# THE NEUMANN PROBLEM FOR MONGE-AMPÈRE TYPE EQUATIONS REVISITED

Feida Jiang and Neil S. Trudinger

*Dedicated to the memory of an incredible kiwi.*

Abstract. This paper concerns *a priori* second order derivative estimates of solutions of the Neumann problem for the Monge-Ampère type equations in bounded domains in $n$ dimensional Euclidean space $\mathbb{R}^n$. We first establish a double normal second order derivative estimate on the boundary under an appropriate notion of domain convexity. Then, assuming a barrier condition for the linearized operator, we provide a complete proof of the global second derivative estimate for elliptic solutions, as previously studied in our earlier work. We also consider extensions to the degenerate elliptic case, in both the regular and strictly regular matrix cases.

## 1. Introduction

In this paper, we revisit the following Neumann problem for Monge-Ampère type equations, considered in [7]:

$$\det[D^2u - A(x, u, Du)] = B(x, u, Du), \qquad \text{in } \Omega, \qquad (1.1)$$

$$D_\nu u = \varphi(x, u), \qquad \text{on } \partial\Omega, \qquad (1.2)$$

where $\Omega \subset \mathbb{R}^n$, $u$ is the unknown scalar function defined on $\bar{\Omega}$, $A$ is a given $n \times n$ symmetric matrix function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, $B$ is a nonnegative scalar valued function on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, $\varphi$ is a scalar valued function defined on $\partial\Omega \times \mathbb{R}$, and $\nu$ denotes the unit inner normal vector field on $\partial\Omega$. As usual, we use $x, z, p$ and $r$ to denote points in $\Omega, \mathbb{R}, \mathbb{R}^n$ and $\mathbb{R}^{n \times n}$ respectively.

A Neumann problem of the form (1.1)-(1.2) arises naturally from the fully non-linear Yamabe problem with prescribed boundary mean curvature in conformal geometry, [7, 9, 15]. In this paper, we complete the proof of second derivative estimates in [7] through upgrading the auxiliary function used there. As a consequence we extend the second derivative estimates in the degenerate case in [14] to embrace general regular matrix functions $A$. We also extend the the strictly regular case in [7] to more general degenerate equations.

We introduce some notation from [7] before stating the main theorems. A matrix $A$, which is twice differentiable with respect to $p$, is called regular, (strictly regular),

in $\Omega$ if $A$ is co-dimension one convex, (strictly co-dimension one convex), with respect to $p$, in the sense that

$$\sum_{i,j,k,l=1}^{n} A_{ij,kl}(x,z,p)\xi_i\xi_j\eta_k\eta_l \geq 0, (\geq c_0 > 0), \tag{1.3}$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$, $\xi, \eta \in \mathbb{R}^n$ and $\xi \cdot \eta = 0$, where $A_{ij,kl} = D^2_{p_k p_l} A_{ij}$ and $c_0$ is a positive constant. A domain $\Omega$ is called uniformly $A$-convex with respect to $u \in C^1(\bar{\Omega})$, if $\partial\Omega \in C^2$ and

$$\sum_{i,j,k=1}^{n} [D_i\nu_j(x) - D_{p_k}A_{ij}(x,u,Du)\nu_k]\tau_i\tau_j \leq -\delta_0|\tau|^2 \tag{1.4}$$

on $\partial\Omega$, for all vectors $\tau = \tau(x)$ tangent to $\partial\Omega$, and some positive constant $\delta_0$. For more information about the definition of uniformly $A$-convex domain, one can refer to the introduction of [7].

For the equation (1.1) with $B > 0$, we have

$$\tilde{F}[u] := \log\det[D^2u - A(\cdot,u,Du)] = \tilde{B}(\cdot,u,Du), \tag{1.5}$$

where $\tilde{B} \triangleq \log B$. We have $\frac{\partial\tilde{F}}{\partial w_{ij}} = w^{ij}$, $\frac{\partial^2\tilde{F}}{\partial w_{ij}\partial w_{kl}} = -w^{ik}w^{jl}$, where $\{w_{ij}\} \triangleq \{u_{ij} - A_{ij}\}$ denotes the augmented Hessian matrix, and $\{w^{ij}\}$ denotes the inverse of the matrix $\{w_{ij}\}$. For convenience, we simply write

$$\mathcal{S} = \sum_{i=1}^{n} w_{ii} \quad \text{and} \quad \mathcal{T} = \sum_{i=1}^{n} w^{ii}. \tag{1.6}$$

We introduce the linearized operators,

$$L = L[u] \triangleq \sum_{i,j=1}^{n} w^{ij}(D_{ij} - D_{p_k}A_{ij}(\cdot,u,Du)D_k), \quad \mathcal{L} = \mathcal{L}[u] \triangleq L - \sum_{k=1}^{n} D_{p_k}\tilde{B}(\cdot,u,Du)D_k. \tag{1.7}$$

We will assume the existence of a nonnegative barrier function $\Phi$ with respect to the linearized operator $L$ in (1.7), in the sense that there exists a barrier function $\Phi \geq 0, \in C^2(\bar{\Omega})$ satisfying

$$L\Phi \geq \mathcal{T} - C_0, \quad \text{in } \Omega, \tag{1.8}$$

$$D_\nu\Phi \geq 0, \quad \text{on } \partial\Omega, \tag{1.9}$$

for some constant $C_0$.

We now formulate a theorem on the global second order derivative estimate for problem (1.1)-(1.2), under the existence of a nonnegative barrier function $\Phi$ satisfying (1.8) and (1.9).

**Theorem 1.1.** *Let* $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ *be an elliptic solution of the Neumann problem (1.1)-(1.2) in a $C^{3,1}$ domain $\Omega \subset \mathbb{R}^n$, which is uniformly $A$-convex with respect to $u$, where $A \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$ is regular, $B > 0, \in C^2(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n)$*

and $\varphi \in C^{2,1}(\partial\Omega \times \mathbb{R})$. *Suppose there exists a barrier function* $\Phi \geq 0, \in C^2(\bar{\Omega})$ *satisfying* (1.8) *and* (1.9). *Then we have the estimate*

$$\sup_{\Omega} |D^2u| \leq C, \tag{1.10}$$

*where* $C$ *is a constant depending on* $n$, $C_0$, $A$, $B$, $\Omega$, $\varphi$, $\delta_0$, $|\Phi|_{1;\Omega}$ *and* $|u|_{1;\Omega}$.

In order to derive an estimate (1.10), which is independent of the positive lower bound of $B$, we let $C_i$, $(i = 1, 2, 3)$, denote positive constants satisfying

$$|DB^{\frac{1}{2(n-1)}}| \leq C_1, \tag{1.11}$$

$$|D^2B^{\frac{1}{n-1}}| \leq C_2, \tag{1.12}$$

and

$$D_{pp}(\log B) \geq -C_3 I, \tag{1.13}$$

in $\mathcal{U} = (\cdot, u, Du)(\Omega)$, where $I$ denotes the identity matrix.

We can then formulate a global second order derivative estimate, (independent of $\inf B$), for problem (1.1)-(1.2) under the conditions (1.11), (1.12), (1.13), which extends the corresponding result in [14] to general regular matrix functions $A$ satisfying the same hypotheses as in Theorem 1.1. We also include the corresponding estimate for strictly regular matrix functions $A$, without the barrier conditions (1.8), (1.9) and condition (1.13).

**Theorem 1.2.** *Under the assumptions of Theorem 1.1, assume also that $B$ satisfies conditions* (1.11), (1.12) *and* (1.13) *for fixed constants* $C_1$, $C_2$ *and* $C_3$. *Then the estimate*

$$\sup_{\Omega} |D^2u| \leq C, \tag{1.14}$$

*holds for constant $C$ depending on* $n$, $C_i$ $(i = 0, 1, 2, 3)$, $A$, $\sup B$, $\Omega$, $\varphi$, $\delta_0$, $|\Phi|_{1;\Omega}$ *and* $|u|_{1;\Omega}$. *If $A$ is strictly regular in $\Omega$, then the barrier conditions* (1.8), (1.9) *and condition* (1.13) *are not necessary and $C$ depends on $c_0$ instead of $C_0$, $C_3$ and* $|\Phi|_{1;\Omega}$.

Note that in Theorems 1.1 and 1.2, we do not assume monotonicity conditions of $A$, $B$ and $\varphi$ with respect to $u$, although these are critical for the construction of barriers $\Phi$ in general domains as in [7].

Some historical results for the Neumann problem for equations of Monge-Ampère type can be found in [4, 7, 8, 10, 11, 13, 14]. The results of this paper, along with their application to full second derivative estimates and existence theorems as in [7], may also be extended to parabolic Monge-Ampère type equations, extending [13], as well as the corresponding problems on Riemannian manifolds, extending [4]. The derivation of second derivative estimates here and in all these papers adapts the approach for the Monge-Ampère equation originating in [10].

This paper is organized as follows. In Section 2, we prove the second order derivative estimates, Theorem 1.1, on uniformly $A$-convex domains. In Section 3, the second order derivative estimates in Theorem 1.2 are treated by combining the techniques in Section 2 and [7, 14].

## 2. Proof of Theorem 1.1

In this section, we consider the global second derivative estimate of the problem (1.1)-(1.2) in uniformly $A$-convex domains for regular matrices $A$, and prove Theorem 1.1.

For the arguments below, we assume the functions $\varphi$, $\nu$ have been smoothly extended to $\bar{\Omega} \times \mathbb{R}$ and $\bar{\Omega}$ respectively. We also assume that near the boundary, $\nu$ is extended to be constant in the normal directions. For convenience in later discussion, we denote $D_{\xi\eta}u \triangleq D_{ij}u\xi_i\eta_j$, $w_{\xi\eta} \triangleq w_{ij}\xi_i\eta_j = D_{ij}u\xi_i\eta_j - A_{ij}\xi_i\eta_j$ for any vectors $\xi$ and $\eta$. As usual, $C$ denotes a constant depending on the known data and may change from line to line in the context.

By differentiation of the equation (1.5) in the direction $\xi$, we have

$$
\begin{aligned}
&w^{ij}(D_{ij}u_\xi - D_\xi A_{ij} - D_z A_{ij} u_\xi - D_{p_l} A_{ij} D_l u_\xi) \\
&= D_\xi \tilde{B} + D_z \tilde{B} u_\xi + D_{p_l} \tilde{B} D_l u_\xi,
\end{aligned}
\tag{2.1}
$$

and a further differentiation in the direction of $\xi$ yields,

$$
\begin{aligned}
&w^{ij}[D_{ij}u_{\xi\xi} - D_{\xi\xi}A_{ij} - (D_{zz}A_{ij})(u_\xi)^2 - (D_{p_k p_l}A_{ij})D_k u_\xi D_l u_\xi \\
&\quad - (D_z A_{ij})u_{\xi\xi} - (D_{p_k}A_{ij})D_k u_{\xi\xi} - 2(D_{\xi z}A_{ij})u_\xi \\
&\quad - 2(D_{\xi p_k}A_{ij})D_k u_\xi - 2(D_{z p_k}A_{ij})(D_k u_\xi)u_\xi] \\
&= w^{ik}w^{jl}D_\xi w_{ij} D_\xi w_{kl} + D_{\xi\xi}\tilde{B} + (D_{zz}\tilde{B})(u_\xi)^2 + (D_{p_k p_l}\tilde{B})D_k u_\xi D_l u_\xi \\
&\quad + 2(D_{\xi z}\tilde{B})u_\xi + 2(D_{\xi p_k}\tilde{B})D_k u_\xi + 2(D_{z p_k}\tilde{B})(D_k u_\xi)u_\xi \\
&\quad + (D_z \tilde{B})u_{\xi\xi} + (D_{p_k}\tilde{B})D_k u_{\xi\xi}.
\end{aligned}
\tag{2.2}
$$

Here and below, we use the standard summation convention that repeated indices indicate summation from 1 to $n$ unless otherwise specified.

Applying the tangential operator $\delta = D - (\nu \cdot D)\nu$ to the boundary condition (1.2), we can get the mixed tangential normal derivative estimate on $\partial\Omega$, namely

$$|D_{\tau\nu}u(x)| \leq C, \quad \text{for } x \in \partial\Omega, \tag{2.3}$$

for any unit tangential vector field $\tau$, where the constant $C$ depends on $\sup_{\bar{\Omega}} |Du|$ and $\Omega$, see [7, 11, 14]. We shall deduce the estimate for $D_{\nu\nu}u$ on $\partial\Omega$. First, the ellipticity of $u$ gives a lower bound of $D_{\nu\nu}u$ in $\bar{\Omega}$, namely

$$D_{\nu\nu}u \geq \sum_{i,j=1}^{n} A_{ij}(\cdot, u, Du)\nu_i\nu_j. \tag{2.4}$$

Next, from the uniform $A$-convexity of $\Omega$ with respect to $u$, we can employ a barrier argument in a boundary strip to obtain an upper bound of $D_{\nu\nu}u$ on $\partial\Omega$, in accordance with the following lemma, which is a refinement of the corresponding estimates in [7, 11, 14]. Here it is convenient to cover simultaneously the degenerate case as it is embraced by the same proof, which is a modification of that in [11].

**Lemma 2.1.** *Let $u \in C^4(\Omega) \cap C^3(\bar{\Omega})$ be an elliptic solution of the Neumann problem (1.1)-(1.2) in a bounded domain $\Omega \subset \mathbb{R}^n$, which is uniformly $A$-convex with respect to $u$. Then we have*

$$D_{\nu\nu}u \leq C, \quad \text{on } \partial\Omega, \tag{2.5}$$

*where the constant $C$ depends on $n$, $A$, $\Omega$, $\varphi$, $\delta_0$, $|u|_{1;\Omega}$ and $|B^{\frac{1}{n-1}}|_{1;\mathcal{U}}$.*

**Proof.** Considering the function

$$h = \nu_k D_k u - \varphi(x, u), \tag{2.6}$$

by calculations, we have from (2.1)

$$|Lh| \leq C[\mathcal{T} + |D\tilde{B}|(1 + |Dh|)], \quad \text{in } \Omega, \tag{2.7}$$

where $L$ is the first operator in (1.7). By using the arithmetic-geometric mean inequality, from (1.1) and the positivity of $B$, we have

$$\Lambda^{\frac{1}{n-1}} \leq \frac{1}{n-1} B^{\frac{1}{n-1}} \mathcal{T}, \tag{2.8}$$

where $\Lambda$ denotes the maximum eigenvalue of the augmented Hessian $w = D^2 u - A$. From (2.6), (2.7), (2.8), (1.2) and $|DB^{\frac{1}{n-1}}| = \frac{1}{n-1} B^{\frac{1}{n-1}} |D\tilde{B}|$, we then obtain

$$\begin{aligned} |Lh| &\leq C|Dh|^{\frac{n-2}{n-1}}\mathcal{T}, \quad \text{for } |Dh| > C', \\ h &= 0, \qquad\qquad \text{on } \partial\Omega, \end{aligned} \tag{2.9}$$

with constants $C$ and $C'$ depending on the same quantities as in the lemma statement, (except that $C'$ is independent of $B$).

Since $\Omega$ is uniformly $A$-convex with respect to $u$ and $A$ is regular with respect to $p$, there exists a defining function $\phi \in C^2(\bar{\Omega})$ satisfying $\phi = 0$ on $\partial\Omega$, $D\phi \neq 0$ on $\partial\Omega$, and $\phi < 0$ in $\Omega$, together with

$$\left[ D_{ij}\phi - \sum_{k=1}^{n} D_{p_k} A_{ij}(\cdot, u, Du) D_k \phi \right]_{n \times n} \geq \delta_1 I \tag{2.10}$$

in a neighbourhood $\mathcal{N}$ of $\partial\Omega$, whenever $D_\nu u \geq \varphi(\cdot, u)$, where $\delta_1$ is a positive constant and $I$ is the identity matrix, with $\mathcal{N}$ and $\delta_1$ depending on $\delta_0$, $A$ and $|u|_{1;\Omega}$. For example, we can take $\phi = -d + t d^2$ near $\partial\Omega$, for a large enough positive constant, where $d := d(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function to $\partial\Omega$. Therefore, we have

$$L\phi \geq \delta_1 \mathcal{T}, \quad \text{in } \mathcal{N} \cap \{d \leq d_0\},$$
$$\phi = 0, \quad \text{on } \partial\Omega, \tag{2.11}$$

for a positive constant $d_0$ depending on $\delta_0$, $A$ and $|u|_{1;\Omega}$. Since the exponent $\frac{n-2}{n-1} < 1$ in (2.9), by taking $-K\phi$ as a barrier function, for sufficiently large constant $K$ and using (2.9) and (2.11), a standard barrier argument, as in the proof of the uniformly convex case in Corollary 14.5 in [2], (under the structure condition (14.32)), leads to the estimate

$$D_\nu h \leq C, \quad \text{on } \partial\Omega. \tag{2.12}$$

Thus, (2.5) follows from (2.12). $\square$

Combining (2.3), (2.4) and (2.5), we conclude that

$$|D_{\nu\xi} u| \leq C, \quad \text{on } \partial\Omega, \tag{2.13}$$

for a unit vector $\xi$ in any direction.

We introduce a perturbation $\tilde{\Phi}$ of the barrier function $\Phi$, given by

$$\tilde{\Phi} = \Phi - a\phi, \tag{2.14}$$

where $a$ is a small positive constant, $\phi \in C^2(\bar{\Omega})$ is a defining function of $\Omega$ satisfying

$$\phi < 0 \quad \text{in } \Omega, \quad \phi = 0 \quad \text{on } \partial\Omega, \quad \text{and} \quad D_\nu \phi = -1 \quad \text{on } \partial\Omega. \tag{2.15}$$

For sufficiently small $a > 0$, the nonnegative function $\tilde{\Phi}$ in (2.14) satisfies

$$L\tilde{\Phi} \geq \frac{1}{2}\mathcal{T} - C_0, \quad \text{in } \Omega,$$
$$D_\nu \tilde{\Phi} \geq a, \quad \text{on } \partial\Omega. \tag{2.16}$$

With the above preparations, we give the proof for the global second order derivative estimate in Theorem 1.1. In order to embrace general regular matrix functions $A$, the argument is a refinement of that in [11] and [7], where the perturbed barrier $\tilde{\Phi}$ is used in its exponential form $e^{\kappa\tilde{\Phi}}$ for some positive constant $\kappa$.

*Proof of Theorem 1.1.* We employ an auxiliary function $v$, given by

$$v = v(\cdot, \xi) = e^{\frac{\alpha}{2}|Du|^2 + e^{\kappa\tilde{\Phi}}}[w_{\xi\xi} - v'(\cdot, \xi)], \tag{2.17}$$

for $x \in \bar{\Omega}$ and $\xi \in \mathbb{S}^{n-1}$, where $\alpha$, $\kappa$ are positive constants to be determined, $\tilde{\Phi}$ is the perturbed barrier function in (2.14), and $v'$ is defined by

$$v'(\cdot, \xi) = 2(\xi \cdot \nu)\xi_i'[D_i\varphi(\cdot, u) - D_k u D_i \nu_k - A_{ij}\nu_j], \tag{2.18}$$

with $\xi' = \xi - (\xi \cdot \nu)\nu$. The term $e^{\kappa\tilde{\Phi}}$ in the auxiliary function $v$ in (2.17) is inspired by that of [12] for the Pogorelov estimate. The strategy of our proof is to estimate $v$ at a maximum point $(x, \xi) \in \bar{\Omega} \times \mathbb{S}^{n-1}$. For this purpose, we divide the discussion into two cases.

**Case 1.** $v$ takes its maximum at an interior point $x_0 \in \Omega$ and a unit vector $\xi$. Let

$$H = \log v = \log(w_{\xi\xi} - v') + \frac{\alpha}{2}|Du|^2 + e^{\kappa\tilde{\Phi}}, \tag{2.19}$$

then the function $H$ also attains its maximum at the point $x_0 \in \Omega$ and the unit vector $\xi$. At the point $x_0$, we have

$$0 = D_i H = \frac{D_i(w_{\xi\xi} - v')}{w_{\xi\xi} - v'} + \alpha D_k u D_{ik} u + \kappa e^{\kappa\tilde{\Phi}} D_i\tilde{\Phi}, \tag{2.20}$$

for $i = 1 \cdots n$, and

$$
\begin{aligned}
0 \geq D_{ij} H = {} & \frac{D_{ij}(w_{\xi\xi} - v')}{w_{\xi\xi} - v'} - \frac{D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v')}{(w_{\xi\xi} - v')^2} \\
& + \alpha(D_{ik}u D_{jk}u + D_k u D_{ijk}u) \\
& + \kappa e^{\kappa\tilde{\Phi}}D_{ij}\tilde{\Phi} + \kappa^2 e^{\kappa\tilde{\Phi}}D_i\tilde{\Phi}D_j\tilde{\Phi}.
\end{aligned}
\tag{2.21}
$$

Consequently, we have at $x_0$,

$$
\begin{aligned}
0 \geq \mathcal{L}H = {} & \mathcal{L}[\log(w_{\xi\xi} - v')] + \frac{\alpha}{2}\mathcal{L}|Du|^2 + \mathcal{L}e^{\kappa\tilde{\Phi}} \\
= {} & \frac{1}{w_{\xi\xi} - v'}\mathcal{L}u_{\xi\xi} - \frac{1}{(w_{\xi\xi} - v')^2}w^{ij}D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v') \\
& - \frac{1}{w_{\xi\xi} - v'}\mathcal{L}(A_{\xi\xi} + v') + \alpha w^{ij}D_{ik}u D_{jk}u + \alpha D_k u \mathcal{L}(D_k u) \\
& + \kappa e^{\kappa\tilde{\Phi}}\mathcal{L}\tilde{\Phi} + \kappa^2 e^{\kappa\tilde{\Phi}}w^{ij}D_i\tilde{\Phi}D_j\tilde{\Phi}.
\end{aligned}
\tag{2.22}
$$

Next, we shall estimate each term on the right hand side of (2.22) at the maximum point $x_0$. Using (2.2) and the regular condition of $A$, (see (3.9) in [17]), we have

$$\mathcal{L}u_{\xi\xi} \geq w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C[(1 + \mathcal{S})\mathcal{T} + \mathcal{S}^2]. \tag{2.23}$$

By Cauchy's inequality, we have

$$w^{ij}D_i(w_{\xi\xi} - v')D_j(w_{\xi\xi} - v') \leq (1+\theta)w^{ij}D_iw_{\xi\xi}D_jw_{\xi\xi} + C(\theta)w^{ij}D_iv'D_jv' \quad (2.24)$$

for any $\theta > 0$, where $C(\theta)$ is a positive constant depending on $\theta$. Since $A_{\xi\xi} + v'$ is a function of $x, z$ and $p$, by direct calculations and (2.1), we have

$$|\mathcal{L}(A_{\xi\xi} + v')| \leq C[(1+\mathcal{S})\mathcal{T} + \mathcal{S}], \quad (2.25)$$

for some constant $C$ depending on $A, B, \varphi$ and $|u|_{1;\Omega}$. Using (2.1), we have

$$|\mathcal{L}(D_ku)| \leq C(1+\mathcal{T}), \quad (2.26)$$

for some constant $C$ depending on $A, B$ and $|u|_{1;\Omega}$. By (2.16), we have

$$\mathcal{L}\tilde{\Phi} \geq \frac{1}{2}\mathcal{T} - \tilde{C}_0, \quad (2.27)$$

for some positive constant $\tilde{C}_0$ depending on $C_0, B, |D\tilde{\Phi}|_{1;\Omega}$ and $|u|_{1;\Omega}$. Inserting (2.23), (2.24), (2.25), (2.26), (2.27), $D_{ik}u = w_{ik} + A_{ik}$ and $w^{ij}w_{jk} = \delta_{ik}$ (here $\delta_{ik}$ is the usual kronecker delta) into (2.22), we get

$$\begin{aligned}
0 \geq &\frac{1}{w_{\xi\xi} - v'}w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - \frac{1+\theta}{(w_{\xi\xi} - v')^2}w^{ij}D_iw_{\xi\xi}D_jw_{\xi\xi} \\
&+ \alpha\mathcal{S} + \kappa e^{\kappa\tilde{\Phi}}\left(\frac{1}{2}\mathcal{T} - \tilde{C}_0\right) - C\left[\frac{1}{w_{\xi\xi} - v'}(1 + \mathcal{S} + \mathcal{T} + \mathcal{S}\mathcal{T} + \mathcal{S}^2) + \alpha(1+\mathcal{T})\right] \\
&+ \kappa^2 e^{\kappa\tilde{\Phi}}w^{ij}D_i\tilde{\Phi}D_j\tilde{\Phi} - \frac{C(\theta)}{(w_{\xi\xi} - v')^2}w^{ij}D_iv'D_jv'.
\end{aligned}$$

$$(2.28)$$

In order to treat the terms with third order derivatives, we need to derive a lower bound for the quantity

$$\mathcal{P} := w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - \frac{1}{w_{11}}w^{ij}D_iw_{\xi\xi}D_jw_{\xi\xi}. \quad (2.29)$$

From the inequality (3.48) in [11], we have

$$\tilde{\mathcal{P}} := w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - \frac{1}{w_{11}}w^{ij}D_\xi w_{i\xi}D_\xi w_{j\xi} \geq 0. \quad (2.30)$$

Using $w_{ij} = u_{ij} - A_{ij}(\cdot, u, Du)$ and by direct calculations, we have

$$\begin{aligned}
&\frac{1}{w_{11}}w^{ij}D_iw_{\xi\xi}D_jw_{\xi\xi} \\
=&\frac{1}{w_{11}}w^{ij}[D_\xi w_{i\xi} + (D_\xi A_{i\xi} - D_iA_{\xi\xi})][D_\xi w_{j\xi} + (D_\xi A_{i\xi} - D_jA_{\xi\xi})] \\
=&\frac{1}{w_{11}}w^{ij}D_\xi w_{i\xi}D_\xi w_{j\xi} + \frac{2}{w_{11}}w^{ij}D_\xi w_{i\xi}(D_\xi A_{j\xi} - D_jA_{\xi\xi}) \\
&+ \frac{1}{w_{11}}w^{ij}(D_\xi A_{i\xi} - D_iA_{\xi\xi})(D_\xi A_{j\xi} - D_jA_{\xi\xi}),
\end{aligned}$$

$$(2.31)$$

and

$$\frac{2}{w_{11}}w^{ij}D_\xi w_{i\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi})$$
$$=\frac{2}{w_{11}}w^{ij}(D_\xi u_{i\xi} - D_\xi A_{i\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi})$$
$$=\frac{2}{w_{11}}w^{ij}[(D_i u_{\xi\xi} - D_i A_{\xi\xi}) - (D_\xi A_{i\xi} - D_i A_{\xi\xi})](D_\xi A_{j\xi} - D_j A_{\xi\xi})$$
$$=\frac{2}{w_{11}}w^{ij}D_i w_{\xi\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}) - \frac{2}{w_{11}}w^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi}).$$

(2.32)

Inserting (2.31) and (2.32) into (2.29), we get

$$\mathcal{P} = \tilde{\mathcal{P}} - \frac{2}{w_{11}}w^{ij}D_i w_{\xi\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}) + \frac{1}{w_{11}}w^{ij}(D_\xi A_{i\xi} - D_i A_{\xi\xi})(D_\xi A_{j\xi} - D_j A_{\xi\xi})$$
$$\geq -\frac{2}{w_{11}}w^{ij}D_i w_{\xi\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}),$$

(2.33)

where (2.30) and $\{w^{ij}\} > 0$ are used in the inequality.

Without loss of generality, we assume that $\{w_{ij}\}$ is diagonal at $x_0$ with the maximum eigenvalue $w_{11}$. We can always assume that $w_{11} > 1$ and is as large as we want; otherwise we are done. Since $v'$ is bounded, $w_{11}$ and $w_{\xi\xi}$ are comparable in the sense that for any $\theta > 0$, there exists a further constant $C(\theta)$ such that

$$|w_{11} - w_{\xi\xi} + v'| < \theta w_{11},$$

(2.34)

if $w_{11} > C(\theta)$. Then (2.34) guarantees the following relationship between $w_{11}$ and $w_{\xi\xi} - v'$,

$$(1 - \theta)w_{11} \leq w_{\xi\xi} - v' \leq (1 + \theta)w_{11}.$$

(2.35)

We now return to the third order derivative terms in (2.28). Using (2.35), we have

$$\frac{1}{w_{\xi\xi} - v'}w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - \frac{1+\theta}{(w_{\xi\xi} - v')^2}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi}$$
$$=\frac{1}{w_{\xi\xi} - v'}\left[w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - \frac{1-\theta}{w_{\xi\xi} - v'}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi}\right] - \frac{2\theta}{(w_{\xi\xi} - v')^2}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi}$$
$$\geq\frac{1}{w_{\xi\xi} - v'}\mathcal{P} - \frac{2\theta}{(w_{\xi\xi} - v')^2}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi}$$
$$\geq -\frac{1}{w_{\xi\xi} - v'}\frac{2}{w_{11}}w^{ij}D_i w_{\xi\xi}(D_\xi A_{j\xi} - D_j A_{\xi\xi}) - \frac{2\theta}{(w_{\xi\xi} - v')^2}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi}$$
$$\geq -\frac{2}{(1-\theta)w_{11}^2}w^{ii}|D_i w_{\xi\xi}||D_\xi A_{i\xi} - D_i A_{\xi\xi}| - \frac{2\theta}{(1-\theta)^2 w_{11}^2}w^{ij}D_i w_{\xi\xi}D_j w_{\xi\xi},$$

(2.36)

where the inequality (2.33) for $\mathcal{P}$ is used in the second inequality. Recalling $D_i H = 0$ in (2.20) and using (2.35), we have

$$
\begin{aligned}
|D_i w_{\xi\xi}| \leq &|\alpha D_k u D_{ik} u + \kappa e^{\kappa\tilde{\Phi}} D_i \tilde{\Phi}|(w_{\xi\xi} - v') + |D_i v'| \\
\leq &(1+\theta) w_{11} |\alpha D_k u (w_{ik} + A_{ik}) + \kappa e^{\kappa\tilde{\Phi}} D_i \tilde{\Phi}| + C(1 + w_{ii}) \qquad (2.37) \\
\leq &(1+\theta)[\alpha C(1 + w_{ii}) + \kappa e^{\kappa\tilde{\Phi}} |D_i \tilde{\Phi}|] w_{11},
\end{aligned}
$$

for $i = 1, \cdots, n$. By a direct calculation, we have

$$
|D_\xi A_{i\xi} - D_i A_{\xi\xi}| \leq C(1 + w_{11}), \qquad (2.38)
$$

for $i = 1, \cdots, n$, where $C$ is a constant depending on $A$ and $|u|_{1;\Omega}$. Using (2.37) and (2.38), we get

$$
\begin{aligned}
&\frac{2}{(1-\theta) w_{11}^2} w^{ii} |D_i w_{\xi\xi}||D_\xi A_{i\xi} - D_i A_{\xi\xi}| \\
&\leq \alpha C(1 + \mathcal{T}) + C\kappa e^{\kappa\tilde{\Phi}} w^{ii} |D_i \tilde{\Phi}|.
\end{aligned} \qquad (2.39)
$$

and

$$
\begin{aligned}
&\frac{2\theta}{(1-\theta)^2 w_{11}^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} \\
&= \frac{2\theta}{(1-\theta)^2 w_{11}^2} w^{ii} (D_i w_{\xi\xi})^2 \\
&\leq \theta C \left[ \alpha^2 \mathcal{S} + \left( \alpha^2 + \kappa^2 e^{2\kappa\tilde{\Phi}} \right) \mathcal{T} \right].
\end{aligned} \qquad (2.40)
$$

Inserting (2.36), (2.39) and (2.40) into (2.28), and using (2.35) again, we then obtain the following inequality provided $w_{11} > C(\theta)$:

$$
\begin{aligned}
0 \geq &\alpha \mathcal{S} + \frac{1}{2} \kappa e^{\kappa\tilde{\Phi}} \mathcal{T} + \kappa^2 e^{\kappa\tilde{\Phi}} w^{ii} |D_i \tilde{\Phi}|^2 - C\kappa e^{\kappa\tilde{\Phi}} w^{ii} |D_i \tilde{\Phi}| \\
&- C(1 + \alpha^2 \theta)\mathcal{S} - C\left[ 1 + \alpha + \left( \alpha^2 + \kappa^2 e^{2\kappa\tilde{\Phi}} \right) \theta \right] \mathcal{T} - \kappa e^{\kappa\tilde{\Phi}} \tilde{C}_0 - C(1 + \alpha).
\end{aligned}
$$
$$(2.41)$$

By Cauchy's inequality, we have

$$
C\kappa w^{ii} |D_i \tilde{\Phi}| \leq \kappa^2 w^{ii} |D_i \tilde{\Phi}|^2 + \frac{C^2}{4} \mathcal{T}. \qquad (2.42)
$$

By taking $\kappa > C^2$ and inserting (2.42) into (2.41), since $\tilde{\Phi} \geq 0$, we get

$$
\begin{aligned}
&[\alpha - C(1 + \alpha^2 \theta)]\mathcal{S} + \left\{ \frac{1}{4}\kappa - C\left[ 1 + \alpha + \left( \alpha^2 + \kappa^2 e^{2\kappa\tilde{\Phi}} \right) \theta \right] \right\} \mathcal{T} \\
&\leq C(1 + \alpha) + \kappa e^{\kappa\tilde{\Phi}} \tilde{C}_0.
\end{aligned} \qquad (2.43)
$$

By successively choosing $\alpha > 2C$, $\kappa > \max\{4C(2+\alpha), C^2\}$ and $\theta = 1/(\alpha^2 + \kappa^2 e^{2\kappa \sup \tilde{\Phi}})$, we thus obtain an estimate $\mathcal{S} \leq C$ at $x_0$, which implies a corresponding estimate for $|D^2 u(x_0)|$.

**Case 2.** We consider the case $x_0 \in \partial\Omega$, namely $v(x, \xi) = e^{\frac{\alpha}{2}|Du|^2 + e^{\kappa\tilde{\Phi}}}[w_{\xi\xi} - v'(\cdot, \xi)]$ attains its maximum over $\bar{\Omega}$ at $x_0 \in \partial\Omega$ and a unit vector $\xi$. We then consider the following two subcases of different directions of $\xi$.

*Subcase (i).* $\xi$ is non-tangential to $\partial\Omega$ at $x_0$. The unit vector $\xi$ can be written as

$$\xi = (\xi \cdot \tau)\tau + (\xi \cdot \nu)\nu, \tag{2.44}$$

where $\tau \in \mathbb{S}^{n-1}$, with $\tau \cdot \nu = 0$, $(\xi \cdot \tau)^2 + (\xi \cdot \nu)^2 = 1$ and $\xi \cdot \nu \neq 0$. By the construction of $v'$ in (2.18), we have at $x_0$,

$$\begin{aligned} w_{\xi\xi} &= (\xi \cdot \tau)^2 w_{\tau\tau} + (\xi \cdot \nu)^2 w_{\nu\nu} + 2(\xi \cdot \tau)(\xi \cdot \nu)w_{\tau\nu} \\ &= (\xi \cdot \tau)^2 w_{\tau\tau} + (\xi \cdot \nu)^2 w_{\nu\nu} + v'(x, \xi). \end{aligned} \tag{2.45}$$

By the construction of $v$ and $v(x_0, \tau) \leq v(x_0, \xi)$, we then have

$$\begin{aligned} v(x_0, \xi) &= (\xi \cdot \tau)^2 v(x_0, \tau) + (\xi \cdot \nu)^2 v(x_0, \nu) \\ &\leq (\xi \cdot \tau)^2 v(x_0, \xi) + (\xi \cdot \nu)^2 v(x_0, \nu), \end{aligned} \tag{2.46}$$

which leads again to

$$v(x_0, \xi) \leq v(x_0, \nu) \leq C, \quad \text{on } \partial\Omega, \tag{2.47}$$

where (2.5) is used in the last inequality.

*Subcase (ii).* $\xi$ is tangential to $\partial\Omega$ at $x_0$. We then have $D_\nu v \leq 0$ at $x_0$. By a direct calculation, we have

$$(w_{\xi\xi} - v')D_\nu\left(\frac{\alpha}{2}|Du|^2 + e^{\kappa\tilde{\Phi}}\right) + D_\nu(w_{\xi\xi} - v') \leq 0, \quad \text{at } x_0. \tag{2.48}$$

From the definition of $v'$ in (2.18), we have

$$v'(x_0, \xi) = 0, \tag{2.49}$$

for tangential $\xi$. Since $\tilde{\Phi} \geq 0$ in $\bar{\Omega}$ and $D_\nu\tilde{\Phi} \geq a > 0$ on $\partial\Omega$, using (2.49), we get from (2.48) that

$$D_\nu u_{\xi\xi} \leq -(\kappa a - \alpha M)w_{\xi\xi} + D_\nu(A_{\xi\xi} + v'), \quad \text{at } x_0, \tag{2.50}$$

where $M$ is a constant defined by

$$M = \max_{x \in \partial\Omega} |D_k u D_{k\nu} u|. \tag{2.51}$$

Here due to (2.13), the constant $M$ in (2.51) is bounded.

On the other hand, by tangentially differentiating the boundary condition twice, we obtain

$$(D_k u)\delta_i\delta_j\nu_k + (\delta_i D_k u)\delta_j\nu_k + (\delta_j D_k u)\delta_i\nu_k + \nu_k\delta_i\delta_j D_k u = \delta_i\delta_j\varphi, \quad \text{on } \partial\Omega. \tag{2.52}$$

Hence at $x_0$, for the tangential direction $\xi$ we have

$$
\begin{aligned}
D_\nu u_{\xi\xi} &\geq \varphi_z D_{ij} u \xi_i \xi_j - 2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j + (\delta_i \nu_j) \xi_i \xi_j D_{\nu\nu} u - C \\
&\geq \varphi_z D_{ij} u \xi_i \xi_j - 2(\delta_i \nu_k) D_{jk} u \xi_i \xi_j - C \\
&\geq (\varphi_z + 2\kappa_0) w_{\xi\xi} - C, \quad \text{at } x_0,
\end{aligned}
\tag{2.53}
$$

where $\kappa_0$ is the minimal curvature of the boundary point over $\partial\Omega$, (2.4) and (2.5) are used in the second inequality and $u_{ij} = w_{ij} + A_{ij}$ is used in the third inequality. Combining (2.50) with (2.53), and using (2.13), we obtain

$$
(\kappa a - \alpha M + \varphi_z + 2\kappa_0) w_{\xi\xi} \leq C, \quad \text{at } x_0.
\tag{2.54}
$$

By choosing $\kappa$ large such that

$$
\kappa \geq \frac{2}{a} \left[ \alpha M - (\inf_\Omega \varphi_z + 2\kappa_0) \right],
\tag{2.55}
$$

we again obtain

$$
v(x_0, \xi) \leq C.
\tag{2.56}
$$

We now conclude from the above two subcases *(i)* and *(ii)* that if $v$ attains its maximum over $\bar{\Omega}$ at a point $x_0 \in \partial\Omega$, then $v(x_0, \xi)$ is bounded from above as in (2.56), which implies the second derivative $D_{\xi\xi} u(x_0)$ is also similarly bounded from above.

Combining **Case 1** and **Case 2**, we obtain the desired estimate (1.10) and complete the proof of Theorem 1.1. $\qquad\qquad\square$

We conclude this section with some remarks.

*Remark* 2.1. In [7], inequality (2.24) is written incorrectly with $\xi$ and $i, j$ interchanged. When we use the correct version (2.30), we need to control the difference $\mathcal{P} - \tilde{\mathcal{P}}$ and this is done through the adjusted auxiliary function (2.17). Rather than indicate all the consequent changes to our previous proof in [7], (and also taking account of Remark 2.3 below), we have for clarity written out the full proof.

*Remark* 2.2. If we assume monotonicity conditions as in [7], namely that $A, B$ and $\varphi$ are non-decreasing with respect to the solution variable $u$, the barrier in (1.8) and (1.9) can be constructed in the form,

$$
\Phi = \frac{1}{\epsilon_1} \left[ e^{K(\bar{u} - u)} - 1 \right]
\tag{2.57}
$$

for some proper constants $\epsilon_1$ and $K$, and an elliptic supersolution $\bar{u} \in C^2(\bar{\Omega})$ satisfying the same boundary condition; see Lemma 2.1 in [7], Lemma 2.2 in [5] or

Lemma 2.1 in [6] for a simpler proof. When $A \equiv 0$, such an elliptic supersolution $\bar{u}$ is readily constructed; see [11]. With the barrier in (2.57), by using the proof of this section, we thus recover Theorems 1.1 and 1.2 in [7] for general regular matrix functions $A$ as stated.

*Remark* 2.3. If we only use the double normal derivative estimate in [7], namely $D_{\nu\nu}u \leq C(1 + M_2)^{\frac{n-2}{n-1}}$ on $\partial\Omega$, we need to adjust the auxiliary function $v$ in (2.17) to be

$$v = e^{\frac{\alpha}{2}|Du - \varphi(x,u)\nu|^2 + e^{\kappa\bar{\Phi}}}[w_{\xi\xi} - v'(\cdot, \xi)], \tag{2.58}$$

where $\varphi$ is the function in (1.2) which has been extended to $\bar{\Omega} \times \mathbb{R}$. The auxiliary function (2.58) is enough for use in the pure tangential derivative estimate on $\partial\Omega$ in subcase (ii) of **Case 2**. In fact, since $|Du - \varphi(x,u)\nu|^2 = |\delta u|^2$ on $\partial\Omega$, differentiating $|Du - \varphi(x,u)\nu|^2$ in the normal direction on $\partial\Omega$, we only get the mixed tangential normal derivatives of $u$, which does not depend on $D_{\nu\nu}u$. So the bound for $D_{\nu\nu}u$ on $\partial\Omega$ will not affect the discussion of the pure tangential estimate on $\partial\Omega$. The same remark is also relevant in the next section.

*Remark* 2.4. If the minimum curvature of $\partial\Omega$, $\kappa_{min} \geq \kappa_0$, for a sufficiently large positive constant $\kappa_0$, it is easy to check that $\Omega$ is uniformly $A$-convex, so that the boundary derivative estimate (2.13) still holds. Also, since $\Omega$ then lies in a small ball $B_R(x_0)$ of radius $R = 1/\kappa_0$ and centre $x_0$, the quadratic function $\Phi(x) = \frac{|x - x_0|^2}{2}$ will serve as a barrier function in the proof of **Case 1**, (analogously to the proof of Lemma 3.3 in [16]), and moreover, in place of (2.17), we can use the simpler auxiliary function

$$v = v(\cdot, \xi) = e^{\frac{\alpha}{2}|Du|^2 + \kappa\Phi}[w_{\xi\xi} - v'(\cdot, \xi)], \tag{2.59}$$

for $\xi \in \mathbb{S}^{n-1}$, where $\alpha, \kappa, v'$ are the same as those in (2.17). We can then use the largeness of $\kappa_0$ in part (ii) of **Case 2**, to complete the proof of estimate (1.10), without any need for the full barrier condition (1.8), (1.9).

## 3. Proof of Theorem 1.2

In this section, under the conditions (1.11), (1.12) and (1.13), we improve the global second order derivative estimate for the problem (1.1)-(1.2) such that the estimate is independent of inf $B$. Both the regular matrix $A$ and the strictly regular matrix $A$ cases are discussed.

If (1.11) holds and $B$ is bounded, it is readily checked that

$$|DB^{\frac{1}{n-1}}| \leq C_1' \tag{3.1}$$

in $\mathcal{U}$, for some positive constant $C_1'$ depending on $n$, $C_1$ and $\sup B$. Then the upper bound (2.5) for $D_{\nu\nu}u$ on $\partial\Omega$ in Lemma 2.1 still holds. From (2.3), (2.4) and (2.5), we can also get

$$|D_{\nu\xi}u| \le C, \quad \text{on } \partial\Omega, \tag{3.2}$$

for a unit vector $\xi$ in any direction.

We are now ready to prove Theorem 1.2. The proofs are the modifications of those in Section 2 for the regular $A$ case and [7] for the strictly regular $A$ case, where (1.11), (1.12) and (1.13) are used properly to avoid the dependence of $\inf B$.

*Proof of Theorem 1.2.* We first deduce some basic inequalities from conditions (1.11), (1.12) and (1.13). By (3.1) and (1.11), we have

$$|D\tilde{B}| \le \tilde{C}B^{-\frac{1}{n-1}}, \quad \text{and} \quad |D\tilde{B}| \le \bar{C}B^{-\frac{1}{2(n-1)}}, \tag{3.3}$$

respectively, where the constant $\tilde{C}$ depends on $n$ and $C_1'$, and the constant $\bar{C}$ depends on $n$ and $C_1$. By (1.11) and (1.12), by direct calculations, we have

$$|D^2\tilde{B}| \le \hat{C}B^{-\frac{1}{n-1}}, \tag{3.4}$$

for some constant $\hat{C}$ depending on $n$, $C_1$ and $C_2$. By (1.13), we have

$$\sum_{k,l}(D_{p_k p_l}\tilde{B})D_k u_\xi D_l u_\xi \ge -C_3\sum_{k=1}^{n}(D_k u_\xi)^2 \ge -C_3'(1+\mathcal{S}^2), \tag{3.5}$$

for some constant $C_3'$ depending on $C_3$, $A$ and $|u|_{1;\Omega}$.

For the regular $A$ case, we will follow the lines of the proof of Theorem 1.1 and make necessary modifications. We still employ the same auxiliary function $v$ in (2.17).

In **Case 1**, $v$ takes its maximum at an interior point $x_0 \in \Omega$ and a unit vector $\xi$. Plugging (3.3), (3.4) and (3.5) into (2.2), then in place of (2.23), we now have

$$\mathcal{L}u_{\xi\xi} \ge w^{ik}w^{jl}D_\xi w_{ij}D_\xi w_{kl} - C[(1+\mathcal{S})\mathcal{T}+\mathcal{S}^2] - C'(1+\mathcal{S})B^{-\frac{1}{n-1}}, \tag{3.6}$$

where the constant $C$ depends also on $C_3'$, and the constant $C'$ depends on $\tilde{C}$ and $\hat{C}$. Similarly, by using the first inequality in (3.3), there are also terms $C(1+\mathcal{S})B^{-\frac{1}{n-1}}$ and $CB^{-\frac{1}{n-1}}$ on the right hand side of (2.25) and (2.26) respectively. In place of (2.27), by the second inequality in (3.3), we have

$$\mathcal{L}\tilde{\Phi} \ge \frac{1}{2}\mathcal{T} - C_0 - \tilde{C}_0 B^{-\frac{1}{2(n-1)}}, \tag{3.7}$$

for some positive constant $\tilde{C}_0$ depending on $\bar{C}$ and $\tilde{\Phi}$. Consequently, in place of (2.28), we have

$$
\begin{aligned}
0 \geq & \frac{1}{w_{\xi\xi} - v'} w^{ik} w^{jl} D_\xi w_{ij} D_\xi w_{kl} - \frac{1+\theta}{(w_{\xi\xi} - v')^2} w^{ij} D_i w_{\xi\xi} D_j w_{\xi\xi} + \alpha \mathcal{S} \\
& + \kappa e^{\kappa \tilde{\Phi}} \left[ \frac{1}{2} \mathcal{T} - C_0 - \tilde{C}_0 B^{-\frac{1}{2(n-1)}} \right] + \kappa^2 e^{\kappa \tilde{\Phi}} w^{ij} D_i \tilde{\Phi} D_j \tilde{\Phi} - \frac{C(\theta)}{(w_{\xi\xi} - v')^2} w^{ij} D_i v' D_j v' \\
& - C \left\{ \frac{1}{w_{\xi\xi} - v'} \left[ 1 + \mathcal{S} + \mathcal{T} + \mathcal{S}\mathcal{T} + \mathcal{S}^2 + (1+\mathcal{S}) B^{-\frac{1}{n-1}} \right] + \alpha \left( B^{-\frac{1}{n-1}} + \mathcal{T} \right) \right\},
\end{aligned}
\tag{3.8}
$$

at the maximum point $x_0 \in \Omega$ of $v$ over $\bar{\Omega}$, for a further constant $C$.

Next, we need to treat the terms in (3.8) involving $B^{-\frac{1}{n-1}}$ and $B^{-\frac{1}{2(n-1)}}$. Without loss of generality, we assume that $\{w_{ij}\}$ is diagonal at $x_0$ with maximum eigenvalue $w_{11}$. We can always assume that $w_{11} > 1$ and is as large as we want; otherwise the proof is completed. From the arithmetic-geometric mean inequality as in (2.8), we then have

$$
\mathcal{T} > (n-1) B^{-\frac{1}{n-1}}.
\tag{3.9}
$$

By using Cauchy's inequality, we also have

$$
\frac{1}{2} \mathcal{T} - C_0 - \tilde{C}_0 B^{-\frac{1}{2(n-1)}} \geq \frac{1}{2} \mathcal{T} - \epsilon B^{-\frac{1}{n-1}} - C(\epsilon),
\tag{3.10}
$$

for any positive constant $\epsilon$, where $C(\epsilon)$ is a positive constant depending on $\epsilon$, $C_0$ and $\tilde{C}_0$. Taking $\epsilon = \frac{n-1}{4}$, from (3.9) and (3.10), we obtain

$$
\frac{1}{2} \mathcal{T} - C_0 - \tilde{C}_0 B^{-\frac{1}{2(n-1)}} \geq \frac{1}{4} \mathcal{T} - C(\epsilon).
\tag{3.11}
$$

Inserting (3.9) and (3.11) into (3.8), and following the same steps as in the proof of Theorem 1.1, we can get an estimate $\mathcal{S} \leq C$ at $x_0$, which implies an estimate for $|D^2 u(x_0)|$.

Since (3.2) still holds, the proof for **Case 2** when $x_0 \in \partial\Omega$ is the same as that of Theorem 1.1. We omit the details.

For the strictly regular $A$ case, we present a proof without the barrier conditions (1.8), (1.9) and condition (1.13) by adapting the proof of Theorem 4.1 in [7]. In order to fit the Neumann case here, we restrict the oblique vector field $\beta$ on $\partial\Omega$ in [7] to be the unit inner normal vector field $\nu$ on $\partial\Omega$. As in [7], we employ the auxiliary function

$$
v = w_{\tau\tau} - K(1 + M_2)\phi,
\tag{3.12}
$$

where $\tau$ is the tangential vector field which is $C^2$ extended to $\bar{\Omega}$, $M_2 = \sup_\Omega |D^2 u|$, $\phi \in C^2(\bar{\Omega})$ is a negative defining function for $\Omega$, and $K$ is a proper chosen constant such that the maximum of $v$ on $\bar{\Omega}$ must occur at an interior point of $\Omega$. Using (3.3) and (3.4), in place of (4.6) in [7], we have

$$w^{ij} A_{ij,kl} u_{k\tau} u_{l\tau} \le C \left[ (1 + M_2)\mathcal{T} + (1 + M_2 + |Du_\tau|^2) B^{-\frac{1}{n-1}} \right], \qquad (3.13)$$

at the interior maximum point $x_0$ of $v$, where the constant $C$ depends also on $C_1$ and $C_2$. Without loss of generality, we choose coordinates so that $w$ is diagonalised at $x_0$ with the maximum eigenvalue $w_{11}$. We can assume that

$$w_{11} > \left[ \frac{2C}{c_0(n-1)} \right]^{n-1} \qquad (3.14)$$

with $C$ being the constant in (3.13), otherwise we have already obtained an upper bound for $u_{\tau\tau}(x_0)$. Similarly to (3.9), using (3.14) we get

$$\mathcal{T} \ge \frac{2C}{c_0} B^{-\frac{1}{n-1}}. \qquad (3.15)$$

Combining (3.13) and (3.15), we get

$$w^{ij} A_{ij,kl} u_{k\tau} u_{l\tau} \le C(1 + M_2)\mathcal{T} + \frac{c_0}{2}|Du_\tau|^2 \mathcal{T}, \qquad (3.16)$$

for a further constant $C$. Using the strictly regular condition of $A$ in (3.16) as in [7], we have

$$c_0|Du_\tau|^2 \mathcal{T} - CM_2 \le C(1 + M_2)\mathcal{T} + \frac{c_0}{2}|Du_\tau|^2 \mathcal{T}, \qquad (3.17)$$

which implies the estimate (4.9) in [7] for $u_{\tau\tau}(x_0)$. The rest of the proof can be completed in the same way as that of Theorem 4.1 in [7]. Note that we only use the strict regularity of $A$ here. The barrier conditions (1.8), (1.9) and condition (1.13) are not used. Therefore, the constant $C$ in the desired estimate (1.14) depends on $c_0$ instead of $C_0$, $C_3$ and $|\Phi|_{1;\Omega}$. $\qquad \square$

*Remark* 3.1. Alternatively, in place of (3.7), we can also use the first inequality in (3.3) to infer

$$\mathcal{L}\tilde{\Phi} \ge \frac{1}{2}\mathcal{T} - C_0 - \tilde{C}_0' B^{-\frac{1}{n-1}} \qquad (3.18)$$

for some constant $\tilde{C}_0'$ depending on $\tilde{C}$ and $\tilde{\Phi}$. Similar to (3.15), by assuming a large enough $w_{11}$ at the interior maximum point $x_0$ of $v$, we have

$$\mathcal{T} > CB^{-\frac{1}{n-1}} \qquad (3.19)$$

for a large enough constant $C$. Then we can obtain a barrier inequality

$$\mathcal{L}\tilde{\Phi} \ge \frac{1}{4}\mathcal{T} - C_0, \quad \text{at } x_0, \qquad (3.20)$$

for the perturbed barrier function $\tilde{\Phi}$, which also suffices in the proof.

*Remark* 3.2. If $B$ is separated in the sense that $B(x, z, p) = f(x, z)g(p)$ with $f > 0$ and $g \geq g_0 > 0$ for some positive constant $g_0$, then condition (1.13) is clearly satisfied and constant $C$ in (1.14) depends on $g_0$ instead of $C_3$.

   If $B$ only depends on $x$, (1.11) holds locally in $\Omega$ if $B^{\frac{1}{n-1}} \in C^{1,1}$ holds globally in $\bar{\Omega}$; see Lemma 3.1 in [1]. In this case, we only need to assume that (1.11) holds near $\partial\Omega$, with $B^{\frac{1}{n-1}} \in C^{1,1}(\bar{\Omega})$. For general $B$ we may also dispense with condition (1.11) if $B^{\frac{1}{n-1}} \in C^{1,1}$ in some strictly larger domain $\mathcal{U}_0$. If we assume just $B^{\frac{1}{n-1}} \in C^{1,1}(\bar{\mathcal{U}})$, by adapting an idea from [3], we will consider the removability of condition (1.11) in the strictly regular case, for more general oblique boundary value problems, in an ensuing work.

*Remark* 3.3. For the degenerate elliptic case when $A = A(x, p)$, $B = B(x, z, p)$ and $\varphi = \varphi(x)$ as in [14], by assuming $B$ is strictly increasing in $z$, the barrier in (1.8) and (1.9) can be constructed as in (2.57) by replacing $\bar{u}$ with a subsolution $\underline{u} \in C^2(\bar{\Omega})$ satisfying the same boundary condition; see [14]. Then following the proof of this section, the second derivative estimate in Theorem 1.1 and the existence of globally $C^{1,1}$ solutions in Theorem 1.2 in [14] are valid for general regular matrix functions $A$, that is the alternative restrictions (a) and (b) in Theorems 1.1 and 1.2 in [14] can be dispensed with.

## References

[1] Z. Błocki, *Regularity of the degenerate Monge-Ampère equation on compact Kähler manifolds*, Math. Z., 244, 153-161, 2003.

[2] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equation of second order*, Second edition, Springer, Berlin, 1983 (reprinted 2001).

[3] P. Guan, $C^2$ *a priori estimates for degenerate Monge-Ampère equations*, Duke Math. J., 86, 323-346, 1997.

[4] A. Hanani, *Le problème de Neumann pour certaines équations du type de Monge-Ampère sur une variété riemannienne*, Canad. J. Math., 52, 757-788, 2000.

[5] F. Jiang, N.S. Trudinger, *On Pogorelov estimates in optimal transportation and geometric optics*, Bull. Math. Sci., 4, 407-431, 2014.

[6] F. Jiang, N.S. Trudinger, *Oblique boundary value problems for augmented Hessian equations* II, Nonlinear Anal., 154, 148-173, 2017.

[7] F. Jiang, N.S. Trudinger, N. Xiang, *On the Neumann problem for Monge-Ampère type equations*, Canad. J. Math., 68, 1334-1361, 2016.

[8] F. Jiang, N. Xiang, J. Xu, *Gradient estimates for Neumann boundary value problem of Monge-Ampère type equations.* Commun. Contemp. Math., 19(4), 1650041, 16pp, 2017.

[9] Q. Jin, A. Li, Y.Y. Li, *Estimates and existence results for a fully nonlinear Yamabe problem on manifolds with boundary*, Calc. Var. PDE., 28, 509-543, 2007.

[10] P.L. Lions, N.S. Trudinger, J. Urbas, *The Neumann problem for equations of Monge-Ampère type*, Proc. Centre Math. Anal., 10, 135-140, 1985.

[11] P.L. Lions, N.S. Trudinger, J. Urbas, *The Neumann problem for equations of Monge-Ampère type*, Comm. Pure Appl. Math., 39, 539-563, 1986.

[12] J. Liu, N.S. Trudinger, *On Pogorelov estimates for Monge-Ampère type equations*, Discrete Contin. Dyn. Syst., 28(3), 1121-1135, 2010.

[13] O.C. Schnürer, K. Smoczyk, *Neumann and second boundary value problems for Hessian and Gauss curvature flows*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 20(6), 1043-1073, 2003.

[14] J. Shi, F. Jiang, *On Neumann problem for the degenerate Monge-Ampère type equations*, Bound. Value Probl., 2021(11), 1-22, 2021.

[15] N.S. Trudinger, *From optimal transportation to conformal geometry, in Geometric Analysis*, Progress in Mathematics, 333, Birkhäuser, 511-520, 2020.

[16] N.S. Trudinger, *On the local theory of prescribed Jacobian equations revisited*, Math. Eng., 3(6), 1-17, 2021.

[17] N.S. Trudinger, X.-J. Wang, *On the second boundary value problem for Monge-Ampère type equations and optimal transportation*, Ann. Scuola Norm. Sup. Pisa Cl. Sci., VIII, 143-174, 2009.

School of Mathematics and Shing-Tung Yau Center of Southeast University

Southeast University

Nanjing 211189

P.R. China

jfd2001@163.com


Mathematical Sciences Institute

The Australian National University

Canberra ACT 0200

Australia

Neil.Trudinger@anu.edu.au