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SIR VAUGHAN FREDERICK RANDAL JONES
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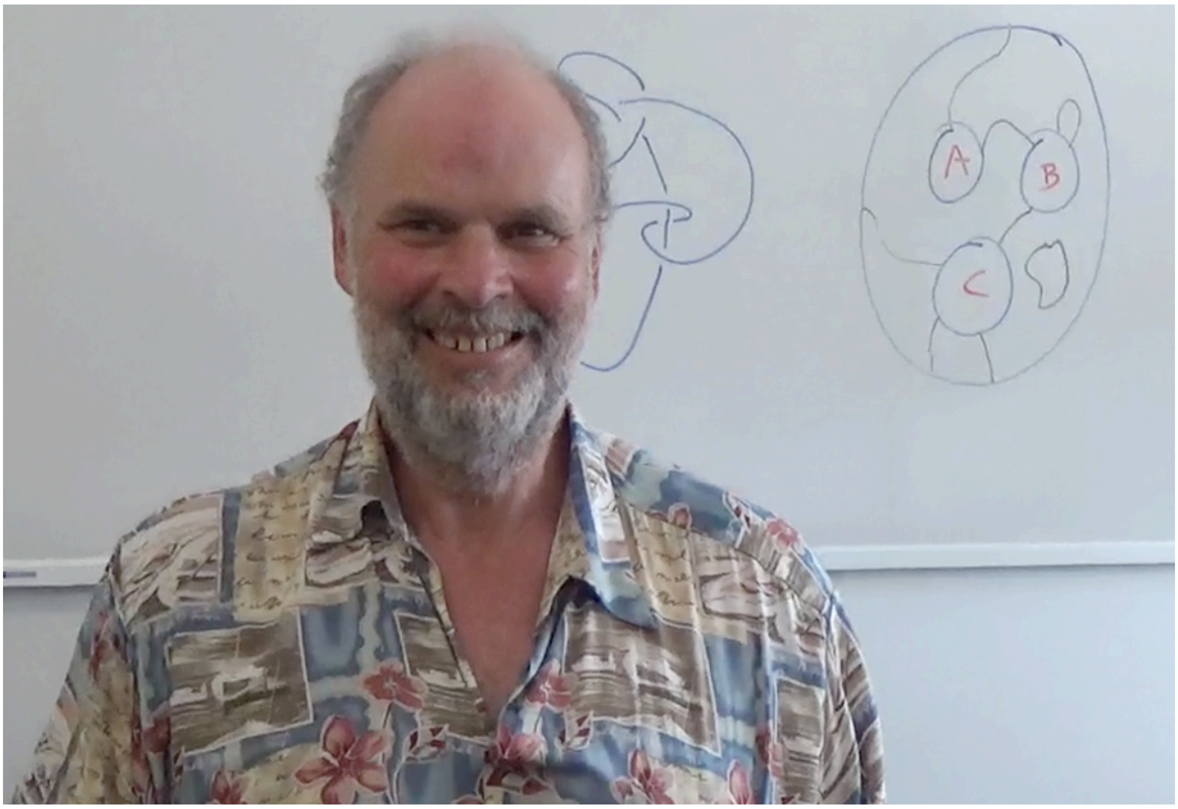

1. Introduction

This volume of the *New Zealand Journal of Mathematics* is dedicated to the memory of Distinguished Professor Sir Vaughan Jones. This introduction contains a brief tribute to Sir Vaughan[1]. Another tribute appears in [**1**].

Vaughan Jones, KNZM, DSc (Geneva), MSc (Auckland), FRS, died unexpectedly after a brief illness at his home in Nashville, Tennessee, USA, on 6 September 2020.

Vaughan was (and still is) undoubtedly New Zealand's most celebrated mathematician. Amongst his numerous awards were the Fields Medal at the Kyoto International Congress of Mathematicians in 1990, honorary doctorates from the University of Auckland (1992), the University of Wales (1993) and Université du Littoral Côte d'Opale (2002), various other medals such as the first NZ Government Science Medal (later renamed the Rutherford Medal) and the Onsager Medal

[1]A previous version of this tribute appeared in [**2**]

of Trondheim University, and fellowships or honorary fellowships of the Royal Society, the Royal Society of New Zealand, the Australian Academy of Sciences, the American Academy of Arts & Sciences, the US National Academy of Sciences, the Norwegian Royal Society of Letters and Sciences and the London Mathematical Society, as well as a recent International Cooperation Award from China.

Born in Gisborne on the last day of 1952, Vaughan completed his primary education at St Peter's School in Cambridge, attended Auckland Grammar School 1966–69, and the University of Auckland 1970–73, graduating Master of Science with first class honours in Mathematics. He then attended the Université de Genève, initially in Physics but later transferring to Mathematics, and completing his Docteur es Sciences degree with a thesis entitled *Actions of Finite Groups on the Hyperfinite* II_1 *Factor*, under the supervision of André Haefliger.

One of us got to teach Vaughan his first course in Analysis (and later taught a course in Knot Theory that Vaughan did not enrol in, so cannot claim help in the discovery of the Jones polynomial). Since Vaughan's death, however, it has been revealed that another member of the outstanding Masters class of 1973 shared some of the ideas from the course with Vaughan, and together they showed that the Alexander polynomial could not be used to distinguish the square and granny knots. On the other hand, the Jones polynomial can distinguish these two knots quickly.

Following a few months as a Junior Lecturer at Auckland, Vaughan moved to Geneva on a Swiss Government Scholarship and a F.W.W. Rhodes Memorial Scholarship. There he won the Vacheron Constantin Prize for his doctoral thesis. Then followed a number of short appointments in the USA, first as E.R.Hedrick Assistant Professor at the University of California, Los Angeles (1980–81), then Visiting Lecturer, Assistant Professor, and Associate Professor at the University of Pennsylvania (1981–84), supported in part by an Alfred P. Sloan Research Fellowship. He spent a large part of his career as a Professor of Mathematics at the University of California, Berkeley, from 1985 to 2012, and then as Stevenson Distinguished Professor of Mathematics at Vanderbilt University, Nashville, Tennessee, where he remained until his death. From 1992 until his death he also had a part-time appointment as a Distinguished Alumnus Professor at the University of Auckland.

Although his appointment as a Distinguished Alumnus Professor was by the University of Auckland, Vaughan made it clear that his preference in filling that role was to support mathematics in the whole of New Zealand. Soon discussions began for the first of the Summer Mathematical Research Workshops. There wasn't a lot of money for that first one, which took place in Huia (to the west of Auckland) in December 1994, but it was a resounding success. Those workshops have continued in January every year since 1996, and around various parts of New Zealand. Vaughan attended every single one of them, and made major contributions by encouraging outstanding mathematicians to contribute lectures, lecturing himself, working hard to ensure that the lectures adhere to the principle that they should take graduate students from where they are to understanding at least some of the frontier of research in the field, teaching participants how to wind-surf, and taking charge of the barbecue at the end of the week. During breaks between talks, Vaughan could be seen talking to a wide range of participants, making even the most timid student feel an important part of the proceedings. To oversee the workshops, together with us four he established the New Zealand Mathematics Research Institute (Inc.), and Vaughan served as chair of its directorate from its founding in 1998 until 2020. Whilst the theme of the first meeting (at Huia) was Topology, Vaughan's vision was to enrich the whole spectrum of mathematics in New Zealand. The reader might like to refer to the list of themes of the meetings in the last 27 years on the NZMRI website.

The principal criteria for a meeting are excellence of the theme and the ability to attract distinguished overseas visitors to New Zealand over the summer. Theme areas have included algebra, analysis, combinatorics, differential equations, geometry, logic, stochastic processes, and many others besides. The 2003 meeting in New Plymouth was a great example, dealing with computational aspects of biology. There were 10 international speakers (slightly more than usual), who were true international experts giving talks aimed at graduate students.

There are usually between 45 and 80 participants. New Zealand-based mathematicians and students are housed free of charge (or given a generous subsidy), and many meals are also included, thanks to Vaughan's view that shared meals encourage the sharing of ideas. The meetings are in many picturesque locations round New Zealand, the idea being that these are attractive both to locals and to the speakers facing long flights overseas in the middle of their winter.

Vaughan would sit at the front and ask a lot of basic questions, making sure that the meeting remained aimed at the graduate students. He would also organise a barbecue dinner at the end of the meeting, and even cook much of the food himself, and generally make sure that things ran smoothly.

In the 1990s context there were virtually no international scientific meetings in New Zealand. Most local mathematicians and statisticians had very little contact with outside ideas. Also there was no significant research funding for the mathematical sciences, until the Marsden Fund was established.

Funding for the workshops has been an ongoing problem and Vaughan was at the vanguard of efforts to raise funds to run the workshops, even contributing a lot of his own money. He was at the forefront of running the New Zealand Institute for Mathematics and its Applications, which was a government-sponsored Centre of Research Excellence from 2002 to 2011. This grew out of the NZMRI, and it sponsored thematic programmes along the lines of international research institutes such as the MSRI at Berkeley, the Newton Institute and the like. Over the life of the NZIMA, over a dozen programmes were sponsored, across the spectrum of mathematics, and a large number of people were supported through these programmes as well as via fellowships and scholarships. The resulting research outcomes were uniformly ranked as excellent according to international reviews. The NZIMA also initiated other activities such as the Mathsreach project, which is widely known in other countries and used as a teaching resource.

We have been very fortunate to witness the amazing growth of mathematics in New Zealand from small seeds over recent years, and our wonderful friend and colleague Vaughan Jones played a visionary and seminal role in this.

2. Vaughan Jones' Research Highlights

What follows is a brief sketch of some of Vaughan's remarkable research contributions. To any reader interested in delving more deeply into any of these areas, we recommend starting with Vaughan's own papers which are easily found. Vaughan Jones was among the most original mathematicians of the last few decades. In the 1980s he discovered entirely new types of *symmetry*, manifesting itself at the deepest quantum levels, and subsequently influencing both pure and applied mathematics. His ground-breaking discoveries led him to create *Subfactor Theory*, and to develop with it a revolutionary approach to *Knot Theory* and 3-*dimensional manifolds*, through completely novel sets of invariants and associated mathematical structures.

Within just a few years of that, Vaughan's newly developed framework and invariants led to the solution of a plethora of longstanding conjectures in knot theory and low dimensional topology, some of which were over 100 years old. His work has been *extraordinarily influential*, both revolutionising some fields and bringing together others, from von Neumann algebras to braid groups, knot theory and 3-manifold invariants, from category theory, algebra and combinatorics to quantum computing, statistical mechanics, algebraic quantum field theory and mathematical biology.

2.1. Theory of subfactors. In 1982 Vaughan discovered an astonishing phenomenon in the theory of von Neumann algebras (*Invent Math.*, 1983). He proved that if $N \subset M$ is an abstract inclusion of von Neumann factors with a trace, then the Murray-von Neumann dimension of M as an N-module, which is a number that he called the *index* of N in M and denoted by $[M : N]$, takes values in the set $\{4\cos^2\pi/n \mid n \geq 3\} \cup [4, \infty)$. By proving this striking result, Vaughan laid down the basis of a new theory, now called the *Theory of Subfactors*. His actual construction of subfactors of index $4\cos^2\pi/n$ constitutes by itself the discovery of an altogether new set of *symmetries* in mathematics. Composition (or *fusion*) of such symmetries gives rise to a completely new type of mathematical *group-like objects*, which are now recognised to naturally occur in many areas of mathematics. These are discussed below.

One of the most innovative techniques Vaughan introduced on this occasion is the so-called *iterated basic construction*, a purely operator algebraic procedure which associates with $N \subset M$ a whole tower of inclusions $N \subset M \subset M_1 \subset ...$, together with idempotents $e_1, e_2, ...$ which satisfy relations reminiscent of the Hecke algebra relations. In more modern terms, this translates into the fact that an inclusion $N \subset M$ can be described as a crossed product type inclusion $N \subset N \times \mathcal{G} = M$, with $\mathcal{G}$ being a new kind of group-like object (of symmetries) acting on N. These objects $\mathcal{G}$ are now called the *standard invariants*, and have an extremely rich combinatorial structure, generalising finitely generated discrete groups, compact Lie groups, homogeneous spaces for finite groups, Hopf algebras and all of the interesting quantum groups. Results of Vaughan's work in 1983-1984 have shown that these objects have a Cayley-type graph and encode generalised Yang-Baxter equations (many not covered by "classical" quantum groups). They give rise to remarkable representations of the braid groups. After an initial axiomatization by Popa (*Invent Math.*, 1994), these standard invariants have been given several alternative interpretations. It was Vaughan who produced the last unifying description of these group-like objects (in the late 1990s) as so-called *planar algebras*[2], a very powerful framework for concrete computations, from which a plethora of new knot invariants and solutions to Yang-Baxter equations emerged.

The "irreducible" group-like objects $\mathcal{G}$ appearing this way have an extremely complex structure, their classification resembles the classification of simple finite groups, but it is far more complicated. Those $\mathcal{G}$ having index ≤ 4 were already classified by 1994, as a result of the work of Jones, Popa, Ocneanu, Izumi and Kawahigashi: for index < 4, they correspond to Coxeter graphs $A_n, n \geq 2$ (one of each); $D_{2n}, n \geq 2$ (one of each); E_6 and E_8 (two of each); while for index $= 4$, they correspond to extended Coxeter graphs E_6^1, E_7^1, E_8^1 (one of each); D_n^1, $n \geq 4$ ($n-1$ of each); A_∞, D_∞ (one of each). During the last ten years, as a result of a tremendous effort by Vaughan and a whole team of young collaborators, this classification has been extended to cover all $\mathcal{G}$ for index ≤ 5. This amazing work is contained in many striking papers, culminating with a beautiful 45-page survey by Jones, Morrison and Snyder, in the Spring 2014 issue of the *Bulletin of the American Math. Society.*

2.2. Knot theory. Shortly after his initial work on subfactors, Vaughan discovered that the iterated basic construction and the standard invariants contain a representation of the braid groups that was previously unknown. Then, in one of the most surprising and ground-breaking discoveries of the last 50 years, he realised that this braid group representation, when taken together with the trace of the ambient von Neumann factors, gives rise to a polynomial which is invariant under the Reidemeister moves of the knots represented by the corresponding braid group elements (*Bull. AMS*, 1985). Hence this polynomial is a knot invariant! In fact within a period of a few days in September/October 1984, the Editors of the *Bulletin of the American Math. Society* received four research announcements each describing the same result: a two-variable polynomial for knots and links that generalised both the Alexander and Jones polynomials.

Vaughan's work in both type II_1 factors and his newly discovered polynomial had reached the wider world, and quickly knot theorists from across the USA and UK had realised that the skein relations for the two different polynomials had a common generalisation. The Editor's note introducing the paper *A new polynomial invariant of knots and links* by P. Freyd, D. Yetter; J. Hoste; W. B. R. Lickorish, K. C. Millett; and A. Ocneanu in *Bull. AMS* 12 (1985), 239–246, is well worth reading. Already in his historic paper, Vaughan was able to derive several surprising applications of his invariant to knot theory. He later proved that the knot invariant is also related to Hecke algebras and their representations, thereby deriving a very simple, conceptual proof of the 2-variable HOMFLY polynomial invariant for knots, which generalised what is now called the 'Jones polynomial'. His article on this in *Annals of Math.* 126 (1987) was recently found to be the most cited article in the *Annals* since the Index-paper of Atiyah and Singer.

A couple of years later, Kauffman gave an alternative diagrammatic description of Vaughan's polynomial, which enabled him and Murasugi to solve several of the

[2]Vaughan always intended his long first paper on planar algebras to be published in this journal. We are pleased to be able to honour this intent by publishing this paper as the first in this volume and thank his widow Wendy for agreeing to this.

Tate conjectures in knot theory. Hence a whole new era in knot theory began with the birth of the Jones polynomial. In particular, the 'complete' Tate conjecture on alternating knots was solved by Thistlethwaite and Menasco using very complicated topological arguments, with the Jones polynomial playing a critical role. Several further questions posed by Vaughan, such as the one about finding a knot with trivial Jones polynomial, continue to be the driving motivation in this newly developed branch of knot theory.

2.3. Three-manifolds. Edward Witten (who also won a Fields Medal in 1990) extended the Jones polynomial and the accompanying theory to 3-manifolds . This followed a suggestion by Vaughan, popularised in notes from a talk at Atiyah's seminar in 1986, namely that the place to look was in quantum gauge field theory. The values of the parameters for which Witten's theory works are precisely the values of interest for subfactors of finite depth! Reshetikin and Turaev have clarified and generalised these discoveries, and Ocneanu has shown that any finite depth subfactor gives a topological quantum field theory (TQFT) in 3 dimensions. The simplest proof of the existence of the Witten invariant (essentially due to Lickorish) is based on the special properties of the 'Jones-Wenzl idempotent', already proved in Vaughan's original subfactor paper. Also the 'volume conjecture' made by Kashaev, Murakami and Murakami is a wonderful example of the influence of Vaughan on knot theory/physics. Based on physical intuition, it was argued that the growth rate of the limit of the coloured Jones polynomial as the number of colours goes to ∞ is given by the hyperbolic volume of the knot complement. This conjecture has been verified for some families of knots, but is known to be quite delicate, requiring subtle analysis to evaluate the limits, even when direct calculation is possible. A large number of people work on this topic, with conferences devoted solely to it.

2.4. Category theory and Khovanov homology. The appearance of Vaughan's braid group representations in a group-like structure led immediately to the discovery of braided tensor categories (by Joyal and Street), which were subsequently investigated by mathematicians and physicists throughout the 1990s (including Longo, Fredenhagen, Frolich, Kawahigashi, M. Izumi, and so on). More recently, the work of Khovanov on 'Categorification of the Jones polynomial' gave general knot invariants by doing a category-based skein theory. In fact Khovanov managed to interpret the Jones polynomial as a homology invariant (namely the Euler characteristic of an associated homology). This *Khovanov homology* approach led to other spectacular developments in recent years, including a new proof (by Rasmussen) of Thom's conjecture on the un-knotting number of a link, which was initially proved by Gauge theory techniques. Also Dror Bar Natan showed that Vaughan's planar algebras are the right language to use in calculating Khovanov homology. He gave a slightly more abstract definition of Khovanov homology, as a planar algebra in an appropriate sense, and because of this, it can be calculated much more rapidly than the original Khovanov homology, even though it is more powerful.

2.5. Braid groups. Given the impact of Vaughan's work, it is hard to imagine nowadays how the theory of braid groups and their representations was before 1984. It was a relatively obscure field, and Vaughan sometimes talked about how little interest there was in it when he found his braid group representations. After the Jones polynomial appeared, however, his discovery of the braid group representation in the tower of subfactors (1982) stimulated significant new interest in braid groups and the discovery of many many other braid group representations. In 2000/2001, Bigelow and Krammer (independently) proved that a particular one of these representations is faithful, so that the braid group is linear (and reported this in ICM 2002 invited talks). A still-unsolved major problem in this field asks whether or not Vaughan's original representation in the Temperley-Lieb algebra is faithful.

2.6. Knots and links from the Thompson groups. During the last few years, Vaughan developed an original procedure for constructing actions of groups of fractions of certain categories. A striking application of his method concerns the Thompson groups F, T and V, which he was able to realise as groups of fractions of categories of forests, thus finding new actions of these groups on many spaces. By representing the category of forests on Conway tangles, he obtained constructions of knots and links from F and T, and showed that any link can be obtained in this way. Also using TQFT, he obtained from this a very interesting new class of unitary representations on Hilbert space, whose coefficients are the TQFT link invariants! This work has been received with a lot interest (and surprise) by the group theory community and other people working on the Thompson groups.

2.7. Algebras. Vaughan's approaches and discoveries have become extremely powerful in many areas of mathematics, and a notable example is in Wenzl's proof of the semisimplicity of the Brauer algebra (1989). The lure of a knot invariant and braid representations provided crucial motivation for people who developed quantum groups, in particular the study of the centralizer algebras. The Temperley-Lieb algebra (more correctly now called the Jones-Temperley-Lieb algebra), which describes the algebraic relations between the Jones idempotents in the standard invariant, has become very popular in algebra. The annular Temperley-Lieb algebra discovered by Vaughan in the mid-1990s (see *l'Enseign. Math.* vol. 38 (2001)) has been studied by a lot of people following Vaughan's work. All of this culminated in his discovery that the standard invariants have a *planar algebra* structure, whose description involves a mixed category/operad structure.

Vaughan's diagrammatic approach to the study of these amazingly rich group-like objects provided a totally new vision of the structure of subfactors. By organising planar pictures in simple ways, some powerful new techniques have been developed for the construction of subfactors and restrictions on their possible invariants, akin to (but much more difficult than) the classification of simple Lie algebras via Coxeter-Dynkin diagrams. Using planar algebras, he and Bisch made a plethora of new discoveries about obstructions on the graphs of subfactors, and on the classification of standard invariants of small dimension. As noted above, the importance of these goes far beyond subfactor theory.

2.8. Other fields. To emphasise the broad impact of Vaughan's work in other fields, we also include the following snippets.

2.8.1. *Combinatorics.* Vaughan's work led directly to a revival in the study of the Tutte polynomial of a graph, and a generalisation of it to signed graphs. It has been shown that computation of the Jones polynomial at almost all values is NP-complete. Another important connection with combinatorics is Jaeger's discovery of a spin model knot invariant based on the Higman-Sims graph, which grew out of an attempt by Pierre de la Harpe and Vaughan to obtain spin models for the Kauffman polynomial.

2.8.2. *Algebraic quantum field theory.* Following Vaughan's breakthrough on subfactors and braid groups in the 1980s, several groups of people (including Frolich, Longo, Roberts, Fredenhagen and Rehren Schroer) noticed a striking connection with the Doplicher Haag and Roberts theory of super-selection sectors. They applied Vaughan's results to super-selection theory in low dimensions, Anyons, the fractional quantum Hall effect, and high temperature superconductivity.

2.8.3. *Conformal field theory.* That the Jones representations of the braid group occur as monodromy in the Wess-Zuminon-Witten model was a startling result of work by Tsuchiya and Kanie. A construction by Vaughan and Wassermann in the early 1990s led to a beautiful direct connection between the theory of subfactors and conformal field theory (CFT). This newly developed area is now pursued by many people (including Wassermann, Longo, Kawahigashi, M. Izumi, F. Xu, and others). Some spectacular discoveries have been made recently in this direction, including new insights into monstrous moonshine, notably by Longo-Kawahigashi (*Annals of Math.* 2004) and Kac-Xu.

2.8.4. *Random matrices.* It was shown by 't-Hooft that in the large N limit for a $U(n)$ gauge theory, the only Feynmann diagrams that contribute are the planar ones. This was adopted as a systematic idea in the study of large random matrices, and it is now becoming apparent that the appropriate language to use in this field is that of Vaughan's planar algebras. If one uses a potential that is in a certain planar algebra, then the partition function (in the large N limit) as well as all expected values lie in the planar algebra. Using these ideas, Vaughan worked with Guionnet and Shlyakhtenko to make connections between free probability, random matrices and von Neumann algebras, and in particular, to construct natural matrix models where the number of matrices is a continuously varying parameter.

2.8.5. *Quantum computing.* Mike Freedman and his Microsoft group have shown that calculation of the Jones polynomial is a universal problem for quantum computing, and proposed building a machine based on Chern Simons field theory. It is quite remarkable that they use Vaughan's original subfactor positivity of the traces (and especially $4\cos^2\pi/5$), as they need the underlying Hilbert space structure. Microsoft is testing materials which can 'reproduce' braid group statistics that bear Jones representations. Vaughan himself wrote an article with Aharanov and Landau showing how to approximate the Jones polynomial using a (conventional) quantum computer.

2.8.6. *Mathematical Biology.* The Jones polynomial has been very useful for molecular biologists, in solving some of the knotting and linking problems for DNA molecules. These scientists now use the Jones polynomial and related polynomials to help them understand DNA recombination mechanisms, on a regular basis.

Marston Conder, Rod Downey, David Gauld and Gaven Martin
With Vaughan, founding Co-Directors of the NZMRI

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[1] Dietmar Bisch, David E Evans, Robion Kirby and Sorin Popa, *Memories of Vaughan Jones*, Notices of the American Mathematical Society.
[2] Marston Conder, Rod Downey, David Gauld and Gaven Martin, *Sir Vaughan Frederick Randal Jones, 31/12/1952 – 6/9/2020*, Newsletter of the New Zealand Mathematical Society, **140**, December 2020, 21–25.