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ON CAPACITABILITY FOR CO-ANALYTIC SETS

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Abstract. It follows from a theorem of Davies [1952] that if A is an analytic subset of 2^ω, λ is positive and the Hausdorff s-measure of A is greater than λ, then there is a compact subset C of A such that the Hausdorff s-measure of C is greater than λ. We exhibit a counterpoint to Davies's theorem: In Gödel's universe of sets, there is a co-analytic subset B of 2^ω which has full Hausdorff dimension such that if C is a closed subset of B then C is countable.

1. Dedication

This paper is dedicated to the memory of Sir Vaughan Frederick Randal Jones. Vaughan was a brilliant mathematician, a generous colleague, and a loyal friend.

2. Preliminaries

2.1. Hausdorff Dimension.

2.1.1. *Standard definitions.*

Definition 1. Let $A \subseteq 2^\omega$ and $s \in [0,1]$.
(1) For $n \in \omega$, $\Lambda^s_n(A)$ is the infimum over all open covers $\{B(\sigma_i) : i \in \omega\}$ of A, in which every σ_i has length greater than n, of $\sum_0^\infty 1/2^{s|\sigma_i|}$. Here, $|\sigma|$ denotes the length of σ.
(2) $\Lambda^s(A)$ is $\lim_{n\to\infty} \Lambda^s_n(A)$.

Definition 2. The *Hausdorff dimension* of A is the supremum of the numbers s such that $\Lambda^s(A) > 0$.

2.1.2. *Effective formulation.*

Definition 3. Let M denote a computable partial function from $2^{<\omega}$ to $2^{<\omega}$.
(1) M is *prefix-free* if for every pair of distinct elements of the domain of M, neither is an initial segment of the other.
(2) The M-complexity of a sequence $\sigma \in 2^{<\omega}$, denote $H_M(\sigma)$, is the shortest length of a sequence τ such that $M(\tau) = \sigma$, if there is such, and is ∞ otherwise.
(3) M is *universal* if for every other such machine M^*, there is a constant C such that for all σ, $H_M(\sigma) < H_{M^*}(\sigma) + C$.
(4) Fix a universal machine U, let H be an abbreviation for H_U and call $H(\sigma)$ the *complexity of* σ.

These recursion theoretic definitions can be naturally applied relative to an arbitrary $B \in 2^\omega$ by replacing "computable" by "computable relative to B." We let $H^B(\sigma)$ refer to the complexity of σ relative to B.

Definition 4. $X \in 2^\omega$ is Martin-Löf random relative to B if there is a constant C such that for all ℓ, $H^B(X\upharpoonright\ell) > \ell - C$.

For any B, the set of X such that X is Martin-Löf random relative to B has full measure. No such X can be computable from B, but there are some which are computable from B', the Turing jump of B, [see Downey and Hirschfeldt, 2010].

Lutz [2000] introduced the effective version of Hausdorff dimension for a subset of 2^ω in terms of computable martingales. Applied to a singleton $\{X\}$, this definition yields an effective dimension for a single $X \in 2^\omega$. Mayordomo [2002] provided an alternate and equivalent characterization, which we take as a definition in the following.

Definition 5. Let $X \in 2^\omega$. The *effective Hausdorff dimension of X relative to B* is the $\liminf_{\ell\to\infty}[\frac{H^B(X\upharpoonright\ell)}{\ell}]$.

Proposition 6. *There is an infinite recursive set $R \subset \omega$ such that for all B and all pairs X and X^*, if X is Martin-Löf random relative to B and $X^*(i)$ is equal to $X(i)$ for all i not in R then the effective Hausdorff dimension of X^* relative to B is equal to 1.*

Proof. Let R be any recursive subset of ω of asymptotic density zero. For example, R could be the set of powers of 2.

For the sake of a contradiction, suppose that X is Martin-Löf random relative to B, that X^* is identical to X except on elements of R and that the effective Hausdorff dimension of X^* relative to B is less than 1. Fix d so that $d < 1$ and there are infinitely many ℓ such that $H^B(X^*\upharpoonright\ell) < d\cdot\ell$.

Let k be given. We may assume that for all $\ell > k$, there are at most $\frac{1-d}{4}\ell$ many elements of R which are less than ℓ. Now, consider numbers $\ell > k$ so that $H^B(X^*\upharpoonright\ell) < d\ell$.

$X\upharpoonright\ell$ can be recovered from $X^*\upharpoonright\ell$ and $X\upharpoonright R\cap\ell$. By assumption, the former satisfies $H^B(X^*\upharpoonright\ell) < d\ell$. For sufficiently large ℓ, the latter has complexity no greater than twice the number of elements of R which are less than ℓ. Hence, the latter has complexity no greater than $2\cdot\frac{1-d}{4}\ell = \frac{1-d}{2}\cdot\ell$. We claim that

$$H^B(X\upharpoonright\ell) < H^B(X^*\upharpoonright\ell) + H^B(X\upharpoonright R) + O(1) < d\ell + \frac{1-d}{2}\ell + O(1) = \frac{1+d}{2}\ell + O(1).$$

It suffices to verify the leftmost inequality above. Consider a machine M which on input ρ searches through the initial segments of ρ for a sequence τ such that the universal machine U halts on input τ. Since the domain of U is prefix free, there can be at most one such τ. Upon finding τ, M views ρ as a concatenation $\tau\sigma$ and attempts to evaluate $U(\sigma)$. If both steps are completed then M replaces the values of $U(\tau)$ on the integers of R with the values of $U(\sigma)$. If $U(\tau)$ is $X^*\upharpoonright\ell$ and $U(\sigma)$ is the sequence whose values are identical with those of $X\upharpoonright R\cap\ell$, then $M(\tau\sigma) = X\upharpoonright\ell$, and so the M-complexity of $X\upharpoonright\ell$ is the sum of the complexities of $X^*\upharpoonright\ell$ and $X\upharpoonright R$. The inequality follows from the universality of U.

Since $d < 1$ and ℓ may be taken arbitrarily large, we have a contradiction to X's satisfying $H^B(X\upharpoonright\ell) \geqslant \ell + O(1)$, as is required for X to be Martin-Löf random relative to B. □

Theorem 7 (Lutz and Lutz [2017]). *For any $A \subseteq 2^\omega$, the Hausdorff dimension of A is the infimum over all B of the supremum over all $X \in A$ of the effective Hausdorff dimension of X relative to B.*

2.1.3. *Capacitability.*

Definition 8. A subset A of 2^ω is *analytic* if it is the continuous image of a Polish space, that is a separable completely metrizable topological space.

A's being analytic is equivalent to there being a closed subset C of $2^\omega \times \omega^\omega$ such that for all $X \in 2^\omega$,

$$X \in A \iff (\exists Y \in \omega^\omega)[(X, Y) \in C].$$

Theorem 9 (Davies [1952], [see Rogers, 1998, Theorem 48]). *Suppose that $s \in [0, 1]$, $A \subseteq 2^\omega$ is analytic, and $\Lambda^s(A) > \lambda > 0$. Then, there is a closed $C \subseteq A$ such that $\Lambda^s(C) > \lambda$.*

When the conclusion to Theorem 9 applies to a set A, we say that A is Λ^s-capacitable.

Proposition 10. *There is a set $A \subseteq 2^\omega$ such that $\Lambda^1(A) = 1$ and A has no uncountable closed subset.*

In other words, Proposition 10 asserts that there is a set $A \subseteq 2^\omega$ of full dimension which is not capacitable. A Bernstein set, one that meets every uncountable closed subset of the real line but that contains none of them, has this property. We can construct such a set by a transfinite recursion of length the cardinality of the continuum. At each step α of the recursion, less than continuum many $X \in 2^\omega$ have been added to A and less than continuum many $Y \in 2^\omega$ have been added to the complement of A. We add another element X to A in order to avoid being contained in the α-th open set whose complement is uncountable and exclude another Y from A to avoid A's containing the α-th uncountable closed set. Since uncountable closed sets have cardinality continuum, there will always be such X and Y available.

2.2. Non-Capacitability and Axiomatic Set Theory.

2.2.1. *AD.* The Axiom of Determinacy (AD) is the assertion that for every subset A of ω^ω, one of the players in the infinite two-player game with payoff set A has a winning strategy. Using a wellordering of ω^ω, one can construct a set that is not determined, so it is standard practice to view AD as applying to the subsets of ω^ω that are obtained without reference to the Axiom of Choice. For example, Martin [1975] showed that all Borel subsets of ω^ω are determined. That all projective sets, or even all sets in $L(\mathbb{R})$, are determined follows from large cardinal hypotheses, [see Martin and Steel, 1988, Woodin, 2010].

One motivation for working with a determinacy hypothesis is that AD implies a host of regularity properties for subsets of ω^ω: Lebesgue measurability, the perfect set property, the property of Baire, to name a few. In the following theorem, Crone, Fishman, and Jackson [2020] add capacitability to this list.

Theorem 11 (Crone et al. [2020], [see also Peng, Wu, and Yu, 2021]). *Assume AD. Let $A \subseteq \mathbb{R}^d$ and $0 \leqslant \delta \leqslant d$, then either the Hausdorff dimension of A is less than or equal to δ or A contains a compact set C such that the Hausdorff dimension of C is greater than or equal to δ.*

In particular, the conclusion of Theorem 11 holds for subsets of $(2^\omega)^d$.

2.2.2. $V = L$. Gödel's Universe of Constructible sets consists of those sets which are generated by iteration of first order definability through the transfinite. L denotes the class of constructible sets and L_α denotes those sets that appear in the first α steps of that recursion. L satisfies the basic axioms of set theory (ZF), the axiom of choice and the generalized continuum hypothesis. $V = L$ is the assertion that every set is constructible, i.e. is an element of L.

Because the class of constructible sets is naturally endowed with a global definable wellordering of its elements, the assumption $V = L$ implies that sets whose existence would normally be demonstrated by abstract applications of the axiom of choice can be explicitly defined. Consequently, if $V = L$ there are projective sets that fail the previously mentioned regularity properties: not Lebesgue measurable, without the perfect set property, or without the property of Baire. C_1, defined as follows, is a canonical example.

Definition 12. C_1 denotes the set of $X \in 2^\omega$ such that there is an ordinal α such that $X \in L_\alpha$ and X can compute a wellordering of ω which is isomorphic to α.

C_1 is well-known within the study of descriptive set theory under the condition that $V = L$. In that setting, by a theorem of Guaspari, Kechris and Sacks (independently), it is an uncountable co-analytic set which has no uncountable closed subset. See Sacks [1976] or Kechris [1975].

Theorem 13. *If $V = L$ then C_1 has Hausdorff dimension equal to 1.*

Proof. By Theorem 7, it suffices to show that for every $B \in 2^\omega$ there is an $X \in C_1$ such that effective Hausdorff dimension of X relative to B is equal to 1. Let X be Martin-Löf random relative to B and recursive in B'. By Proposition 6, let $R \subset \omega$ be such that any X^* which is equal to X on all arguments not in R satisfies $H^B(X^*) = 1$. Let Y be such that $Y \in C_1$ and B' is recursive relative to Y. It would be sufficient to let Y be the master-code for $L_{\beta+1}$, where β is the least ordinal such that $B \in L_\beta$, [see Jensen, 1972]. Then, let X^* be the result of replacing the values of X on arguments of R so that the sequence of replaced values is the sequence of values of Y. That is, if i_n is the nth element of R, then $X^*(i_n) = Y(n)$. Note that $Y \geqslant_T X^*$. Since R is recursive, $X^* \geqslant_T Y$. For every $\beta > \omega$, L_β is closed under Turing equivalence. Thus, least β such that $X^* \in L_\beta$ is the same as least β such that $Y \in L_\beta$. Since Y can compute a representation of this ordinal, so can X^*. Hence $X^* \in C_1$, as required to finish the proof. □

Thus, if $V = L$ then there is a co-analytic set of full dimension all of whose closed subsets are countable (and therefore of dimension zero), which is an extreme counterexample to capacitability.

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