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GLOBAL WELL-POSEDNESS FOR THE 3D MAGNETO-MICROPOLAR EQUATIONS WITH FRACTIONAL DISSIPATION

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Abstract. This paper focus on the Cauchy problem of the 3D incompressible magneto-micropolar equations with fractional dissipation in the Sobolev space. Liu, Sun and Xin obtained the global solutions to the 3D magneto-micropolar equations with $\alpha = \beta = \gamma = \frac{5}{4}$. Deng and Shang established the global well-posedness of the 3D magneto-micropolar equations in the case of $\alpha \geq \frac{5}{4}$, $\alpha + \beta \geq \frac{5}{2}$ and $\gamma \geq 2 - \alpha \geq \frac{3}{4}$. In this paper, we establish the global well-posedness of the 3D magneto-micropolar equations with $\alpha = \beta = \frac{5}{4}$ and $\gamma = \frac{1}{2}$, which improves the results of Liu-Sun-Xin and Deng-Shang by reducing the value of γ to $\frac{1}{2}$.

1. Introduction

In this paper, we are concerned with the global well-posedness to the 3D magneto-micropolar equations with fractional dissipation

$$\begin{cases} \partial_t u + u \cdot \nabla u + (\mu + \chi)\Lambda^{2\alpha} u = -\nabla \pi + b \cdot \nabla b + 2\chi \nabla \times w, \\ \partial_t w + u \cdot \nabla w + 4\chi w - \kappa \nabla \nabla \cdot w + \eta \Lambda^{2\gamma} w = 2\chi \nabla \times u, \\ \partial_t b + u \cdot \nabla b + \nu \Lambda^{2\beta} b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), b(x,0) = b_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t)$, $w = w(x,t)$, $b = b(x,t)$ and $\pi = \pi(x,t)$ represent the velocity of the fluid, the microrotational velocity, the magnetic field and the hydrostatic pressure, respectively. The parameters μ, χ and $\frac{1}{\nu}$ are the kinematic viscosity, vortex viscosity and magnetic Reynolds number, respectively. κ and η are angular viscosities. The parameters α, β, γ are nonnegative constants, and $\Lambda = (-\Delta)^{\frac{1}{2}}$ denotes the Zygmund operator defined via Fourier transform, namely

$$\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi), \quad \forall s \geq 0.$$

The magneto-micropolar system (1.1) is closely related to many classical fluid equations. When the magnetic field disappears, namely $b = 0$, then the system (1.1) reduces to the 3D incompressible micropolar equations which has been investigated extensively with many interesting results. The existence of weak solutions was proved by Galdi and Rionero in [**5**]. Yamaguchi [**26**] obtained the existence of global

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strong solutions in a bounded domain. Recently, the hyperdissipation micropolar equations have received significant attention in mathematical fluid dynamics (see for example [**3, 7, 17, 21**]).

When $w = 0$ and $\chi = 0$, the system (1.1) becomes the generalized magnetohydrodynamics (MHD) equations which model the motion of electrically conducting fluid. The global well-posedness of the MHD equations have been investigated extensively and important progress has been made. Readers may refer to [**12, 13, 22, 23, 25**] and the references therein.

For the magneto-micropolar equations (1.1) in $\mathbb{R}^2$, Regmi and Wu [**15**] established the global regularity for magneto-micropolar equations with partial dissipation. Ma [**11**] extended the results of Regmi and Wu to other mixed partial viscosities cases. Yamazaki [**27**] established the global regularity of solutions for 2D magneto-micropolar equations with $\alpha = \beta = 1$, $\gamma = 0$. Recently, Shang and Zhao obtained the global regularity with $\alpha = 0$, $\beta > 1$, $\gamma = 1$ in [**19**]. Shang and Wu [**18**] dealt with the global regularity problem with $1 < \alpha < 2$, $0 < \beta < 1$, $\alpha+\beta \geq 2$, $\gamma = 0$ or $\alpha = 2$, $\beta = \gamma = 0$, or $\alpha > 0$, $\beta = \gamma = 1$. Many more exciting results on the global regularity for magneto-micropolar equations with partial dissipation are available for the 2D case (see for example [**9, 14, 16, 31, 32**]).

For the 3D case of the system (1.1), Yuan [**28**] first established the regularity of weak solutions and blow-up criteria for smooth solutions in the whole space. Gala extended the results of Yuan in [**28**] to the Morrey-Campanato spaces in [**4**]. The regularity criterion for the 3D magneto-micropolar fluid equations in Triebel-Lizorkin space was proved in [**34**]. For more blow-up criteria of smooth solutions and the regularity criteria of weak solutions readers refer to [**24, 29, 30, 33**]. Recently, Li-Shang [**8**] and Tan-Wu [**20**] established the existence of 3D small global smooth solutions in the case of $\alpha = \beta = \gamma = 1$. Liu, Sun and Xin [**10**] obtained the global existence and uniqueness of solutions for the case $\alpha = \beta = \gamma = \frac{1}{2} + \frac{n}{4}$ with $n \geq 3$. Very recently, Deng and Shang [**2**] established the global well-posedness of magneto-micropolar equations with $\alpha \geq \frac{1}{2} + \frac{n}{4}$, $\alpha + \gamma \geq \max\{2, \frac{n}{2}\}$, $\alpha + \beta \geq 1 + \frac{n}{2}$ when $n \geq 3$.

In this paper, we consider the system (1.1) in the case of $\alpha = \beta = \frac{5}{4}$ and $\gamma = \frac{1}{2}$. We will show the global well-posedness of the following system

$$\begin{cases} \partial_t u + u\cdot\nabla u + (\mu+\chi)\Lambda^{\frac{5}{2}} u = -\nabla\pi + b\cdot\nabla b + 2\chi\nabla\times w, \\ \partial_t w + u\cdot\nabla w + 4\chi w - \kappa\nabla\nabla\cdot w + \eta\Lambda w = 2\chi\nabla\times u, \\ \partial_t b + u\cdot\nabla b + \nu\Lambda^{\frac{5}{2}} b = b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(x,0) = u_0(x), w(x,0) = w_0(x), b(x,0) = b_0(x). \end{cases} \tag{1.2}$$

More precisely, we establish the following main result.

Theorem 1.1. *Assume the initial data $(u_0, w_0, b_0) \in H^2(\mathbb{R}^3)$ with $\nabla\cdot u_0 = \nabla\cdot b_0 = 0$. Then the system (1.2) has a unique global solution (u, w, b) satisfying*

$$(u,w,b) \in L^\infty(0,T;H^2(\mathbb{R}^3)),\ (u,b) \in L^2(0,T;H^{\frac{13}{4}}(\mathbb{R}^3)),\ w \in L^2(0,T;H^{\frac{5}{2}}(\mathbb{R}^3)).$$

for any $T > 0$.

Remark 1.2. Theorem 1.1 improves the results in [**10**] and [**2**] by reducing the value of γ to $\gamma = \frac{1}{2}$. In fact, when $n = 3$, Liu etc. [**10**] obtained the well-posedness of solutions to the system (1.1) with $\alpha = \beta = \gamma = \frac{5}{4}$. Deng and Shang [**2**] established the global well-posedness for the case of $\alpha \geq \frac{5}{4}$, $\alpha + \beta \geq \frac{5}{2}$ and $\gamma \geq 2 - \alpha \geq \frac{3}{4}$. In Theorem 1.1 we only need $\gamma = \frac{1}{2}$.

Remark 1.3. Notice that the value of $\alpha = \beta = \frac{5}{4}$ and $\gamma = \frac{1}{2}$ here are the minimum, therefore Theorem 1.1 is also valid for the case of $\alpha = \beta \geq \frac{5}{4}$ and $\gamma \geq \frac{1}{2}$.

Remark 1.4. When the magnetic field $b = 0$, system (1.2) reduces to the micropolar equations, the result of Theorem 1.1 also holds true for the micropolar equations.

Throughout the paper, C stands for a generic positive constant which may be different from line to line. In the following, for notational convenience, we use $\|\cdot\|_X$ to denote $\|\cdot\|_{X(\mathbb{R}^3)}$. Furthermore, we use $\|(u,w,b)(t)\|_X^p$ to denote $\|u(t)\|_X^p + \|w(t)\|_X^p + \|b(t)\|_X^p$.

2. Preliminaries

In this section we present several elementary lemmas which are needed in the proof of Theorem 1.1. The first contains two calculus inequalities involving fractional differential operators. We can find the details in [**6**] for example.

Lemma 2.1. *Suppose that $s > 0$ and $p \in (1,\infty)$. Let f, g be two functions such that $\Lambda^s f \in L^{p_1}$, $g \in L^{p_2}$, $\Lambda^{s-1} g \in L^{p_3}$ and $\nabla f \in L^{p_4}$, then there exists a constant C such that*

$$\|[\Lambda^s, f]g\|_{L^p} \leq C(\|\Lambda^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + \|\Lambda^{s-1} g\|_{L^{p_3}} \|\nabla f\|_{L^{p_4}})$$

with $p_2, p_4 \in [1,\infty]$, $p_1, p_3 \in (1,\infty)$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4},$$

where $[\Lambda^s, f]g := \Lambda^s(fg) - f(\Lambda^s g)$ and $\Lambda^s = (-\Delta)^{\frac{s}{2}}$. In particular, we have the following form of commutator estimate

$$\|[\Lambda^{s-1}\partial_i, f]g\|_{L^p} \leq C(\|\Lambda f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).$$

Then we recall the known Gagliardo–Nirenberg inequality as follows [**1**, Theorem 2.44].

Lemma 2.2. *(Gagliardo–Nirenberg inequality) Let $1 < q, r \leq \infty$ and $0 < m < n < \infty$. Then a constant C exists such that*

$$\|f\|_{\dot{W}_p^m} \leq C \|f\|_{L^q}^{\alpha} \|f\|_{\dot{W}_r^n}^{1-\alpha} \quad \text{with} \quad \frac{1}{p} = \frac{\alpha}{q} + \frac{1-\alpha}{r} \quad \text{and} \quad \alpha = 1 - \frac{m}{n}.$$

3. Proof of the Main Theorem

This section is devoted to the proof of Theorem 1.1. We first prove the global existence part. To this end, the crucial piece is the global a priori H^2 bound for (u,w,b). If we obtain this global bound, then the existence part of Theorem 1.1 can be proved by the standard Friedrichs method, so we omit the details here. Thus our main effort is to establish the global bounds for $\|(u,w,b)(t)\|_{H^2(\mathbb{R}^3)}$ with any given $t > 0$.

3.1. The L^2 estimate. Taking the L^2 inner products of the system $(1.2)_{1,2,3}$ with (u,w,b) respectively, adding the results and integrating by parts, it follows by $\nabla\cdot u=0$ and $\nabla\cdot b=0$ that

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u,w,b)(t)\|_{L^2}^2+(\mu+\chi)\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2+4\chi\|w\|_{L^2}^2+\kappa\|\nabla\cdot w\|_{L^2}^2\\&\quad+\eta\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2+\nu\|\Lambda^{\frac{5}{4}}b\|_{L^2}^2\\&=4\chi\int_{\mathbb{R}^3}\nabla\times u\cdot w\mathrm{d}x,\end{aligned}\tag{3.1}$$

where we have used the facts

$$\int_{\mathbb{R}^3}b\cdot\nabla b\cdot u\mathrm{d}x+\int_{\mathbb{R}^3}b\cdot\nabla u\cdot b\mathrm{d}x=0,\quad \int_{\mathbb{R}^3}\nabla\times w\cdot u\mathrm{d}x=\int_{\mathbb{R}^3}\nabla\times u\cdot w\mathrm{d}x.$$

Then applying Hölder inequality, Lemma 2.2 and Young's inequality, we derive that

$$\begin{aligned}4\chi\int_{\mathbb{R}^3}\nabla\times u\cdot w\mathrm{d}x&\le 4\chi\|\nabla u\|_{L^2}\|w\|_{L^2}\\&\le 4\chi\|u\|_{L^2}^{\frac{1}{5}}\|\Lambda^{\frac{5}{4}}u\|_{L^2}^{\frac{4}{5}}\|w\|_{L^2}\\&\le 2\chi\|w\|_{L^2}^2+\frac{(\mu+\chi)}{2}\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2+C\|u\|_{L^2}^2.\end{aligned}\tag{3.2}$$

Inserting (3.2) into (3.1), it thus follows from the Gronwall's inequality that

$$\begin{aligned}&\|(u,w,b)(t)\|_{L^2}^2+(\mu+\chi)\int_0^t\|\Lambda^{\frac{5}{4}}u(\tau)\|_{L^2}^2d\tau+4\chi\int_0^t\|w(\tau)\|_{L^2}^2\mathrm{d}\tau\\&\quad+2\kappa\int_0^t\|\nabla\cdot w(\tau)\|_{L^2}^2\mathrm{d}\tau+2\eta\int_0^t\|\Lambda^{\frac{1}{2}}w(\tau)\|_{L^2}^2\mathrm{d}\tau+2\nu\int_0^t\|\Lambda^{\frac{5}{4}}b(\tau)\|_{L^2}^2\mathrm{d}\tau\\&\le C.\end{aligned}\tag{3.3}$$

3.2. The $H^{\frac{3}{4}}$ estimate. Applying $\Lambda^{\frac{3}{4}}$ to the system $(1.2)_{1,3}$, dotting the resultant by $(\Lambda^{\frac{3}{4}}u,\Lambda^{\frac{3}{4}}b)$ respectively, integrating over $\mathbb{R}^3$ and adding them up, by the divergence free conditions $\nabla\cdot u=0$ and $\nabla\cdot b=0$, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^{\frac{3}{4}}u,\Lambda^{\frac{3}{4}}b)(t)\|_{L^2}^2+(\mu+\chi)\|\Lambda^2u\|_{L^2}^2+\nu\|\Lambda^2b\|_{L^2}^2\\&=\int_{\mathbb{R}^3}[\Lambda^{\frac{3}{4}},b\cdot\nabla]b\cdot\Lambda^{\frac{3}{4}}u\mathrm{d}x+2\chi\int_{\mathbb{R}^3}\Lambda^{\frac{3}{4}}(\nabla\times w)\cdot\Lambda^{\frac{3}{4}}u\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}[\Lambda^{\frac{3}{4}},u\cdot\nabla]u\cdot\Lambda^{\frac{3}{4}}u\mathrm{d}x+\int_{\mathbb{R}^3}[\Lambda^{\frac{3}{4}},b\cdot\nabla]u\cdot\Lambda^{\frac{3}{4}}b\mathrm{d}x\\&\quad-\int_{\mathbb{R}^3}[\Lambda^{\frac{3}{4}},u\cdot\nabla]b\cdot\Lambda^{\frac{3}{4}}b\mathrm{d}x\\&:=I_1+I_2+I_3+I_4+I_5,\end{aligned}\tag{3.4}$$

where the following property has been applied

$$\int_{\mathbb{R}^3}b\cdot\nabla\Lambda^{\frac{3}{4}}b\cdot\Lambda^{\frac{3}{4}}u\mathrm{d}x+\int_{\mathbb{R}^3}b\cdot\nabla\Lambda^{\frac{3}{4}}u\cdot\Lambda^{\frac{3}{4}}b\mathrm{d}x=0.$$

By Hölder inequality, together with Lemma 2.1, Sobolev embedding theorem and Young's inequality, we find

$$\begin{aligned} I_1 &\le C\|[\Lambda^{\frac{3}{4}}, b\cdot\nabla]b\|_{L^2}\|\Lambda^{\frac{3}{4}}u\|_{L^2} \\ &\le C\|\Lambda^{\frac{3}{4}}b\|_{L^{12}}\|\nabla b\|_{L^{\frac{12}{5}}}\|\Lambda^{\frac{3}{4}}u\|_{L^2} \\ &\le C\|\Lambda^2 b\|_{L^2}\|\Lambda^{\frac{5}{4}}b\|_{L^2}\|\Lambda^{\frac{3}{4}}u\|_{L^2} \\ &\le \frac{\nu}{6}\|\Lambda^2 b\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}}b\|_{L^2}^2\|\Lambda^{\frac{3}{4}}u\|_{L^2}^2. \end{aligned} \tag{3.5}$$

Arguing similarly to above inequality (3.5), it can be derived that

$$\begin{aligned} I_3 &\le C\|[\Lambda^{\frac{3}{4}}, u\cdot\nabla]u\|_{L^2}\|\Lambda^{\frac{3}{4}}u\|_{L^2} \\ &\le \frac{\mu+\chi}{8}\|\Lambda^2 u\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2\|\Lambda^{\frac{3}{4}}u\|_{L^2}^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} I_4 &\le C\|[\Lambda^{\frac{3}{4}}, b\cdot\nabla]u\|_{L^2}\|\Lambda^{\frac{3}{4}}b\|_{L^2} \\ &\le \frac{\mu+\chi}{8}\|\Lambda^2 u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^2 b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2)\|\Lambda^{\frac{3}{4}}b\|_{L^2}^2, \end{aligned} \tag{3.7}$$

$$\begin{aligned} I_5 &\le C\|[\Lambda^{\frac{3}{4}}, u\cdot\nabla]b\|_{L^2}\|\Lambda^{\frac{3}{4}}b\|_{L^2} \\ &\le \frac{\mu+\chi}{8}\|\Lambda^2 u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^2 b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2)\|\Lambda^{\frac{3}{4}}b\|_{L^2}^2. \end{aligned} \tag{3.8}$$

By Hölder and Young's inequalities, it follows that

$$\begin{aligned} I_2 &\le 2\chi\|\Lambda^2 u\|_{L^2}\|\Lambda^{\frac{1}{2}}w\|_{L^2} \\ &\le \frac{\mu+\chi}{8}\|\Lambda^2 u\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2. \end{aligned} \tag{3.9}$$

Inserting the estimates (3.5)-(3.9) into (3.4), it thus leads to

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^{\frac{3}{4}}u, \Lambda^{\frac{3}{4}}b)(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^2 u\|_{L^2}^2 + \nu\|\Lambda^2 b\|_{L^2}^2 \\ &\le C(\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2)\|(\Lambda^{\frac{3}{4}}u, \Lambda^{\frac{3}{4}}b)\|_{L^2}^2 + C\|\Lambda^{\frac{1}{2}}w\|_{L^2}^2. \end{aligned}$$

Then Gronwall's inequality and (3.3) imply that

$$\|(\Lambda^{\frac{3}{4}}u, \Lambda^{\frac{3}{4}}b)(t)\|_{L^2}^2 + (\mu+\chi)\int_0^t \|\Lambda^2 u(\tau)\|_{L^2}^2 \mathrm{d}\tau + \nu\int_0^t \|\Lambda^2 b(\tau)\|_{L^2}^2 \mathrm{d}\tau \le C. \tag{3.10}$$

3.3. The H^1 estimate. Applying Λ to both sides of system $(1.2)_{1,2,3}$ and taking the L^2 inner products with $(\Lambda u, \Lambda w, \Lambda b)$ respectively, adding them up together with

$\nabla \cdot u = 0$ and $\nabla \cdot b = 0$, we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda u, \Lambda w, \Lambda b)(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + 4\chi\|\Lambda w\|_{L^2}^2 + \kappa\|\Lambda\nabla\cdot w\|_{L^2}^2 \\
&\quad + \eta\|\Lambda^{\frac{3}{2}} w\|_{L^2}^2 + \nu\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3}[\Lambda, b\cdot\nabla]b\cdot\Lambda u\mathrm{d}x + 2\chi\int_{\mathbb{R}^3}\Lambda(\nabla\times w)\cdot\Lambda u\mathrm{d}x - \int_{\mathbb{R}^3}[\Lambda, u\cdot\nabla]u\cdot\Lambda u\mathrm{d}x \\
&\quad + 2\chi\int_{\mathbb{R}^3}\Lambda(\nabla\times u)\cdot\Lambda w\mathrm{d}x - \int_{\mathbb{R}^3}[\Lambda, u\cdot\nabla]w\cdot\Lambda w\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3}[\Lambda, b\cdot\nabla]u\cdot\Lambda b\mathrm{d}x - \int_{\mathbb{R}^3}[\Lambda, u\cdot\nabla]b\cdot\Lambda b\mathrm{d}x \\
&:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6 + J_7, \qquad (3.11)
\end{aligned}$$

where we also take advantage of the cancellation identity

$$\int_{\mathbb{R}^3} b\cdot\nabla\Lambda b\cdot\Lambda u\mathrm{d}x + \int_{\mathbb{R}^3} b\cdot\nabla\Lambda u\cdot\Lambda b\mathrm{d}x = 0.$$

Use an argument similar to that used in deriving the estimate (3.5) to obtain that

$$\begin{aligned}
J_1 &\le C\|[\Lambda, b\cdot\nabla]b\|_{L^2}\|\Lambda u\|_{L^2} \\
&\le C\|\Lambda b\|_{L^{12}}\|\nabla b\|_{L^{\frac{12}{5}}}\|\Lambda u\|_{L^2} \\
&\le C\|\Lambda^{\frac{9}{4}} b\|_{L^2}\|\Lambda^{\frac{5}{4}} b\|_{L^2}\|\Lambda u\|_{L^2} \\
&\le \frac{\nu}{6}\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} b\|_{L^2}^2\|\Lambda u\|_{L^2}^2. \qquad (3.12)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
J_3 &\le C\|[\Lambda, u\cdot\nabla]u\|_{L^2}\|\Lambda u\|_{L^2} \\
&\le \frac{\mu+\chi}{6}\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} u\|_{L^2}^2\|\Lambda u\|_{L^2}^2, \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
J_6 &\le C\|[\Lambda, b\cdot\nabla]u\|_{L^2}\|\Lambda b\|_{L^2} \\
&\le \frac{\mu+\chi}{6}\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}} b\|_{L^2}^2)\|\Lambda b\|_{L^2}^2 \qquad (3.14)
\end{aligned}$$

and

$$\begin{aligned}
J_7 &\le C\|[\Lambda, u\cdot\nabla]b\|_{L^2}\|\Lambda b\|_{L^2} \\
&\le \frac{\mu+\chi}{6}\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}} b\|_{L^2}^2)\|\Lambda b\|_{L^2}^2. \qquad (3.15)
\end{aligned}$$

In view of the Hölder and Young's inequalities, one has

$$\begin{aligned}
J_2 + J_4 &= 4\chi\int_{\mathbb{R}^3}\Lambda(\nabla\times u)\cdot\Lambda w\mathrm{d}x \\
&\le 4\chi\|\Lambda^2 u\|_{L^2}\|\Lambda w\|_{L^2} \\
&\le (\mu+\chi)\|\Lambda^2 u\|_{L^2}^2 + C\|\Lambda w\|_{L^2}^2. \qquad (3.16)
\end{aligned}$$

By Hölder inequality, together with Sobolev embedding theorem and Young's inequality, it can be shown that

$$\begin{aligned} J_5 &\leq C\|[\Lambda, u\cdot\nabla]w\|_{L^2}\|\Lambda w\|_{L^2} \\ &\leq C\|\Lambda u\|_{L^6}\|\nabla w\|_{L^3}\|\Lambda w\|_{L^2} \\ &\leq C\|\Lambda^2 u\|_{L^2}\|\Lambda^{\frac{3}{2}} w\|_{L^2}\|\Lambda w\|_{L^2} \\ &\leq \frac{\eta}{2}\|\Lambda^{\frac{3}{2}} w\|_{L^2}^2 + C\|\Lambda^2 u\|_{L^2}^2\|\Lambda w\|_{L^2}^2. \end{aligned} \tag{3.17}$$

Collecting the estimates (3.12)-(3.17) into (3.11), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda u, \Lambda w, \Lambda b)(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{9}{4}} u\|_{L^2}^2 + 8\chi\|\Lambda w\|_{L^2}^2 + 2\kappa\|\Lambda\nabla\cdot w\|_{L^2}^2 \\ &\quad + \eta\|\Lambda^{\frac{3}{2}} w\|_{L^2}^2 + \nu\|\Lambda^{\frac{9}{4}} b\|_{L^2}^2 \\ &\leq C(1 + \|\Lambda^{\frac{5}{4}} u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}} b\|_{L^2}^2 + \|\Lambda^2 u\|_{L^2}^2)\|(\Lambda u, \Lambda w, \Lambda b)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^2 u\|_{L^2}^2. \end{aligned}$$

Then Gronwall's inequality, together with (3.3) and (3.10) imply that

$$\begin{aligned} &\|(\Lambda u, \Lambda w, \Lambda b)(t)\|_{L^2}^2 + (\mu+\chi)\int_0^t \|\Lambda^{\frac{9}{4}} u(\tau)\|_{L^2}^2 \mathrm{d}\tau + 8\chi\int_0^t \|\Lambda w(\tau)\|_{L^2}^2 \mathrm{d}\tau \\ &\quad + 2\kappa\int_0^t \|\Lambda\nabla\cdot w(\tau)\|_{L^2}^2 \mathrm{d}\tau + \eta\int_0^t \|\Lambda^{\frac{3}{2}} w(\tau)\|_{L^2}^2 \mathrm{d}\tau + \nu\int_0^t \|\Lambda^{\frac{9}{4}} b(\tau)\|_{L^2}^2 \mathrm{d}\tau \\ &\leq C. \end{aligned} \tag{3.18}$$

3.4. The H^2 estimate. Applying Λ^2 to both sides of system $(1.2)_{1,2,3}$, taking the L^2 inner products of the resulting equations with $(\Lambda^2 u, \Lambda^2 w, \Lambda^2 b)$ respectively, and using the divergence free property, we deduce by adding them up

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^2 u, \Lambda^2 w, \Lambda^2 b)(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{13}{4}} u\|_{L^2}^2 + 4\chi\|\Lambda^2 w\|_{L^2}^2 + \kappa\|\Lambda^2\nabla\cdot w\|_{L^2}^2 \\ &\quad + \eta\|\Lambda^{\frac{5}{2}} w\|_{L^2}^2 + \nu\|\Lambda^{\frac{13}{4}} b\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} [\Lambda^2, b\cdot\nabla]b\cdot\Lambda^2 u \mathrm{d}x + 2\chi\int_{\mathbb{R}^3} \Lambda^2(\nabla\times w)\cdot\Lambda^2 u \mathrm{d}x - \int_{\mathbb{R}^3} [\Lambda^2, u\cdot\nabla]u\cdot\Lambda^2 u \mathrm{d}x \\ &\quad + 2\chi\int_{\mathbb{R}^3} \Lambda^2(\nabla\times u)\cdot\Lambda^2 w \mathrm{d}x - \int_{\mathbb{R}^3} [\Lambda^2, u\cdot\nabla]w\cdot\Lambda^2 w \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3} [\Lambda^2, b\cdot\nabla]u\cdot\Lambda^2 b \mathrm{d}x - \int_{\mathbb{R}^3} [\Lambda^2, u\cdot\nabla]b\cdot\Lambda^2 b \mathrm{d}x \\ &:= K_1 + K_2 + K_3 + K_4 + K_5 + K_6 + K_7. \end{aligned} \tag{3.19}$$

Arguing like in deriving (3.5), it follows that

$$\begin{aligned} K_1 &\leq C\|[\Lambda^2, b\cdot\nabla]b\|_{L^2}\|\Lambda^2 u\|_{L^2} \\ &\leq C\|\Lambda^2 b\|_{L^{12}}\|\nabla b\|_{L^{\frac{12}{5}}}\|\Lambda^2 u\|_{L^2} \\ &\leq C\|\Lambda^{\frac{13}{4}} b\|_{L^2}\|\Lambda^{\frac{5}{4}} b\|_{L^2}\|\Lambda^2 u\|_{L^2} \\ &\leq \frac{\nu}{6}\|\Lambda^{\frac{13}{4}} b\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} b\|_{L^2}^2\|\Lambda^2 u\|_{L^2}^2. \end{aligned} \tag{3.20}$$

Similarly, we obtain

$$K_3 \le C\|[\Lambda^2, u\cdot\nabla]u\|_{L^2}\|\Lambda^2 u\|_{L^2} \le \frac{\mu+\chi}{10}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2\|\Lambda^2 u\|_{L^2}^2, \tag{3.21}$$

$$K_6 \le C\|[\Lambda^2, b\cdot\nabla]u\|_{L^2}\|\Lambda^2 b\|_{L^2} \le \frac{\mu+\chi}{10}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^{\frac{13}{4}}b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2)\|\Lambda^2 b\|_{L^2}^2 \tag{3.22}$$

and

$$K_7 \le C\|[\Lambda^2, u\cdot\nabla]b\|_{L^2}\|\Lambda^2 b\|_{L^2} \le \frac{\mu+\chi}{10}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + \frac{\nu}{6}\|\Lambda^{\frac{13}{4}}b\|_{L^2}^2 + C(\|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2)\|\Lambda^2 b\|_{L^2}^2. \tag{3.23}$$

Thanks to Hölder inequality, Lemma 2.2 and Young's inequality, we have

$$\begin{aligned}K_2 + K_4 &= 4\chi\int_{\mathbb{R}^3}\Lambda^2(\nabla\times u)\cdot\Lambda^2 w\mathrm{d}x\\ &\le 4\chi\|\Lambda^3 u\|_{L^2}\|\Lambda^2 w\|_{L^2}\\ &\le 4\chi\|\Lambda^2 u\|_{L^2}^{\frac{1}{5}}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^{\frac{4}{5}}\|\Lambda^2 w\|_{L^2}\\ &\le \frac{\mu+\chi}{10}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + C\|(\Lambda^2 u, \Lambda^2 w)\|_{L^2}^2.\end{aligned} \tag{3.24}$$

By Hölder inequality and Lemma 2.1, together with Sobolev embedding theorem and Young's inequality, we arrive at

$$\begin{aligned}K_5 &\le \int_{\mathbb{R}^3}[\Lambda^2\partial_i, u]w\cdot\Lambda^2 w\mathrm{d}x\\ &\le C\|[\Lambda^2\partial_i, u]w\|_{L^2}\|\Lambda^2 w\|_{L^2}\\ &\le C\|\nabla u\|_{L^{12}}\|\Lambda^2 w\|_{L^{\frac{12}{5}}}\|\Lambda^2 w\|_{L^2} + C\|\Lambda^3 u\|_{L^{\frac{12}{5}}}\|w\|_{L^{12}}\|\Lambda^2 w\|_{L^2}\\ &\le C\|\Lambda^{\frac{9}{4}}u\|_{L^2}\|\Lambda^2 w\|_{L^2}^{\frac{3}{2}}\|\Lambda^{\frac{5}{2}}w\|_{L^2}^{\frac{1}{2}} + C\|\Lambda^{\frac{13}{4}}u\|_{L^2}\|w\|_{L^2}^{\frac{1}{6}}\|\Lambda^{\frac{3}{2}}w\|_{L^2}^{\frac{5}{6}}\|\Lambda^2 w\|_{L^2}\\ &\le \frac{\eta}{2}\|\Lambda^{\frac{5}{2}}w\|_{L^2}^2 + \frac{\mu+\chi}{10}\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + C(1 + \|w\|_{L^2}^2 + \|\Lambda^{\frac{9}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}w\|_{L^2}^2)\|\Lambda^2 w\|_{L^2}^2.\end{aligned} \tag{3.25}$$

Substituting the estimates (3.20)-(3.25) into (3.19), we eventually obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|(\Lambda^2 u, \Lambda^2 w, \Lambda^2 b)(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{13}{4}}u\|_{L^2}^2 + 8\chi\|\Lambda^2 w\|_{L^2}^2 + 2\kappa\|\Lambda^2\nabla\cdot w\|_{L^2}^2\\ &\quad + \eta\|\Lambda^{\frac{5}{2}}w\|_{L^2}^2 + \nu\|\Lambda^{\frac{13}{4}}b\|_{L^2}^2\\ &\le C(1 + \|w\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}u\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}w\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b\|_{L^2}^2 + \|\Lambda^{\frac{9}{4}}u\|_{L^2}^2)\|(\Lambda^2 u, \Lambda^2 w, \Lambda^2 b)\|_{L^2}^2.\end{aligned}$$

Then Gronwall's inequality, together with (3.3) and (3.18) yield

$$\begin{aligned}&\|(\Lambda^2 u, \Lambda^2 w, \Lambda^2 b)(t)\|_{L^2}^2 + (\mu+\chi)\int_0^t\|\Lambda^{\frac{13}{4}}u(\tau)\|_{L^2}^2\mathrm{d}\tau + 8\chi\int_0^t\|\Lambda^2 w(\tau)\|_{L^2}^2\mathrm{d}\tau\\ &\quad + 2\kappa\int_0^t\|\Lambda^2\nabla\cdot w(\tau)\|_{L^2}^2\mathrm{d}\tau + \eta\int_0^t\|\Lambda^{\frac{5}{2}}w(\tau)\|_{L^2}^2\mathrm{d}\tau + \nu\int_0^t\|\Lambda^{\frac{13}{4}}b(\tau)\|_{L^2}^2\mathrm{d}\tau\\ &\le C.\end{aligned} \tag{3.26}$$

Finally, we prove the uniqueness. Assume that (u_1, w_1, b_1, π_1) and (u_2, w_2, b_2, π_2) are two solutions for the system (1.2) with the same initial data. We define

$$\bar{u} \triangleq u_1 - u_2,\ \bar{w} \triangleq w_1 - w_2,\ \bar{b} \triangleq b_1 - b_2,\ \bar{\pi} \triangleq \pi_1 - \pi_2,$$

thus the differences $(\bar{u}, \bar{w}, \bar{b}, \bar{\pi})$ between these two solutions satisfy the following equations

$$\begin{cases} \partial_t \bar{u} + u_1 \cdot \nabla \bar{u} + \bar{u} \cdot \nabla u_2 + (\mu + \chi)\Lambda^{\frac{5}{2}} \bar{u} = -\nabla \bar{\pi} + b_1 \cdot \nabla \bar{b} + \bar{b} \cdot \nabla b_2 + 2\chi \nabla \times \bar{w}, \\ \partial_t \bar{w} + u_1 \cdot \nabla \bar{w} + \bar{u} \cdot \nabla w_2 + 4\chi \bar{w} - \kappa \nabla \nabla \cdot \bar{w} + \eta \Lambda \bar{w} = 2\chi \nabla \times \bar{u}, \\ \partial_t \bar{b} + u_1 \cdot \nabla \bar{b} + \bar{u} \cdot \nabla b_2 + \nu \Lambda^{\frac{5}{2}} \bar{b} = b_1 \cdot \nabla \bar{u} + \bar{b} \cdot \nabla u_2, \\ \nabla \cdot \bar{u} = \nabla \cdot \bar{w} = \nabla \cdot \bar{b} = 0, \\ \bar{u}(x,0) = \bar{w}(x,0) = \bar{b}(x,0) = 0. \end{cases} \tag{3.27}$$

Taking the L^2 inner products of the system $(3.27)_{1,2,3}$ with $(\bar{u}, \bar{w}, \bar{b})$ respectively, integrating by parts we obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\bar{u}, \bar{w}, \bar{b})(t)\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + 4\chi\|\bar{w}\|_{L^2}^2 + \kappa\|\nabla \cdot \bar{w}\|_{L^2}^2 + \eta\|\Lambda^{\frac{1}{2}}\bar{w}\|_{L^2}^2 \\ &\quad + \nu\|\Lambda^{\frac{5}{4}}\bar{b}\|_{L^2}^2 \\ &= \int_{\mathbb{R}^3} \bar{b} \cdot \nabla b_2 \cdot \bar{u}\mathrm{d}x + 2\chi\int_{\mathbb{R}^3} \nabla \times \bar{w} \cdot \bar{u}\mathrm{d}x - \int_{\mathbb{R}^3} \bar{u} \cdot \nabla u_2 \cdot \bar{u}\mathrm{d}x + 2\chi\int_{\mathbb{R}^3} \nabla \times \bar{u} \cdot \bar{w}\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} \bar{u} \cdot \nabla w_2 \cdot \bar{w}\mathrm{d}x + \int_{\mathbb{R}^3} \bar{b} \cdot \nabla u_2 \cdot \bar{b}\mathrm{d}x - \int_{\mathbb{R}^3} \bar{u} \cdot \nabla b_2 \cdot \bar{b}\mathrm{d}x \\ &:= N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7. \end{aligned} \tag{3.28}$$

Applying Hölder inequality, Sobolev embedding theorem and Young's inequality, it then follows that

$$\begin{aligned} N_1 + N_7 &\leq 2\|\bar{b}\|_{L^2}\|\nabla b_2\|_{L^{\frac{12}{5}}}\|\bar{u}\|_{L^{12}} \\ &\leq 2\|\bar{b}\|_{L^2}\|\Lambda^{\frac{5}{4}} b_2\|_{L^2}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2} \\ &\leq \frac{(\mu+\chi)}{8}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} b_2\|_{L^2}^2\|\bar{b}\|_{L^2}^2. \end{aligned} \tag{3.29}$$

Similarly, it can be shown that

$$\begin{aligned} N_3 &\leq C\|\bar{u}\|_{L^{12}}\|\nabla u_2\|_{L^{\frac{12}{5}}}\|\bar{u}\|_{L^2} \\ &\leq \frac{(\mu+\chi)}{8}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} u_2\|_{L^2}^2\|\bar{u}\|_{L^2}^2, \end{aligned} \tag{3.30}$$

$$\begin{aligned} N_6 &\leq C\|\bar{b}\|_{L^{12}}\|\nabla u_2\|_{L^{\frac{12}{5}}}\|\bar{b}\|_{L^2} \\ &\leq \frac{\nu}{2}\|\Lambda^{\frac{5}{4}}\bar{b}\|_{L^2}^2 + C\|\Lambda^{\frac{5}{4}} u_2\|_{L^2}^2\|\bar{b}\|_{L^2}^2. \end{aligned} \tag{3.31}$$

Again applying Hölder inequality, Lemma 2.2 and Young's inequality, it allows us to obtain

$$\begin{aligned} N_2 + N_4 &\le 4\chi\|\nabla \times \bar{u}\|_{L^2}\|\bar{w}\|_{L^2} \\ &\le 4\chi\|\bar{u}\|_{L^2}^{\frac{1}{5}}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^{\frac{4}{5}}\|\bar{w}\|_{L^2} \\ &\le \frac{(\mu+\chi)}{8}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + C\|(\bar{u},\bar{w})\|_{L^2}^2 \end{aligned} \tag{3.32}$$

and

$$\begin{aligned} N_5 &\le C\|\bar{u}\|_{L^6}\|\nabla w_2\|_{L^3}\|\bar{w}\|_{L^2} \\ &\le C\|\bar{u}\|_{L^2}^{\frac{1}{5}}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^{\frac{4}{5}}\|\Lambda^{\frac{3}{2}}w_2\|_{L^2}\|\bar{w}\|_{L^2} \\ &\le \frac{(\mu+\chi)}{8}\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + C(1+\|\Lambda^{\frac{3}{2}}w_2\|_{L^2}^2)\|(\bar{u},\bar{w})\|_{L^2}^2. \end{aligned} \tag{3.33}$$

Putting the estimates (3.29)-(3.33) into (3.28), we finally arrive at

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|(\bar{u},\bar{w},\bar{b})\|_{L^2}^2 + (\mu+\chi)\|\Lambda^{\frac{5}{4}}\bar{u}\|_{L^2}^2 + 8\chi\|\bar{w}\|_{L^2}^2 + 2\kappa\|\nabla\cdot\bar{w}\|_{L^2}^2 \\ &\quad + 2\eta\|\Lambda^{\frac{1}{2}}\bar{b}\|_{L^2}^2 + \nu\|\Lambda^{\frac{5}{4}}\bar{b}\|_{L^2}^2 \\ &\le C(1+\|\Lambda^{\frac{5}{4}}u_2\|_{L^2}^2 + \|\Lambda^{\frac{3}{2}}w_2\|_{L^2}^2 + \|\Lambda^{\frac{5}{4}}b_2\|_{L^2}^2)\|(\bar{u},\bar{w},\bar{b})\|_{L^2}^2. \end{aligned} \tag{3.34}$$

By the Gronwall's inequality, (3.3) and (3.18), we get the uniqueness immediately. Thus the proof of Theorem 1.1 is completed.

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