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THE NORMING SET OF A SYMMETRIC 3-LINEAR FORM ON THE PLANE WITH THE l_1-NORM

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Abstract. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $T \in \mathcal{L}(^nE)$ if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$, where $\mathcal{L}(^nE)$ denotes the space of all continuous n-linear forms on E. For $T \in \mathcal{L}(^nE)$, we define

$$\mathrm{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. We classify Norm(T) for every $T \in \mathcal{L}_s(^3l_1^2)$.

1. Introduction

In 1961 Bishop and Phelps [**2**] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, specially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [**1**], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [**3**] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials. Jiménez-Sevilla and Payá [**4**] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}, n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^nE)$ the Banach space of all continuous n-linear forms on E endowed with the norm $\|T\| = \sup_{(x_1,\cdots,x_n)\in S_E\times\cdots\times S_E} |T(x_1, \cdots, x_n)|$. $\mathcal{L}_s(^nE)$ denote the closed subspace of all continuous symmetric n-linear forms on E. An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of T if $\|x_1\| = \cdots = \|x_n\| = 1$ and $|T(x_1, \ldots, x_n)| = \|T\|$.

For $T \in \mathcal{L}(^nE)$, we define

$$\mathrm{Norm}(T) = \Big\{(x_1, \ldots, x_n) \in E^n : (x_1, \ldots, x_n) \text{ is a norming point of } T\Big\}.$$

Norm(T) is called the *norming set* of T. Notice that $(x_1, \ldots, x_n) \in \mathrm{Norm}(T)$ if and only if $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \mathrm{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$. Indeed, if

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$(x_1, \ldots, x_n) \in \text{Norm}(T)$, then

$$|T(\epsilon_1 x_1, \ldots, \epsilon_n x_n)| = |\epsilon_1 \cdots \epsilon_n T(x_1, \ldots, x_n)| = |T(x_1, \ldots, x_n)| = \|T\|,$$

which shows that $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$. If $(\epsilon_1 x_1, \ldots, \epsilon_n x_n) \in \text{Norm}(T)$ for some $\epsilon_k = \pm 1$ $(k = 1, \ldots, n)$, then

$$(x_1, \ldots, x_n) = \Big(\epsilon_1(\epsilon_1 x_1), \ldots, \epsilon_n(\epsilon_n x_n)\Big) \in \text{Norm}(T).$$

The following examples show that $\text{Norm}(T) = \emptyset$ or an infinite set.
Examples. (a) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = \sum_{i=1}^{\infty} \frac{1}{2^i} x_i y_i \in \mathcal{L}_s(^2 c_0).$$

We claim that $\text{Norm}(T) = \emptyset$. Obviously, $\|T\| = 1$. Assume that $\text{Norm}(T) \neq \emptyset$. Let $\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) \in \text{Norm}(T)$. Then,

$$1 = \Big|T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big)\Big| \leq \sum_{i=1}^{\infty} \frac{1}{2^i}|x_i|\ |y_i| \leq \sum_{i=1}^{\infty} \frac{1}{2^i} = 1,$$

which shows that $|x_i| = |y_i| = 1$ for all $i \in \mathbb{N}$. Hence, $(x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}} \notin c_0$. This is a contradiction. Therefore, $\text{Norm}(T) = \emptyset$.

(b) Let

$$T\Big((x_i)_{i\in\mathbb{N}},\ (y_i)_{i\in\mathbb{N}}\Big) = x_1 y_1 \in \mathcal{L}_s(^2 c_0).$$

Then,

$$\text{Norm}(T) = \Big\{\Big((\pm 1, x_2, x_3, \ldots),\ (\pm 1, y_2, y_3, \ldots)\Big) \in c_0 \times c_0 : |x_j| \leq 1,\ |y_j| \leq 1 \text{ for } j \geq 2\Big\}.$$

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^n E)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^n E)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^n E)$, we define

$$\text{Norm}(P) = \Big\{x \in E : x \text{ is a norming point of } P\Big\}.$$

$\text{Norm}(P)$ is called the *norming set* of P. Notice that $\text{Norm}(P) = \emptyset$ or a finite set or an infinite set.

Kim [**6**] classify $\text{Norm}(P)$ for every $P \in \mathcal{P}(^2 l_\infty^2)$, where $l_\infty^2 = \mathbb{R}^2$ with the supremum norm.

If $\text{Norm}(T) \neq \emptyset$, $T \in \mathcal{L}(^n E)$ is called a *norm attaining* n-linear form and if $\text{Norm}(P) \neq \emptyset$, $P \in \mathcal{P}(^n E)$ is called a *norm attaining* n-homogeneoue polynomial (see [**3**]).

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [**9**].

It seems to be natural and interesting to study about $\text{Norm}(T)$ for $T \in \mathcal{L}(^n E)$. For $m \in \mathbb{N}$, let $l_1^m := \mathbb{R}^m$ with the the l_1-norm and $l_\infty^2 = \mathbb{R}^2$ with the supremum norm. Notice that if $E = l_1^m$ or l_∞^2 and $T \in \mathcal{L}(^n E)$, $\text{Norm}(T) \neq \emptyset$ since S_E is compact. Kim ([**5**], [**7**], [**8**]) classified Norm(T) for every $T \in \mathcal{L}({}^2 l_\infty^2), \mathcal{L}_s({}^2 l_\infty^2)$ or $\mathcal{L}_s({}^2 l_1^2)$.

In this paper, we classify Norm(T) for every $T \in \mathcal{L}_s({}^3 l_1^2)$.

2. Results

Let $T((x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)) = ax_1x_2x_3 + by_1y_2y_3 + c(x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2) + d(y_1x_2x_3 + y_2x_1x_3 + y_3x_1x_2) \in \mathcal{L}_s({}^3 l_1^2)$ for some $a, b, c, d \in \mathbb{R}$. For simplicity, we denote $T = (a, b, c, d)$.

Theorem 2.1. *Let $T = (a, b, c, d) \in \mathcal{L}_s({}^3 l_1^2)$ for some $a, b, c, d \in \mathbb{R}$. Then,*

$$\|T\| = \max\Big\{|a|,\ |b|,\ |c|,\ |d|\Big\}.$$

Proof. Let $M := \max\Big\{|a|,\ |b|,\ |c|,\ |d|\Big\}$. Let $(x_j, y_j) \in S_{l_1^2}$ for $j = 1, 2, 3$. It follows that

$$\begin{aligned}
&|T((x_1, y_1), (x_2, y_2), (x_3, y_3))| \\
&\le |a|\,|x_1x_2x_3| + |b|\,|y_1y_2y_3| + |c|\,|x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2| \\
&\qquad + |d|\,|y_1x_2x_3 + y_2x_1x_3 + y_3x_1x_2| \\
&\le M\Big(|x_1x_2x_3| + |y_1y_2y_3| + |x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2| + |y_1x_2x_3 + y_2x_1x_3 + y_3x_1x_2|\Big) \\
&\le M\Big(|x_1||x_2||x_3| + |y_1||y_2||y_3| + |x_1||y_2||y_3| + |x_2||y_1||y_3| + |x_3||y_1||y_2| + |y_1||x_2||x_3| \\
&\qquad + |y_2||x_1|\,|x_3| + |y_3|\,|x_1|\,|x_2|\Big) \\
&= M\Big(|x_1| + |y_1|\Big)\Big(|x_2| + |y_2|\Big)\Big(|x_3| + |y_3|\Big) = M \\
&= \max\Big\{\Big|T\Big((1,0),(1,0),(1,0)\Big)\Big|,\ \Big|T\Big((0,1),(0,1),(0,1)\Big)\Big|,\ \Big|T\Big((1,0),(0,1),(0,1)\Big)\Big|, \\
&\qquad \Big|T\Big((1,0),(1,0),(0,1)\Big)\Big|\Big\} \\
&\le \|T\|.
\end{aligned}$$

Therefore, $\|T\| = M$. □

Notice that if $\|T\| = 1$, then $|a| \le 1, |b| \le 1, |c| \le 1$ and $|d| \le 1$.

Lemma 2.2. *Let $n, m \ge 2$. Let $T \in \mathcal{L}({}^m l_1^n)$ with*

$$T\Big(\Big(x_1^{(1)}, \ldots, x_n^{(1)}\Big), \ldots, \Big(x_1^{(m)}, \ldots, x_n^{(m)}\Big)\Big) = \sum_{1 \le i_k \le n,\ 1 \le k \le m} a_{i_1 \cdots i_m}\ x_{i_1}^{(1)} \cdots x_{i_m}^{(m)}$$

for some $a_{i_1 \cdots i_m} \in \mathbb{R}$. Suppose that $\Big(t_1^{(1)}, \ldots, t_n^{(1)}\Big), \ldots, \Big(t_1^{(m)}, \ldots, t_n^{(m)}\Big) \in$ Norm(T). If $|a_{i_1' \cdots i_m'}| < \|T\|$ for some $1 \le i_k' \le n,\ 1 \le k \le m$, then $t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)} = 0$.

Proof. Assume that $t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)} \ne 0$. It follows that

$$\begin{aligned}
\|T\| &= \Big|T\Big(\Big(t_1^{(1)}, \ldots, t_n^{(1)}\Big), \ldots, \Big(t_1^{(m)}, \ldots, t_n^{(m)}\Big)\Big)\Big| \\
&= \Big|\sum_{1 \le i_k \ne i_k' \le n,\ 1 \le k \le m} a_{i_1 \cdots i_m}\ t_{i_1}^{(1)} \cdots t_{i_m}^{(m)} + a_{i_1' \cdots i_m'}\, t_{i_1'}^{(1)} \cdots t_{i_m'}^{(m)}\Big|
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{1\le i_k\ne i_k'\le n,\ 1\le k\le m} \left|a_{i_1\cdots i_m}\right|\left|t_{i_1}^{(1)}\right|\cdots\left|t_{i_m}^{(m)}\right| + \left|a_{i_1'\cdots i_m'}\right|\left|t_{i_1'}^{(1)}\right|\cdots\left|t_{i_m'}^{(m)}\right| \\
&< \|T\| \sum_{1\le i_k\ne i_k'\le n,\ 1\le k\le m} \left|t_{i_1}^{(1)}\right|\cdots\left|t_{i_m}^{(m)}\right| + \|T\|\left|t_{i_1'}^{(1)}\right|\cdots\left|t_{i_m'}^{(m)}\right| \\
&= \|T\|\Big(\sum_{1\le j\le n}\left|t_j^{(1)}\right|\Big)\cdots\Big(\sum_{1\le j\le n}\left|t_j^{(m)}\right|\Big) \\
&= \|T\|,
\end{aligned}
$$

which is a contradiction. Therefore the statement follows. □

Lemma 2.3. *Let $T=(a,b,c,d)\in\mathcal{L}_s({}^3l_1^2)$ for some $a,b,c,d\in\mathbb{R}$. Then,*

(a) $\Big((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)\Big)\in \mathrm{Norm}(T)$ *if and only if* $\Big(\pm(x_1,y_1),\ \pm(x_2,y_2),\ \pm(x_3,y_3)\Big)\in \mathrm{Norm}(T)$;

(b) $\Big((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)\Big)\in \mathrm{Norm}(T')$ *if and only if* $\Big((x_1,-y_1),\ (x_2,-y_2),\ (x_3,-y_3)\Big)\in \mathrm{Norm}(T)$, *where* $T'=(a,-b,c,-d)$;

(c) $\Big((y_1,x_1),\ (y_2,x_2),\ (y_3,x_3)\Big)\in \mathrm{Norm}(\tilde{T})$ *if and only if* $\Big((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)\Big)\in \mathrm{Norm}(T)$, *where* $\tilde{T}=(b,a,d,c)$.

Proof. This is obvious. □

Kim [**8**] classified $\mathrm{Norm}(T)$ for every $T\in\mathcal{L}({}^2l_1^2)$.

Theorem 2.4. ([**8**]). *Let* $T\Big((x_1,y_1),\ (x_2,y_2)\Big)=ax_1x_2+by_1y_2+cx_1y_2+dx_2y_1\in \mathcal{L}({}^2l_1^2)$ *be such that* $\|T\|=1$ *with* $a\ge|b|$ *and* $a\ge c\ge d\ge 0$. *Then we have the following:*

Case 1. *If* $1=a>|b|, 1>c\ge d$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ (\pm1,0)\Big)\Big\}.$$

Case 2. $(1=a=|b|, 1>c\ge d)$ *or* $(1=a>|b|, 1=c>d)$

If $1=a=|b|, 1>c\ge d$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ (\pm1,0)\Big),\Big((0,\pm1),\ (0,\pm1)\Big)\Big\}.$$

If $1=a>|b|, 1=c>d$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ \pm(t,1-t)\Big):0\le t\le 1\Big\}.$$

Case 3. $(1=a=c=d>|b|)$ *or* $(1=a=|b|, 1=c>d)$

If $1=a=c=d>|b|$, *then*

$$\mathrm{Norm}(T)=\Big\{\pm\Big((1,0),\ \pm(t,1-t)\Big),\pm\Big(\pm(t,1-t),(1,0)\Big):0\le t\le 1\Big\}.$$

If $1=a=b, 1=c>d$, *then*

$$\mathrm{Norm}(T)=\Big\{\Big(\pm(1,0),\ \pm(t,1-t)\Big),\pm\Big(\pm(t,1-t),(0,1)\Big):0\le t\le 1\Big\}.$$

If $1 = a = -b, 1 = c > d$, *then*

$$\text{Norm}(T) = \Big\{\Big(\pm(1,0),\ \pm(t,1-t)\Big), \pm\Big(\pm(t,t-1),(0,1)\Big) : 0 \le t \le 1\Big\}.$$

Case 4. $1 = a = |b| = c = d$

If $b = 1$, *then*

$$\text{Norm}(T) = \Big\{\Big(\pm(t,1-t),\ \pm(s,1-s)\Big) : 0 \le t, s \le 1\Big\}.$$

If $b = -1$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&\Big(\pm(0,1),\ \pm(t,t-1)\Big), \Big(\pm(t,t-1), \pm(0,1)\Big),\\ &\Big(\pm(1,0),\ \pm(t,1-t)\Big), \Big(\pm(t,1-t), \pm(1,0)\Big) : 0 \le t \le 1\Big\}.\end{aligned}$$

We are in the position to prove the main result of this paper.

Let $T((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)) = ax_1x_2x_3 + by_1y_2y_3 + c(x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2) + d(y_1x_2x_3 + y_2x_1x_3 + y_3x_1x_2) \in \mathcal{L}_s({}^3l_1^2)$. By Lemma 2.3, we may assume that $a \ge b \ge 0$.

Theorem 2.5. *Let* $T((x_1,y_1),\ (x_2,y_2),\ (x_3,y_3)) = ax_1x_2x_3 + by_1y_2y_3 + c(x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2) + d(y_1x_2x_3 + y_2x_1x_3 + y_3x_1x_2) \in \mathcal{L}_s({}^3l_1^2)$ *be such that* $\|T\| = 1$ *with* $a \ge b \ge 0$. *Then we have the following 11 cases:*

Case 1. *If* $a = 1, b < 1, |c| < 1$ *and* $|d| < 1$, *then*

$$\text{Norm}(T) = \Big\{\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big)\Big\}.$$

Case 2. *If* $|c| = 1, a < 1, b < 1$ *and* $|d| < 1$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&\Big((\pm1,0),\ (0,\pm1),\ (0,\pm1)\Big),\ \Big((0,\pm1),\ (\pm1,0),\ (0,\pm1)\Big),\\ &\Big((0,\pm1),\ (0,\pm1),\ (\pm1,0)\Big)\Big\}.\end{aligned}$$

Case 3. *If* $|d| = 1, a < 1, b < 1$ *and* $|c| < 1$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&\Big((\pm1,0),\ (\pm1,0),\ (0,\pm1)\Big),\ \Big((\pm1,0),\ (0,\pm1),\ (\pm1,0)\Big),\\ &\Big((0,\pm1),\ (\pm1,0),\ (\pm1,0)\Big)\Big\}.\end{aligned}$$

Case 4. *If* $a = b = 1, |c| < 1$ *and* $|d| < 1$, *then*

$$\text{Norm}(T) = \Big\{\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (0,\pm1),\ (0,\pm1)\Big)\Big\}.$$

Case 5. *If* $a = |c| = 1, b < 1$ *and* $|d| < 1$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (0,\pm1),\ (\pm1,0)\Big),\\ &\Big((0,\pm1),\ (\pm1,0),\ (0,\pm1)\Big),\ \Big((\pm1,0),\ (0,\pm1),\ (0,\pm1)\Big)\Big\}.\end{aligned}$$

Case 6. *If* $a = |d| = 1, b < 1$ *and* $|c| < 1$, *then*

$$\begin{aligned}\text{Norm}(T) = \Big\{&\Big((\pm1,0),\ (\pm1,0),\ \pm(t,(1-t)\,\text{sign}(d))\Big),\\ &\Big((\pm1,0),\ \pm(t,(1-t)\,\text{sign}(d)),\ (\pm1,0)\Big),\end{aligned}$$

$$\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm1,0),\ (\pm1,0)\Big) : 0\le t\le 1\Big\}.$$

Case 7. *If* $|c|=|d|=1$ *and* $a<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm1,0),\ (0,\pm1),\ \pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c))\Big),\\ &\Big((\pm1,0),\ \pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c)),\ (0,\pm1)\Big),\\ &\Big(\pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c)),\ (\pm1,0),\ (0,\pm1)\Big),\\ &\Big((0,\pm1),\ (\pm1,0),\ \pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c))\Big),\\ &\Big((0,\pm1),\ \pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c)),\ (\pm1,0)\Big),\\ &\Big(\pm(t\,\mathrm{sign}(d),(1-t)\,\mathrm{sign}(c)),\ (0,\pm1),\ (\pm1,0)\Big) : 0\le t\le 1\Big\}.\end{aligned}$$

Case 8. *If* $a=b=|c|=1$ *and* $|d|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (0,\pm1),\ \pm(t,(1-t)\,\mathrm{sign}(c))\Big)\\ &\Big((0,\pm1),\ \pm(t,(1-t)\,\mathrm{sign}(c)),\ (0,\pm1)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(c)),\ (0,\pm1),\ (0,\pm1)\Big) : 0\le t\le 1\Big\}.\end{aligned}$$

Case 9. *If* $a=b=|d|=1$ *and* $|c|<1$, *then*

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((0,\pm1),\ (0,\pm1),\ (0,\pm1)\Big),\ \Big((\pm1,0),\ (\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d))\Big)\\ &\Big((\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm1,0)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm1,0),\ (\pm1,0)\Big) : 0\le t\le 1\Big\}.\end{aligned}$$

Case 10. *If* $a=|c|=|d|=1$ *and* $b<1$, *then the following holds.*

If $c=1$, *then*

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d)),\ \pm(s,(1-s)\,\mathrm{sign}(d)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm1,0),\ \pm(s,(1-s)\,\mathrm{sign}(d)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ \pm(s,(1-s)\,\mathrm{sign}(d),\ (\pm1,0)\Big) : 0\le t,s\le 1\Big\}.\end{aligned}$$

If $c=-1$, *then*

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm1,0),\ (0,\pm1),\ \pm(t,(t-1)\,\mathrm{sign}(d)),\\ &\Big((\pm1,0),\ \pm(t,(t-1)\,\mathrm{sign}(d)),\ (0,\pm1)\Big),\\ &\Big(\pm(t,(t-1)\,\mathrm{sign}(d)),\ (\pm1,0),\ (0,\pm1)\Big),\\ &\Big((0,\pm1),\ (\pm1,0),\ \pm(t,(t-1)\,\mathrm{sign}(d))\Big),\\ &\Big((0,\pm1),\ \pm(t,(t-1)\,\mathrm{sign}(d)),\ (\pm1,0)\Big),\\ &\Big(\pm(t,(t-1)\,\mathrm{sign}(d)),\ (0,\pm1),\ (\pm1,0)\Big),\end{aligned}$$

$$\Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,(1-t)\operatorname{sign}(d))\Big),$$
$$\Big((\pm 1,0),\ \pm(t,(1-t)\operatorname{sign}(d)),\ (\pm 1,0)\Big),$$
$$\Big(\pm(t,(1-t)\operatorname{sign}(d)),\ (\pm 1,0),\ (\pm 1,0)\Big) : 0\le t\le 1\Big\}.$$

Case 11. *If $a=b=|c|=|d|=1$, then the following holds.*

If $-c=d=1$, then

$$\begin{aligned}\operatorname{Norm}(T)=\Big\{&\Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,1-t)\Big),\Big((\pm 1,0),\ \pm(t,1-t),\ (\pm 1,0)\Big),\\ &\Big(\pm(t,1-t),\ (\pm 1,0),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\\ &\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\Big(\pm(t,t-1),\ (0,\pm 1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ \pm(s,s-1),\ (0,\pm 1)\ \Big) : 0\le t,s\le 1\Big\}.\end{aligned}$$

If $c=-1=d$, then

$$\begin{aligned}\operatorname{Norm}(T)=\Big\{&\Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((\pm 1,0),\ \pm(t,t-1),\ (\pm 1,0)\Big),\\ &\Big(\pm(t,t-1),\ (\pm 1,0),\ (\pm 1,0)\Big),\Big(\pm(t,t-1),\ (\pm 1,0),\ (0,\pm 1)\Big),\\ &\Big((\pm 1,0),\ \pm(t,t-1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ \pm(s,s-1)\Big),\Big(\pm(t,t-1),\ \pm(s,s-1),\ (0,\pm 1)\ \Big) :\\ &0\le t,s\le 1\Big\}.\end{aligned}$$

If $c=1=-d$, then

$$\begin{aligned}\operatorname{Norm}(T)=\Big\{&\Big((0,\pm 1),\ (0,\pm 1),\ \pm(t,t-1)\Big),\Big((0,\pm 1),\ \pm(t,1-t),\ (0,\pm 1)\Big),\\ &\Big(\pm(t,1-t),\ (0,\pm 1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big((\pm 1,0),\ \pm(t,t-1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((\pm 1,0),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ (\pm 1,0),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ \pm(s,s-1),\ (\pm 1,0)\ \Big) : 0\le t,s\le 1\Big\}.\end{aligned}$$

If $c = d = 1$, *then*

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(t, 1-t),\ \pm(s, 1-s),\ \pm(w, 1-w)\Big) : 0 \le t, s, w \le 1\Big\}.$$

Proof. Since $1 = \|T\| = \max\Big\{a, b, |c|, |d|\Big\}$, $a = 1, b = 1, c = 1$, or $d = 1$. Hence, we may consider the 11 cases separately. Let $\Big((x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)\Big) \in \mathrm{Norm}(T)$. By Lemma 2.3, we may assume that $x_j \ge 0$ for $j = 1, 2, 3$.

Case 1. Suppose that $a = 1, b < 1, |c| < 1$ and $|d| < 1$.

It is obvious that $\Big((\pm1, 0),\ (\pm1, 0),\ (\pm1, 0)\Big) \in \mathrm{Norm}(T)$. By Lemma 2.2,

$$1 = \Big|T\Big((x_1, y_1), (x_2, y_2), (x_3, y_3)\Big)\Big| = |x_1|\,|x_2|\,|x_3|.$$

Hence, $|x_j| = 1, y_j = 0$ for $j = 1, 2, 3$. Therefore $\mathrm{Norm}(T) = \Big\{\Big((\pm1, 0),\ (\pm1, 0),\ (\pm1, 0)\Big)\Big\}$.

Case 2. Suppose that $|c| = 1, a < 1, b < 1$ and $|d| < 1$.

By Lemma 2.2, we have $0 = x_1x_2x_3 = y_1y_2y_3 = y_1x_2x_3 = y_2x_1x_3 = y_3x_1x_2$ and $1 = \Big|x_1y_2y_3 + x_2y_1y_3 + x_3y_1y_2\Big|$. We split into three subcases.

Suppose that $x_1 = 0$. Then $|y_1| = 1$ and $x_2x_3 = 0$. If $x_2 = 0$, then $|y_2| = 1$ and $1 = |x_3|\,|y_1|\,|y_2| = |x_3|$. Hence, $\Big((0, \pm1),\ (0, \pm1),\ (\pm1, 0)\Big) \in \mathrm{Norm}(T)$. If $x_3 = 0$, then $|y_3| = 1$ and $1 = |x_2|\,|y_1|\,|y_3| = |x_2|$. As a result we have $\Big((0, \pm1),\ (\pm1, 0),\ (0, \pm1)\Big) \in \mathrm{Norm}(T)$.

Suppose that $x_2 = 0$. Then $|y_2| = 1$ and $x_1x_3 = 0$. If $x_1 = 0$, then $|y_1| = 1$ and $1 = |x_3|\,|y_1|\,|y_2| = |x_3|$. Therefore we have $\Big((0, \pm1),\ (0, \pm1),\ (\pm1, 0)\Big) \in \mathrm{Norm}(T)$. If $x_3 = 0$, then $|y_3| = 1$ and $1 = |x_1|\,|y_2|\,|y_3| = |x_1|$. Hence, $\Big((\pm1, 0),\ (0, \pm1),\ (0, \pm1)\Big) \in \mathrm{Norm}(T)$.

Suppose that $x_3 = 0$. Then $|y_3| = 1$ and $x_1x_2 = 0$. If $x_1 = 0$, then $|y_1| = 1$ and $1 = |x_2|\,|y_1|\,|y_3| = |x_2|$. We conclude that $\Big((0, \pm1),\ (\pm1, 0),\ (0, \pm1)\Big) \in \mathrm{Norm}(T)$. If $x_2 = 0$, then $|y_2| = 1$ and $1 = |x_1|\,|y_2|\,|y_3| = |x_1|$. Then we have $\Big((\pm1, 0),\ (0, \pm1),\ (0, \pm1)\Big) \in \mathrm{Norm}(T)$. Therefore,

$$\mathrm{Norm}(T) = \Big\{\Big((\pm1, 0),\ (0, \pm1),\ (0, \pm1)\Big),\ \Big((0, \pm1),\ (\pm1, 0),\ (0, \pm1)\Big),\ \Big((0, \pm1),\ (0, \pm1),\ (\pm1, 0)\Big)\Big\}.$$

Case 3. Suppose that $|d| = 1, a < 1, b < 1$ and $|c| < 1$.

Let $\tilde{T} = (b, a, d, c) \in \mathcal{L}_s({}^3l_1^2)$. By case 2 we know that,

$$\mathrm{Norm}(\tilde{T}) = \Big\{\Big((\pm1, 0),\ (0, \pm1),\ (0, \pm1)\Big),\ \Big((0, \pm1),\ (\pm1, 0),\ (0, \pm1)\Big),\ \Big((0, \pm1),\ (0, \pm1),\ (\pm1, 0)\Big)\Big\}.$$

By Lemma 2.3 we have

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ (\pm1,0),\ (0,\pm1)\Big),\ \Big((\pm1,0),\ (0,\pm1),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (\pm1,0),\ (\pm1,0)\Big)\Big\}.$$

Case 4. Suppose that $a=b=1, |c|<1$ and $|d|<1$.

By Lemma 2.2 we have $0=x_1y_2y_3=x_2y_1y_3=x_3y_1y_2=y_1x_2x_3=y_2x_1x_3=y_3x_1x_2$ and $1=\Big|x_1x_2x_3+y_1y_2y_3\Big|$. Then $x_{j_0}=0$ or $y_{j_0}=0$ for some $j_0\in\{1,2,3\}$. We now consider two subcases.

Suppose that $x_{j_0}=0$ for some $j_0\in\{1,2,3\}$. Then $1=\Big|x_1x_2x_3+y_1y_2y_3\Big|=|y_1|\,|y_2|\,|y_3|$. Hence, $1=|y_j|$ and $x_j=0$ for all $j=1,2,3$. As a result we have $\Big((0,\pm1),\ (0,\pm1),\ (0,\pm1)\Big)\in\mathrm{Norm}(T)$.

Suppose that $y_{j_0}=0$ for some $j_0\in\{1,2,3\}$. Then $1=\Big|x_1x_2x_3+y_1y_2y_3\Big|=|x_1|\,|x_2|\,|x_3|$. Therefore, $1=|x_j|$ and $y_j=0$ for all $j=1,2,3$. We conclude that $\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big)\in\mathrm{Norm}(T)$. Therefore,

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (0,\pm1),\ (0,\pm1)\Big)\Big\}.$$

Case 5. Suppose that $a=|c|=1, b<1$ and $|d|<1$.

By Lemma 2.2 we know that $0=y_1y_2y_3=y_1x_2x_3=y_2x_1x_3=y_3x_1x_2$ and $1=\Big|x_1x_2x_3+c(x_1y_2y_3+x_2y_1y_3+x_3y_1y_2)\Big|$. If $y_1=0.$, then we have $|x_1|=1$ and $0=x_2y_3$. If $x_2=0$, then $|y_2|=1$ and $1=|y_2|\,|y_3|$. As a result we have $|y_2|=|y_3|=1$ and $x_2=x_3=0$. Therefore, $\Big((\pm1,0),\ (0,\pm1),\ (0,\pm1)\Big)\in\mathrm{Norm}(T)$.

If $y_3=0$, then $|x_3|=1$ and $y_2=0$. We conclude that, $\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big)\in\mathrm{Norm}(T)$. Since T is symmetric,

$$\mathrm{Norm}(T)=\Big\{\Big((\pm1,0),\ (\pm1,0),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (0,\pm1),\ (\pm1,0)\Big),\ \Big((0,\pm1),\ (\pm1,0),\ (0,\pm1)\Big),\ \Big((\pm1,0),\ (0,\pm1),\ (0,\pm1)\Big)\Big\}.$$

Case 6. Suppose that $a=|d|=1, b<1$ and $|c|<1$.

By Lemma 2.2 we have $0=y_1y_2y_3=x_1y_2y_3=x_2y_1y_3=x_3y_1y_2$ and $1=\Big|x_1x_2x_3+d(y_1x_2x_3+y_2x_1x_3+y_3x_1x_2)\Big|$. Consider the following three subcases.

Suppose that $y_1=0$. Then $|x_1|=1$ and $0=y_2y_3$. If $y_2=0$, then $|x_2|=1$ and $1=|x_3+dy_3|$. Hence, $(x_3,y_3)=\pm(t,(1-t)\,\mathrm{sign}(d))$ for some $0\le t\le 1$. As a result, $\Big((\pm1,0),\ (\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d))\Big)\in\mathrm{Norm}(T)$. If $y_3=0$, then $|x_3|=1$ and $1=|x_2+dy_2|$. Therefore, $(x_2,y_2)=\pm(t,(1-t)\,\mathrm{sign}(d))$ for some $0\le t\le 1$. We now have $\Big((\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm1,0)\Big)\in\mathrm{Norm}(T)$.

Suppose that $y_2=0$. Then $|x_2|=1$ and $0=y_1y_3$. If $y_1=0$, then $|x_1|=1$ and $1=|x_3+dy_3|$. Now we have $(x_3,y_3)=\pm(t,(1-t)\,\mathrm{sign}(d))$ for some $0\le t\le 1$. Hence, $\Big((\pm1,0),\ (\pm1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d))\Big)\in\mathrm{Norm}(T)$. If $y_3=0$, then $|x_3|=1$

and $1 = |x_2 + dy_2|$. As a result we have $(x_2, y_2) = \pm(t, (1-t)\,\mathrm{sign}(d))$ for some $0 \le t \le 1$. We now have that $\Big((\pm 1, 0),\ \pm(t, (1-t)\,\mathrm{sign}(d)),\ (\pm 1, 0)\Big) \in \mathrm{Norm}(T)$.

Suppose that $y_3 = 0$. Then $|x_3| = 1$ and $0 = y_1 y_2$. If $y_1 = 0$, then $|x_1| = 1$ and $1 = |x_2 + dy_2|$. Therefore, $(x_2, y_2) = \pm(t, (1-t)\,\mathrm{sign}(d))$ for some $0 \le t \le 1$. Now we have that $\Big((\pm 1, 0),\ \pm(t, (1-t)\,\mathrm{sign}(d)),\ (\pm 1, 0)\Big) \in \mathrm{Norm}(T)$. If $y_2 = 0$, then $|x_2| = 1$ and $1 = |x_1 + dy_1|$. Therefore, $(x_1, y_1) = \pm(t, (1-t)\,\mathrm{sign}(d))$ for some $0 \le t \le 1$. We conclude that $\Big(\pm(t, (1-t)\,\mathrm{sign}(d)),\ (\pm 1, 0),\ (\pm 1, 0)\Big) \in \mathrm{Norm}(T)$. We conclude that

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\Big((\pm 1, 0),\ (\pm 1, 0),\ \pm(t, (1-t)\,\mathrm{sign}(d))\Big),\\ &\Big((\pm 1, 0),\ \pm(t, (1-t)\,\mathrm{sign}(d)),\ (\pm 1, 0)\Big),\\ &\Big(\pm(t, (1-t)\,\mathrm{sign}(d)),\ (\pm 1, 0),\ (\pm 1, 0)\Big) : 0 \le t \le 1\Big\}.\end{aligned}$$

Case 7. Suppose that $|c| = |d| = 1$ and $a < 1$.

By Lemma 2.2 we know that $0 = x_1x_2x_3 = y_1y_2y_3$.

Pick $j_0 \in \{1, 2, 3\}$ such that $x_{j_0} = 0$. Since T is symmetric, we may assume that $j_0 = 1$. Then $|y_1| = 1$. Hence, $y_2 = 0$ or $y_3 = 0$. If $y_2 = 0$, then $|x_2| = 1$ and $1 = |cx_2y_1y_3 + dx_2x_3y_1| = |cy_3 + dx_3|$. Therefore, $(x_3, y_3) = \pm(t\,\mathrm{sign}(c), (1-t)\,\mathrm{sign}(d))$ for some $0 \le t \le 1$. As a result we have $\Big((0, \pm 1),\ (\pm 1, 0),\ \pm(t\,\mathrm{sign}(c), (1-t)\,\mathrm{sign}(d))\Big) \in \mathrm{Norm}(T)$ for all $0 \le t \le 1$. If $y_3 = 0$, then $|x_3| = 1$ and $1 = |cx_3y_1y_2 + dx_2x_3y_1| = |cy_2 + dx_2|$. We can see that $(x_2, y_2) = \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c))$ for some $0 \le t \le 1$. Hence, $\Big((0, \pm 1, 0),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c)),\ (\pm 1, 0)\Big) \in \mathrm{Norm}(T)$ for all $0 \le t \le 1$. Therefore,

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\Big((\pm 1, 0),\ (0, \pm 1),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c))\Big),\\ &\Big((\pm 1, 0),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c)),\ (0, \pm 1)\Big),\\ &\Big(\pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c)),\ (\pm 1, 0),\ (0, \pm 1)\Big),\\ &\Big((0, \pm 1),\ (\pm 1, 0),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c))\Big),\\ &\Big((0, \pm 1),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c)),\ (\pm 1, 0)\Big),\\ &\Big(\pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c)),\ (0, \pm 1),\ (\pm 1, 0)\Big) : 0 \le t \le 1\Big\}.\end{aligned}$$

Case 8. Suppose that $a = b = |c| = 1$ and $|d| < 1$.

By Lemma 2.2 we have $0 = x_2x_3y_1 = x_1x_3y_2 = x_1x_2y_3$. We consider the following three subcases.

Suppose that $x_2 = 0$. Then $|y_2| = 1$ and $x_1x_3 = 0$. If $x_1 = 0$, then $|y_1| = 1$ and $1 = |y_3 + cx_3|$. Hence, $(x_3, y_3) = \pm(t, (1-t)\,\mathrm{sign}(c))$ for some $0 \le t \le 1$. As a result, $\Big((0, \pm 1),\ (0, \pm 1),\ \pm(t\,\mathrm{sign}(d), (1-t)\,\mathrm{sign}(c))\Big) \in \mathrm{Norm}(T)$ for all $0 \le t \le 1$.

If $x_3 = 0$, then $|y_3| = 1$ and $1 = |y_2 + cx_2|$. Therefore, $(x_2, y_2) = \pm(t, (1 - t)\operatorname{sign}(c))$ for some $0 \le t \le 1$. We conclude that $\Big((0, \pm 1),\ \pm(t \operatorname{sign}(d), (1-t)\operatorname{sign}(c)),\ (0, \pm 1)\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$.

If $y_1 = 0$, then $|x_1| = 1$ and $x_3y_2 = 0$. If $x_3 = 0$, then $|y_3| = 1$ and $1 = |y_2 + cx_2|$. We can see that $(x_2, y_2) = \pm(t, (1 - t)\operatorname{sign}(c))$ for some $0 \le t \le 1$. Therefore, $\Big((\pm 1, 0),\ \pm(t \operatorname{sign}(d), (1 - t)\operatorname{sign}(c)),\ (0, \pm 1)\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$. If $y_2 = 0$, then $|x_2| = 1$ and $1 = |x_3|$. Hence, $\Big((\pm 1, 0),\ (\pm 1, 0),\ (\pm 1, 0)\Big) \in \operatorname{Norm}(T)$. From here we have

$$\begin{aligned}\operatorname{Norm}(T) = \Big\{ &\Big((\pm 1, 0),\ (\pm 1, 0),\ (\pm 1, 0)\Big),\ \Big((0, \pm 1),\ (0, \pm 1),\ \pm(t, (1 - t)\operatorname{sign}(c))\Big) \\ &\Big((0, \pm 1),\ \pm(t, (1 - t)\operatorname{sign}(c)),\ (0, \pm 1)\Big), \\ &\Big(\pm(t, (1 - t)\operatorname{sign}(c)),\ (0, \pm 1),\ (0, \pm 1)\Big) : 0 \le t \le 1\Big\}.\end{aligned}$$

Case 9. Suppose that $a = b = |d| = 1$ and $|c| < 1$.

By Lemma 2.2 we have $0 = x_1y_2y_3 = x_2y_1y_3 = x_3y_1y_2$. We consider the following three subcases.

Suppose that $x_1 = 0$. Then $|y_1| = 1$ and $x_2y_3 = 0$. If $x_2 = 0$, then $|y_2| = 1, x_3 = 0$ and $|y_3| = 1$. Hence, $\Big((0, \pm 1),\ (0, \pm 1),\ (0, \pm 1)\Big) \in \operatorname{Norm}(T)$. If $y_3 = 0$, then $|x_3| = 1, y_2 = 0$ and $|x_2| = 1$. As a result, $\Big((0, \pm 1),\ (\pm 1, 0),\ (\pm 1, 0)\Big) \in \operatorname{Norm}(T)$.

Suppose that $y_2 = 0$. Then $|x_2| = 1$ and $y_1y_3 = 0$. If $y_1 = 0$, then $|x_1| = 1$ and $|x_3 + dy_3| = 1$. Therefore, $\Big((\pm 1, 0),\ (\pm 1, 0),\ \pm(t \operatorname{sign}(d), (1-t)\operatorname{sign}(c))\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$. If $y_3 = 0$, then $|x_3| = 1$ and $|x_1 + dy_1| = 1$. We can see that $\Big(\pm(t \operatorname{sign}(d), (1 - t)\operatorname{sign}(c)),\ (\pm 1, 0),\ (\pm 1, 0)\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$.

Suppose that $y_3 = 0$. Then $|x_3| = 1$ and $y_1y_2 = 0$. If $y_1 = 0$, then $|x_1| = 1$ and $|x_2 + dy_2| = 1$. Hence, $\Big((\pm 1, 0),\ \pm(t \operatorname{sign}(d), (1 - t)\operatorname{sign}(c)),\ (\pm 1, 0)\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$. If $y_2 = 0$, then $|x_2| = 1$ and $|x_3 + dy_3| = 1$. We conclude that $\Big((\pm 1, 0),\ (\pm 1, 0),\ \pm(t \operatorname{sign}(d), (1 - t)\operatorname{sign}(c))\Big) \in \operatorname{Norm}(T)$ for all $0 \le t \le 1$. As a result we have

$$\begin{aligned}\operatorname{Norm}(T) = \Big\{ &\Big((0, \pm 1),\ (0, \pm 1),\ (0, \pm 1)\Big),\ \Big((\pm 1, 0),\ (\pm 1, 0),\ \pm(t, (1 - t)\operatorname{sign}(d))\Big) \\ &\Big((\pm 1, 0),\ \pm(t, (1 - t)\operatorname{sign}(d)),\ (\pm 1, 0)\Big), \\ &\Big(\pm(t, (1 - t)\operatorname{sign}(d)),\ (\pm 1, 0),\ (\pm 1, 0)\Big) : 0 \le t \le 1\Big\}.\end{aligned}$$

Case 10. Suppose that $a = |c| = |d| = 1$ and $b < 1$.

In that case we have $0 = y_1y_2y_3$. Suppose that $y_1 = 0$. Then $|x_1| = 1$ and

$$1 = |x_2x_3 + cy_2y_3 + d(x_3y_1 + x_2y_3)|.$$

If $c = 1$, then $\Big((\pm 1, 0),\ \pm(t, (1 - t)\operatorname{sign}(d)),\ \pm(s, (1 - s)\operatorname{sign}(d))\Big) \in \operatorname{Norm}(T)$ for $0 \le t, s \le 1$.

Let $c=-1$. If $d=1$, by Theorem 2.4, $\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big)\in \mathrm{Norm}(T)$ for $0\le t\le 1$. If $d=-1$, then, by case 4 of Theorem 2.4, $\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,1-t)\Big)\in \mathrm{Norm}(T)$ for $0\le t\le 1$. Since T is symmetric we know that

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm 1,0),\ \pm(t,(1-t)\,\mathrm{sign}(d)),\ \pm(s,(1-s)\,\mathrm{sign}(d)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ (\pm 1,0),\ \pm(s,(1-s)\,\mathrm{sign}(d)\Big),\\ &\Big(\pm(t,(1-t)\,\mathrm{sign}(d)),\ \pm(s,(1-s)\,\mathrm{sign}(d),\ (\pm 1,0)\Big):0\le t,s\le 1\Big\}.\end{aligned}$$

Case 11. Suppose that $a=b=|c|=|d|=1$.

We first consider $c=-1$ or $d=-1$.
We claim that there is a $j_0\in\{1,2,3\}$ such that $x_{j_0}y_{j_0}=0$.

Assume the contrary. Without loss of generality we may assume that $c=-1$ because if $d=-1$, we may consider $\tilde{T}=(b,a,d,c)$. By Lemma 2.3, we may assume that $x_j>0$ and $y_j\neq 0$ for all $j=1,2,3$. Since

$$\begin{aligned}1&=\Big|T\Big((x_1,y_1),(x_2,y_2),(x_3,y_3)\Big)\Big|\\ &=\Big|x_1x_2x_3+y_1y_2y_3-(x_1y_2y_3+x_2y_1y_3+x_3y_1y_2)+d(y_1x_2x_3+y_2x_1x_3+y_3x_1x_2)\Big|\\ &=|x_1x_2x_3|+|y_1y_2y_3|+|x_1y_2y_3|+|x_2y_1y_3|+|x_3y_1y_2|+|y_1x_2x_3|+|y_2x_1x_3|+|y_3x_1x_2|,\end{aligned}$$

which shows that $y_1y_2<0, y_2y_3<0$ and $y_1y_3<0$. Then

$$0<(y_1y_2y_3)^2=(y_1y_2)(y_2y_3)(y_1y_3)<0,$$

which is a contradiction, so the claim follows.

Let $c=-1$. Since T is symmetric, we may assume that $j_0=1$. Hence, $x_1y_1=0$. If $x_1=0$, then $|y_1|=1$ and

$$1=\Big|T\Big((0,1),(x_2,y_2),(x_3,y_3)\Big)\Big|=\Big|y_2y_3-(x_2y_3+x_3y_2)+dx_2x_3\Big|.$$

If $d=1$, then, by case 4 of Theorem 2.4, $\Big((0,\pm 1),\ \pm(t,t-1),\ \pm(s,s-1)\Big)\in \mathrm{Norm}(T)$ for $0\le t,s\le 1$. Since T is symmetric,

$$\begin{aligned}\mathrm{Norm}(T)=\Big\{&\Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,1-t)\Big),\Big((\pm 1,0),\ \pm(t,1-t),\ (\pm 1,0)\Big),\\ &\Big(\pm(t,1-t),\ (\pm 1,0),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\\ &\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\Big(\pm(t,t-1),\ (0,\pm 1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ \pm(s,s-1),\ (0,\pm 1)\ \Big):0\le t,s\le 1\Big\}.\end{aligned}$$

If $d=-1$, then, by case 4 of Theorem 2.4,

$$\Big((0,\pm 1),\ (0,\pm 1),\ \pm(t,t-1)\Big),\ \Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,1-t)\Big)\in \mathrm{Norm}(T)$$

for $0 \le t \le 1$. If $y_1 = 0$, then $|x_1| = 1$ and

$$1 = \Big|T\Big((0,1),(x_2,y_2),(x_3,y_3)\Big)\Big| = \Big|x_2x_3 - y_2y_3 + d(x_3y_2 + x_2y_3)\Big|.$$

By case 4 of Theorem 2.4,

$$\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t{-}1)\mathrm{sign}(d)\Big),\ \Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,1{-}t)\mathrm{sign}(d)\Big) \in \mathrm{Norm}(T)$$

for $0 \le t \le 1$. Since T is symmetric we conclude that

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\Big((\pm 1,0),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((\pm 1,0),\ \pm(t,t-1),\ (\pm 1,0)\Big),\Big(\pm(t,t-1),\\ &(\pm 1,0),\ (\pm 1,0)\Big),\Big(\pm(t,t-1),\ (\pm 1,0),\ (0,\pm 1)\Big),\\ &\Big((\pm 1,0),\ \pm(t,t-1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ \pm(s,s-1)\Big),\Big(\pm(t,t-1),\ \pm(s,s-1),\ (0,\pm 1)\Big):\\ &0 \le t,s \le 1\Big\}.\end{aligned}$$

If $c = 1$ and $d = -1$, we may consider $\tilde{T} = (b,a,d,c)$. By the above,

$$\begin{aligned}\mathrm{Norm}(T) = \Big\{&\Big((0,\pm 1),\ (0,\pm 1),\ \pm(t,t-1)\Big),\Big((0,\pm 1),\ \pm(t,1-t),\ (0,\pm 1)\Big),\\ &\Big(\pm(t,1-t),\ (0,\pm 1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big((\pm 1,0),\ \pm(t,t-1),\ (0,\pm 1)\Big),\Big((\pm 1,0),\ (0,\pm 1),\ \pm(t,t-1)\Big),\\ &\Big(\pm(t,t-1),\ (0,\pm 1),\ (\pm 1,0)\Big),\Big((0,\pm 1),\ \pm(t,t-1),\ (\pm 1,0)\Big),\\ &\Big((0,\pm 1),\ (\pm 1,0),\ \pm(t,t-1)\Big),\Big((\pm 1,0),\ \pm(t,t-1),\ \pm(s,s-1)\Big),\\ &\Big(\pm(t,t-1),\ (\pm 1,0),\ \pm(s,s-1)\Big),\Big(\pm(t,t-1),\ \pm(s,s-1),\ (\pm 1,0)\Big):\\ &0 \le t,s \le 1\Big\}.\end{aligned}$$

Finally assume that $c = d = 1$. It is then trivial that

$$\mathrm{Norm}(T) = \Big\{\Big(\pm(t,1-t),\ \pm(s,1-s),\ \pm(w,1-w)\Big) : 0 \le t,s,w \le 1\Big\}$$

which completes the proof. □

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