ where A is as in Example 3.3. □

Case 7: $\omega_1, \omega_2 \in KFIF \cup KFKIF$. We proceed similarly to Cases 3, 4, and 6. Observe that we may write

$$\omega_1 = k_{x_1} f_{y_1} \sigma_{z_1} i_1 f_*$$

with $x_1 \geq y_1$, $y_1 \leq z_1$, where either $\sigma_{z_1} = i_{z_1} \in I$ (in case $\omega_1 \in KFIF$) or $\sigma_{z_1} = k_{z_1} \in K$ where $z_1 > y_1$ (in case $\omega_1 \in KFKIF \backslash KFIF$). Similarly, we may write ω_2 as

$$\omega_2 = k_{x_2} f_{y_2} \rho_{z_2} i_1 f_*$$

where $x_2 \geq y_2$, $y_2 \leq z_2$, and either $\rho_{z_2} = i_{z_2}$ or else $\rho_{z_2} = k_{z_2}$ and $z_2 > y_2$. Since $\omega_1 \neq \omega_2$, there are four sub-cases: either $z_1 \neq z_2$; or $z_1 = z_2$ but $\sigma_{z_1} \neq \rho_{z_2}$; or $y_1 \neq y_2$; or $x_1 \neq x_2$. In each of the four sub-cases below, we denote $\sigma_1 = i_1$ and $\sigma_n = i_n$ if $\sigma_{z_1} = i_{z_1}$; and $\sigma_1 = k_1$ and $\sigma_n = k_n$ if $\sigma_{z_1} = k_{z_1}$. Similarly we allow ρ_1, ρ_n to denote either i_1, i_n or k_1, k_n respectively as implied by the value of ρ_{z_2}.

Sub-Case (a): Suppose $z_1 < z_2$. Consider the separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{z_1} = \tau_s$ and $\tau_{z_1+1} = ... = \tau_n = \tau_u$. Since $y_1 \leq z_1$, we have $\omega_1 = k_{x_1} f_1 \sigma_1 i_1 f_* = k_{x_1} f_1 i_1 f_*$, so $\omega_1 = f_1 i_1 f_*$ or $\omega_1 = k_n f_1 i_1 f_*$, depending on the value of x_1. On the other hand, considering all possible values of x_2, y_2, and $\rho_{z_2} = \rho_n$, we compute

$$\omega_2 = k_{x_2} f_{y_2} \rho_n i_1 f_* \in \{f_1 i_n f_*, f_1 k_n i_* f_*, f_n i_n f_*, k_n f_1 k_n i_* f_*, k_n f_1 i_n f_*\}.$$

It follows that $\omega_1 A \neq \omega_2 A$, where A is the initial set from Example 3.3.

Sub-Case (b): Suppose $z_1 = z_2$, but $\sigma_{z_1} = k_{z_1}$ with $z_1 > y_1$, while $\rho_{z_2} = i_{z_2}$. We take the separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{y_1} = \tau_s$ and $\tau_{y_1+1} = ... = \tau_n = \tau_u$. We have $\omega_1 = k_{x_1} f_1 k_n i_1 f_* \in \{f_1 k_n i_* f_*, k_n f_1 k_n i_* f_*\}$, while since $z_2 = z_1 > y_1$, we have $\omega_2 = k_{x_2} f_{y_2} i_n i_1 f_* = k_{x_2} f_{y_2} i_n f_* \in \{f_1 i_n f_*, f_n i_n f_*, k_n f_1 i_n f_*\}$. So $\omega_1 A \neq \omega_2 A$, taking A from Example 3.3.

Sub-Case (c): Suppose $y_1 < y_2$, and take the separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{y_1} = \tau_s$ and $\tau_{y_1+1} = ... = \tau_n = \tau_u$. We have

$$\omega_1 = k_{x_1} f_1 \sigma_{z_1} i_1 f_* \in \{f_1 i_1 f_*, f_1 i_n f_*, f_1 k_n i_* f_*, k_n f_1 i_1 f_*, k_n f_1 i_n f_*, k_n f_1 k_n i_* f_*\},$$

whereas since $z_2 \geq y_2$, we have $\omega_2 = k_{x_2} f_n \rho_n i_1 f_* = k_{x_2} f_n i_n f_* = f_n i_n f_*$. So $\omega_1 A \neq \omega_2 A$, taking A from Example 3.3.

Sub-Case (d): Suppose $y_1 = y_2$, but $x_1 < x_2$, and take the separating space $(\mathbb{R}, \tau_1, ..., \tau_n)$ where $\tau_1 = ... = \tau_{x_1} = \tau_s$ and $\tau_{x_1+1} = ... = \tau_n = \tau_u$ with the initial set A from Example 3.3. Since $y_1 = y_2 \leq x_1$, we have $\omega_1 = k_1 f_1 \sigma_{z_1} i_1 f_* \in \{f_1 i_1 f_*, f_1 i_n f_*, f_1 k_n i_* f_*\}$ and $\omega_2 = k_n f_1 \rho_{z_2} i_1 f_* \in \{k_n f_1 i_1 f_*, k_n f_1 i_n f_*, k_n f_1 k_n i_* f_*\}$, so $\omega_1 A \neq \omega_2 A$.

Case 8: $\omega_1, \omega_2 \in KFIK \cup KFKIK$. In this case take the same separating space as in Case 7, but for an initial set take $f_n A$ where A is the initial set from Case 7. We are done if $\omega_1 f_n A \neq \omega_2 f_n A$; but this follows from Case 7 because $\omega_1 f_n, \omega_2 f_n \in KFIF \cup KFKIF$.

Case 9: $\omega_1, \omega_2 \in KFKI \cup KFIKI$. Take the same separating space as in Cases 7 and 8, and for an initial set take cA where A is the initial set from Case 8. Then since $\omega_1 c, \omega_2 c \in KFIK \cup KFKIK$, we have $\omega_1 cA \neq \omega_2 cA$ by Case 8.

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