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NUMERICAL RADIUS POINTS OF $\mathcal{L}(^m l_\infty^n : l_\infty^n)$

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Abstract. For $n \geq 2$ and a real Banach space E, $\mathcal{L}(^n E : E)$ denotes the space of all continuous n-linear mappings from E to itself. Let

$$\Pi(E) = \Big\{ \ [x^*, (x_1, \ldots, x_n)] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n \ \Big\}.$$

For $T \in \mathcal{L}(^n E : E)$, we define

$$\mathrm{Nrad}(T) = \Big\{ \ [x^*, (x_1, \ldots, x_n)] \in \Pi(E) : |x^*(T(x_1, \ldots, x_n))| = v(T) \ \Big\},$$

where $v(T)$ denotes the numerical radius of T. T is called *numerical radius peak mapping* if there is $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ that satisfies $\mathrm{Nrad}(T) = \Big\{ \ \pm [x^*, (x_1, \ldots, x_n)] \ \Big\}$.

In this paper we classify $\mathrm{Nrad}(T)$ for every $T \in \mathcal{L}(^2 l_\infty^2 : l_\infty^2)$ in connection with the set of the norm attaining points of T. We also characterize all numerical radius peak mappings in $\mathcal{L}(^m l_\infty^n : l_\infty^n)$ for $n, m \geq 2$, where $l_\infty^n = \mathbb{R}^n$ with the supremum norm.

1. Introduction

In 1961 Bishop and Phelps [**2**] initiated and showed that the set of norm attaining functionals on a Banach space is dense in the dual space. Shortly after, attention was paid to possible extensions of this result to more general settings, especially bounded linear operators between Banach spaces. The problem of denseness of norm attaining functions has moved to other types of mappings like multilinear forms or polynomials. The first result about norm attaining multilinear forms appeared in a joint work of Aron, Finet and Werner [**1**], where they showed that the Radon-Nikodym property is sufficient for the denseness of norm attaining multilinear forms. Choi and Kim [**3**] showed that the Radon-Nikodym property is also sufficient for the denseness of norm attaining polynomials and investigated the denseness of numerical radius attaining multilinear mappings and polynomials on a Banach space. Jiménez-Sevilla and Payá [**4**] studied the denseness of norm attaining multilinear forms and polynomials on preduals of Lorentz sequence spaces.

Let $n \in \mathbb{N}$ and $n \geq 2$. We write S_E for the unit sphere of a Banach space E. We denote by $\mathcal{L}(^n E : E)$ the Banach space of all continuous n-linear mappings from E into itself endowed with the norm $\|T\| = \sup_{(x_1, \cdots, x_n) \in S_E \times \cdots \times S_E} \|T(x_1, \cdots, x_n)\|$. $\mathcal{L}_s(^n E : E)$ denotes the closed subspace of all continuous symmetric n-linear mappings on E. We let

$$\Pi(E) = \Big\{ \ [x^*, (x_1, \ldots, x_n)] : x^*(x_j) = \|x^*\| = \|x_j\| = 1 \text{ for } j = 1, \ldots, n \ \Big\}.$$

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An element $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ is called a *numerical radius point* of $T \in \mathcal{L}(^nE : E)$ if $|x^*(T(x_1, \ldots, x_n))| = v(T)$, where the numerical radius of T is defined by

$$v(T) = \sup_{[y^*, (y_1, \ldots, y_n)] \in \Pi(E)} \Big| y^* \Big(T(y_1, \ldots, y_n) \Big) \Big|.$$

For $T \in \mathcal{L}(^nE : E)$, we define

$$\mathrm{Nrad}(T) = \Big\{ [x^*, (x_1, \ldots, x_n)] \in \Pi(E) : |x^*(T(x_1, \ldots, x_n))| = v(T) \Big\}.$$

$\mathrm{Nrad}(T)$ is called the *set of numerical radius points* of T. Notice that $[x^*, (x_1, \ldots, x_n)] \in \mathrm{Nrad}(T)$ if and only if $[-x^*, (-x_1, \ldots, -x_n)] \in \mathrm{Nrad}(T)$.

T is called *numerical radius peak mapping* if there is $[x^*, (x_1, \ldots, x_n)] \in \Pi(E)$ such that $\mathrm{Nrad}(T) = \Big\{ \pm [x^*, (x_1, \ldots, x_n)] \Big\}$.

An element $(x_1, \ldots, x_n) \in E^n$ is called a *norming point* of $L \in \mathcal{L}(^nE)$ if $\|x_1\| = \ldots = \|x_n\| = 1$ and $|L(x_1, \ldots, x_n)| = \|L\|$. We then define

$$\mathrm{Norm}(L) = \Big\{ (x_1, \ldots, x_n) \in S_E \times \ldots \times S_E : |L(x_1, \ldots, x_n)| = \|L\| \Big\}.$$

$\mathrm{Norm}(L)$ is called the *norming set* of L.

A mapping $P : E \to \mathbb{R}$ is a continuous n-homogeneous polynomial if there exists a continuous n-linear form L on the product $E \times \cdots \times E$ such that $P(x) = L(x, \ldots, x)$ for every $x \in E$. We denote by $\mathcal{P}(^nE)$ the Banach space of all continuous n-homogeneous polynomials from E into $\mathbb{R}$ endowed with the norm $\|P\| = \sup_{\|x\|=1} |P(x)|$. An element $[x^*, x] \in \pi(E)$ is called a *numerical radius point* of $P \in \mathcal{P}(^nE : E)$ if $|x^*(P(x))| = v(P)$, where the numerical radius of P is defined by

$$v(P) = \sup_{[y^*, y] \in \Pi(E)} \Big| y^*(P(y)) \Big|.$$

We define

$$\mathrm{Nrad}(P) = \Big\{ [x^*, x] \in \Pi(E) : |x^*(P(x))| = v(P) \Big\}.$$

$\mathrm{Nrad}(P)$ is called the *set of numerical radius points* of P. Notice that $[x^*, x] \in \mathrm{Nrad}(P)$ if and only if $[-x^*, -x] \in \mathrm{Nrad}(P)$.

An element $x \in E$ is called a *norming point* of $P \in \mathcal{P}(^nE)$ if $\|x\| = 1$ and $|P(x)| = \|P\|$. For $P \in \mathcal{P}(^nE)$, we define

$$\mathrm{Norm}(P) = \Big\{ x \in S_E : |P(x)| = \|P\| \Big\}.$$

$\mathrm{Norm}(P)$ is called the *norming set* of P.

Kim in [**6**] classified $\mathrm{Norm}(P)$ for every $P \in \mathcal{P}(^2l^2_\infty)$, where $l^n_\infty = \mathbb{R}^n$ with the supremum norm. Kim in [**5**] also classified $\mathrm{Norm}(T)$ for every $T \in \mathcal{L}(^2l^2_\infty)$.

If $T \in \mathcal{L}(^nE)$ or $\mathcal{L}(^nE : E)$ and $\mathrm{Norm}(T) \neq \emptyset$, T is called a *norm attaining* and if $T \in \mathcal{L}(^nE : E)$ and $\mathrm{Nrad}(T) \neq \emptyset$, T is called a *numerical radius attaining.* Similarly, if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and $\mathrm{Norm}(P) \neq \emptyset$, P is called a *norm attaining* and if $P \in \mathcal{P}(^nE)$ or $\mathcal{P}(^nE : E)$ and $\mathrm{Nrad}(P) \neq \emptyset$, P is called a *numerical radius attaining.* (See [**3**])

For more details about the theory of multilinear mappings and polynomials on a Banach space, we refer to [**7**].

In this paper we classify $\mathrm{Nrad}(T)$ for every $T \in \mathcal{L}(^2 l_\infty^2 : l_\infty^2)$ in connection with $\mathrm{Norm}(T)$. We also characterize all numerical radius peak multilinear mappings in $\mathcal{L}(^m l_\infty^n : l_\infty^n)$ for $n, m \geq 2$.

2. Results

Throughout the paper we let E be a Banach space and $n, m \in \mathbb{N}, n, m \geq 2$. We denote $l_\infty^m = \mathbb{R}^m$ with the supremum norm.

For $k = 1, \ldots, m$, we let

$$\begin{aligned}\mathcal{W}_{n,m}(k) = \Big\{ \Big(&(w_1^{(1)}, \ldots, w_{k-1}^{(1)}, 1, w_{k+1}^{(1)}, \ldots, w_{k+1}^{(1)}, \ldots, w_m^{(1)}), \ldots, \\ &(w_1^{(n)}, \ldots, w_{k-1}^{(n)}, 1, w_{k+1}^{(n)}, \ldots, w_m^{(n)}) \Big) \\ &: w_j^{(i)} = \pm 1 \text{ for } 1 \leq i \leq n, 1 \leq j \neq k \leq m \Big\}.\end{aligned}$$

Note that for $1 \leq k \leq m$, $\mathcal{W}_{n,m}(k)$ has $2^{(m-1)n}$-elements in $S_{l_\infty^m} \times \cdots \times S_{l_\infty^m}$. Let S be a non-empty subset of a real Banach space E. Let

$$\mathrm{conv}(S) = \Big\{ \sum_{j=1}^k t_j a_j : 0 \leq t_j \leq 1, \sum_{j=1}^k t_j = 1, a_j \in S \text{ for } k \in \mathbb{N} \text{ and } 1 \leq j \leq k \Big\}.$$

We call $\mathrm{conv}(S)$ the convex hull of S. Recall that the Krein-Milman Theorem says that every non-empty compact convex subset of a Hausdorff locally convex space is the closed convex hull of its set of extreme points. Hence, the unit ball of l_∞^m is the closed convex hull of the set of its extreme points.

Theorem 2.1. *Let $n, m \geq 2$ and $T \in \mathcal{L}(^n l_\infty^m)$. Then, $\|T\| = \sup_{W \in \mathcal{W}_{n,m}(k)} |T(W)|$ for $1 \leq k \leq m$.*

Proof. Write

$$\mathrm{ext}\, B_{l_\infty^m} = \{a_1, \ldots, a_{2^m}\},$$

where $|e_j^*(a_l)| = 1$ for all $1 \leq j \leq m$ and $1 \leq l \leq 2^m$. By the Krein-Milman Theorem we have

$$B_{l_\infty^m} = \overline{\mathrm{conv}}\Big(\{a_1, \ldots, a_{2^m}\}\Big).$$

Let $(x_1^{(j)}, \ldots, x_m^{(j)}) \in B_{l_\infty^m}$ $(1 \leq j \leq n)$. There exists $t_1^{(j)}, \ldots, t_{2^m}^{(j)} \in \mathbb{R}$ such that

$$|t_1^{(j)}| + \ldots + |t_{2^m}^{(j)}| \leq 1 \text{ and } (x_1^{(j)}, \ldots, x_m^{(j)}) = t_1^{(j)} a_1 + \cdots + t_{2^m}^{(j)} a_{2^m} \ (1 \leq j \leq n).$$

It follows that

$$\begin{aligned}
&\Big|T\Big((x_1^{(1)},\ldots,x_m^{(1)}),\ldots,(x_1^{(n)},\ldots,x_m^{(n)})\Big)\Big|\\
=&\Big|T\Big(t_1^{(1)}a_1+\cdots+t_{2^m}^{(1)}a_{2^m},\ldots,t_1^{(n)}a_1+\cdots+t_{2^m}^{(n)}a_{2^m}\Big)\Big|\\
\leq&\sum_{1\leq j_k\leq 2^m,1\leq k\leq n}|t_{j_1}^{(1)}|\cdots|t_{j_n}^{(n)}|\;|T(a_{j_1},\ldots,a_{j_n})|\\
=&\sum_{1\leq j_k\leq 2^m,1\leq k\leq n}|t_{j_1}^{(1)}|\cdots|t_{j_n}^{(n)}|\;\Big|T\Big(sign(e_k^*(a_{j_1}))a_{j_1},\ldots,sign(e_k^*(a_{j_n}))a_{j_n}\Big)\Big|\\
\leq&\Big(\sum_{1\leq j_1\leq 2^m}|t_{j_1}^{(1)}|\Big)\cdots\Big(\sum_{1\leq j_n\leq 2^m}|t_{j_n}^{(n)}|\Big)\sup_{W\in\mathcal{W}_{n,m}(k)}|T(W)|\\
\leq&\sup_{W\in\mathcal{W}_{n,m}(k)}|T(W)|,
\end{aligned}$$

which completes the proof. □

We can now present explicit formulae for the numerical radius $v(T)$ for every $T\in\mathcal{L}(^n l_\infty^m : l_\infty^m)$.

Theorem 2.2. *Let $T\in\mathcal{L}(^n l_\infty^m : l_\infty^m)$ with $T=(T_1,\ldots,T_m)$ for some $T_k\in\mathcal{L}(^n l_\infty^m)$ $(k=1,\ldots,m)$. Then*

(1) $v(T)=\|T\|=\max\Big\{\|T_k\|:1\leq k\leq m\Big\}$.

(2) $v(T)=\max\{I_k,J_k:1\leq k\leq m\}=\max\{I_k:1\leq k\leq m\}$, *where*

$$\begin{aligned}
I_k=\sup\Big\{\Big|&e_k^*\Big(T\Big((x_1^{(1)},\ldots,x_{k-1}^{(1)},1,x_k^{(1)},\ldots,x_m^{(1)}),\ldots,\\
&(x_1^{(n)},\ldots,x_{k-1}^{(n)},1,x_k^{(n)},\ldots,x_m^{(n)})\Big)\Big)\Big|:|x_l^{(j)}|\leq 1,1\leq j\leq n,1\leq l\neq k\leq m\Big\},
\end{aligned}$$

$$\begin{aligned}
J_k=\sup\Big\{\sum_{1\leq l\neq k\leq m}\Big|&\epsilon_l z_l T_l\Big((\epsilon_1\,\mathrm{sign}(z_1),\ldots,\epsilon_{k-1}\,\mathrm{sign}(z_{k-1}),1,\epsilon_{k+1}\,\mathrm{sign}(z_{k+1}),\\
&\ldots,\epsilon_m\,\mathrm{sign}(z_m)),\ldots,(\epsilon_1\,\mathrm{sign}(z_1),\ldots,\epsilon_{k-1}\,\mathrm{sign}(z_{k-1}),1,\epsilon_{k+1}\,\mathrm{sign}(z_{k+1}),\\
&\ldots,\epsilon_m\,\mathrm{sign}(z_m))\Big)+z_kT_k\Big((\epsilon_1\,\mathrm{sign}(z_1),\ldots,\epsilon_{k-1}\,\mathrm{sign}(z_{k-1}),1,\epsilon_{k+1}\,\mathrm{sign}(z_{k+1}),\\
&\ldots,\epsilon_m\,\mathrm{sign}(z_m)),\ldots,(\epsilon_1\,\mathrm{sign}(z_1),\ldots,\epsilon_{k-1}\,\mathrm{sign}(z_{k-1}),1,\epsilon_{k+1}\,\mathrm{sign}(z_{k+1}),\\
&\ldots,\epsilon_m\,\mathrm{sign}(z_m))\Big)\Big|:|z_1|+\cdots+|z_m|=1,z_k\geq 0,\epsilon_l=\pm 1,1\leq l\neq k\leq m\Big\}.
\end{aligned}$$

Proof. Notice that $v(T)=\max\{I_k,J_k:1\leq k\leq m\}$. By Theorem 2.1,

$$\begin{aligned}
\|T_k\|&=\sup_{W_j\in\mathcal{W}_{n,m}(k),1\leq j\leq n}|T_k(W_1,\ldots,W_n)|\\
&\leq\sup_{[e_k^*,(X_1,\ldots,X_n)]\in\Pi(l_\infty^m)}|T_k(X_1,\ldots,X_n)|=I_k\leq\|T_k\|
\end{aligned}$$

for every $1\leq k\leq m$. Hence, $I_k=\|T_k\|$ for $k=1,\ldots,m$. It follows that

$$v(T)\geq\quad\max\{I_k:1\leq k\leq m\}=\max\Big\{\|T_k\|:1\leq k\leq m\Big\}\geq\|T\|\geq v(P),$$

which concludes the proof. □

Kim in [**5**] classified $\mathrm{Norm}(L)$ for every $L \in \mathcal{L}(^2 l_\infty^2)$. We classify $\mathrm{Nrad}(T)$ for every $T \in \mathcal{L}(^2 l_\infty^2 : l_\infty^2)$ in connection with $\mathrm{Norm}(T)$.

Theorem 2.3. *Let $T \in \mathcal{L}(^2 l_\infty^2 : l_\infty^2)$ with $T = (T_1, T_2)$ for some $T_k \in \mathcal{L}(^2 l_\infty^2)$ $(k = 1, 2)$. The the following assertions hold:*

Case 1. *If $\|T_1\| > \|T_2\|$, then*

$$\mathrm{Nrad}(T) = \Big\{ \pm [e_1^*, (X, Y)] \in \Pi(l_\infty^2) : (X, Y) \in \mathrm{Norm}(T_1) \Big\}.$$

Case 2. $\|T_1\| = \|T_2\|$.

Subcase 1. *If $((1, 1), (1, 1)), ((1, -1), (1, -1)) \notin \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$, then*

$$\mathrm{Nrad}(T) = \Big\{ \pm [e_1^*, (X, Y)] \in \Pi(l_\infty^2) : (X, Y) \in \mathrm{Norm}(T_1) \Big\} \cup \Big\{ \pm [e_2^*, (X, Y)] \in \Pi(l_\infty^2) : (X, Y) \in \mathrm{Norm}(T_2) \Big\}.$$

Subcase 2. $((1, 1), (1, 1)) \notin \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$ *and* $((1, -1), (1, -1)) \in \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$. *Let*

$$\mathcal{F} = \Big\{ \pm [e_1^*, (X, Y)] \in \Pi(l_\infty^2) : (X, Y) \in \mathrm{Norm}(T_1) \Big\} \cup \Big\{ \pm [e_2^*, (X, Y)] \in \Pi(l_\infty^2) : (X, Y) \in \mathrm{Norm}(T_2) \Big\}.$$

If $T_1((1, -1), (1, -1)) \cdot T_2((1, -1), (1, -1)) \geq 0$, *then* $\mathrm{Nrad}(T) = \mathcal{F}$.
If $T_1((1, -1), (1, -1)) \cdot T_2((1, -1), (1, -1)) < 0$, *then*

$$\mathrm{Nrad}(T) = \mathcal{F} \cup \Big\{ \pm [z e_1^* + (z - 1) e_2^*, ((1, -1), (1, -1))] : 0 < z < 1 \Big\}.$$

Subcase 3. $((1, -1), (1, -1)) \notin \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$ *and* $((1, 1), (1, 1)) \in \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$.

If $T_1((1, 1), (1, 1)) \cdot T_2((1, 1), (1, 1)) < 0$, *then* $\mathrm{Nrad}(T) = \mathcal{F}$.
If $T_1((1, 1), (1, 1)) \cdot T_2((1, 1), (1, 1)) \geq 0$, *then*

$$\mathrm{Nrad}(T) = \mathcal{F} \cup \Big\{ \pm [z e_1^* + (1 - z) e_2^*, ((1, 1), (1, 1))] : 0 < z < 1 \Big\}.$$

Subcase 4. $((1, -1), (1, -1)), ((1, 1), (1, 1)) \in \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$.

If $T_1((1, -1), (1, -1)) \cdot T_2((1, -1), (1, -1)) \geq 0$ *and* $T_1((1, 1), (1, 1)) \cdot T_2((1, 1), (1, 1)) < 0$, *then* $\mathrm{Nrad}(T) = \mathcal{F}$.

If $T_1((1, -1), (1, -1)) \cdot T_2((1, -1), (1, -1)) \geq 0$ *and* $T_1((1, 1), (1, 1)) \cdot T_2((1, 1), (1, 1)) \geq 0$, *then*

$$\mathrm{Nrad}(T) = \mathcal{F} \cup \Big\{ \pm [z e_1^* + (1 - z) e_2^*, ((1, 1), (1, 1))] : 0 < z < 1 \Big\}.$$

If $T_1((1, -1), (1, -1)) \cdot T_2((1, -1), (1, -1)) < 0$ *and* $T_1((1, 1), (1, 1)) \cdot T_2((1, 1), (1, 1)) < 0$, *then*

$$\mathrm{Nrad}(T) = \mathcal{F} \cup \Big\{ \pm [z e_1^* + (z - 1) e_2^*, ((1, -1), (1, -1))] : 0 < z < 1 \Big\}.$$

If $T_1((1,-1),(1,-1))\cdot T_2((1,-1),(1,-1)) < 0$ *and* $T_1((1,1),(1,1))\cdot T_2((1,1),(1,1)) \geq 0$*, then*

$$\mathrm{Nrad}(T) = \mathcal{F} \cup \Big\{ \pm [ze_1^* + (z-1)e_2^*, ((1,-1),(1,-1))], \\ \pm [ze_1^* + (1-z)e_2^*, ((1,1),(1,1))] : 0 < z < 1 \Big\}.$$

Case 3. *If* $\|T_2\| > \|T_1\|$*, then*

$$\mathrm{Nrad}(T) = \Big\{ \pm [e_2^*, (X,Y)] \in \Pi(l_\infty^2) : (X,Y) \in \mathrm{Norm}(T_2) \Big\}.$$

Proof. Case 1. Suppose that $\|T_1\| > \|T_2\|$. We claim the following.

Claim. $\mathrm{Nrad}(T) = \Big\{ \pm [e_1^*, (X,Y)] \in \Pi(l_\infty^2) : (X,Y) \in \mathrm{Norm}(T_1) \Big\}$.

Notice that $[e_1^*, (X,Y)] \in \mathrm{Nrad}(T)$ for every $(X,Y) \in \mathrm{Norm}(T_1)$. Indeed, by Theorem 2.2,

$$|e_1^*(T(X,Y))| = |T_1(X,Y)| = \|T_1\| = \|T\| = v(T).$$

Hence we have

$$\Big\{ \pm [e_1^*, (X,Y)] \in \Pi(l_\infty^2) : (X,Y) \in \mathrm{Norm}(T_1) \Big\} \subseteq \mathrm{Nrad}(T).$$

Let $[z^*, (X,Y)] \in \mathrm{Nrad}(T)$. Write $z^* = z_1e_1^* + z_2e_2^*$ for some $(z_1,z_2) \in S_{l_1^2}$. We will show that $z^* = \pm e_1^*$ and $(X,Y) \in \mathrm{Norm}(T_1)$. We claim that $z_2 = 0$. Assume that $z_2 \neq 0$. By Theorem 2.2, it follows that

$$\begin{aligned} \|T_1\| = \|T\| = v(T) = |z^*(T(X,Y))| &\leq |z_1|\ |T_1(X,Y)| + |z_2|\ |T_2(X,Y)| \\ &\leq |z_1|\ \|T_1\| + |z_2|\ \|T_2\| < |z_1|\ \|T_1\| + |z_2|\ \|T_1\| = \|T_1\|, \end{aligned}$$

which is a contradiction. Hence, $z^* = \pm e_1^*$. Without loss of generality we may assume that $z^* = e_1^*$. Notice that $(X,Y) \in \mathrm{Norm}(T_1)$. Indeed, by Theorem 2.2,

$$\|T_1\| = \|T\| = v(T) = |e_1^*(T(X,Y))| = |T_1(X,Y)|.$$

Therefore $[z^*, (X,Y)] = [e_1^*, (X,Y)]$ for some $(X,Y) \in \mathrm{Norm}(T_1)$. As a result

$$\mathrm{Nrad}(T) \subseteq \Big\{ \pm [e_1^*, (X,Y)] \in \Pi(l_\infty^2) : (X,Y) \in \mathrm{Norm}(T_1) \Big\}.$$

Case 2. Suppose that $\|T_1\| = \|T_2\|$.

Subcase 1. Assuming $((1,1),(1,1)), ((1,-1),(1,-1)) \notin \mathrm{Norm}(T_1) \cap \mathrm{Norm}(T_2)$, we claim the following.

Claim. $\mathrm{Nrad}(T) = \mathcal{F}$.

By a similar argument in the proof of Case 1, $\mathcal{F} \subseteq \mathrm{Nrad}(T)$. Let $[z^*, (X,Y)] \in \mathrm{Nrad}(T)$. Write $z^* = z_1e_1^* + z_2e_2^*$ for some $(z_1,z_2) \in S_{l_1^2}$. We will show that $z_1z_2 = 0$. Assume that $z_1z_2 \neq 0$. By Theorem 2.2, it follows that

$$\begin{aligned} \|T_1\| = v(T) = |z^*(T(X,Y))| &= |z_1|\ |T_1(X,Y)| + |z_2|\ |T_2(X,Y)| \\ &\leq |z_1|\ \|T_1\| + |z_2|\ \|T_2\| \leq |z_1|\ \|T_1\| + |z_2|\ \|T_1\| = \|T_1\|, \end{aligned}$$

which shows that $\|T_j\| = |T_j(X,Y)|$ $(j = 1,2)$. Hence, $(X,Y) \in \text{Norm}(T_1) \cap \text{Norm}(T_2)$. Write $X = (u_1, v_1)$ and $Y = (u_2, v_2)$ for some $(u_2, v_2) \in S_{l^2_\infty}$. Since $[z^*, (X,Y)] \in \Pi(l^2_\infty)$, for $j = 1,2$,

$$1 = z_1 u_j + z_2 v_j \leq |z_1|\ |u_j| + |z_2|\ |v_j| \leq |z_1| + |z_2| = 1,$$

which implies that $|u_j| = |v_j| = 1$ for $j = 1,2$. Without loss of generality, we may assume that $u_1 = v_1 = 1$. Since $((1,1),(1,1)),((1,-1),(1,-1)) \notin \text{Norm}(T_1) \cap \text{Norm}(T_2)$, we have either $(X = (1,1), Y = (1,-1))$ or $(X = (1,-1), Y = (1,1))$.

If $X = (1,1), Y = (1,-1)$, then

$$1 = z^*(X) = z^*(Y) = z_1 - z_2 = z_1 + z_2,$$

so we have $z_2 = 0$. This is a contradiction. If $X = (1,-1), Y = (1,1)$, then

$$1 = z^*(X) = z^*(Y) = z_1 + z_2 = z_1 - z_2,$$

and so $z_2 = 0$. This is also a contradiction. Therefore, $z_1 z_2 = 0$. If $z_1 = 0$, then $z^* = \pm e_2^*$ and $(X,Y) \in \text{Norm}(T_2)$. If $z^* = e_2^*$, then $[z^*, (X,Y)] = [e_2^*, (X,Y)] \in \mathcal{F}$. If $z^* = -e_2^*$, then $[z^*, (X,Y)] = -[e_2^*, (-X,-Y)] \in \mathcal{F}$ because $(-X,-Y) \in \text{Norm}(T_2)$. Hence, $\text{Nrad}(T) \subseteq \mathcal{F}$. We have shown the claim.

Subcase 2. Assume that $((1,1),(1,1)) \notin \text{Norm}(T_1) \cap \text{Norm}(T_2)$ and $((1,-1),(1,-1)) \in \text{Norm}(T_1) \cap \text{Norm}(T_2)$. We claim the following.

Claim. If $T_1((1,-1),(1,-1)) \cdot T_2((1,-1),(1,-1)) \geq 0$, then $\text{Nrad}(T) = \mathcal{F}$.

By a similar argument in the proof of Case 1,

$$\mathcal{F} \cup \Big\{ \pm [z e_1^* + (z-1) e_2^*, ((1,-1),(1,-1))] : 0 < z < 1 \Big\} \subseteq \text{Nrad}(T).$$

Let $[z^*, (X,Y)] \in \text{Nrad}(T)$. Write $z^* = z_1 e_1^* + z_2 e_2^*$ for some $(z_1, z_2) \in S_{l_1^2}$. Suppose that $z_1 z_2 = 0$. If $z_1 = 0$, then $z^* = \pm e_2^*$ and $(X,Y) \in \text{Norm}(T_2)$. If $z^* = e_2^*$, then $[z^*, (X,Y)] = [e_2^*, (X,Y)] \in \mathcal{F}$. If $z^* = -e_2^*$, then $[z^*, (X,Y)] = -[e_2^*, (-X,-Y)] \in \mathcal{F}$ because $(-X,-Y) \in \text{Norm}(T_2)$. Suppose that $z_1 z_2 \neq 0$. Since $1 = z^*(X) = z^*(Y)$, $z^* = \pm(z_0 e_1 + (z_0 - 1) e_2)$ for some $0 < z_0 < 1$. By the same argument as in the proof of Subcase 1 and our hypothesis, we have either $(X = (1,1), Y = (1,-1))$, $(X = (1,-1), Y = (1,1))$, or $(X = (1,-1), Y = (1,-1))$. Using the same argument as in the proof of Subcase 1 we know that $X = (1,-1)$ and $Y = (1,-1)$. We will show that $T_1((1,-1),(1,-1)) \cdot T_2((1,-1),(1,-1)) < 0$. Assume that

$$T_1((1,-1),(1,-1)) \cdot T_2((1,-1),(1,-1)) \geq 0.$$

By Theorem 2.2, it follows that

$$\begin{aligned} \|T_1\| = v(T) &= |z^*(T((1,-1),(1,-1)))| \\ &= |z_0 T_1((1,-1),(1,-1)) + (z_0 - 1) T_2((1,-1),(1,-1))| \\ &< |z_0|\ \|T_1\| + |z_0 - 1|\ \|T_2\| \leq |z_0|\ \|T_1\| + |z_0 - 1|\ \|T_1\| = \|T_1\|, \end{aligned}$$

which is impossible. Therefore we have $T_1((1,-1),(1,-1)) \cdot T_2((1,-1),(1,-1)) < 0$. Notice that

$$\begin{aligned} [z^*, (X,Y)] &= \pm[z_0 e_1^* + (z_0 - 1) e_2^*, ((1,-1),(1,-1))] \\ &\subseteq \mathcal{F} \cup \Big\{ \pm [z e_1^* + (z-1) e_2^*, ((1,-1),(1,-1))] : 0 < z < 1 \Big\}. \end{aligned}$$

Therefore, we have shown that if $T_1((1,-1),(1,-1))\cdot T_2((1,-1),(1,-1))\geq 0$, then $\mathrm{Nrad}(T)=\mathcal{F}$ and that if $T_1((1,-1),(1,-1))\cdot T_2((1,-1),(1,-1))<0$, then

$$\mathrm{Nrad}(T)=\mathcal{F}\cup\Big\{\ \pm[ze_1^*+(z-1)e_2^*,((1,-1),(1,-1))]:0<z<1\ \Big\}.$$

Subcase 3. Assume that $((1,-1),(1,-1))\notin\mathrm{Norm}(T_1)\cap\mathrm{Norm}(T_2)$ and $((1,1),(1,1))\in\mathrm{Norm}(T_1)\cap\mathrm{Norm}(T_2)$.

By analogous arguments as in the proof of Subcase 2, we conclude that if $T_1((1,1),(1,1))\cdot T_2((1,1),(1,1))<0$, then $\mathrm{Nrad}(T)=\mathcal{F}$ and that if $T_1((1,1),(1,1))\cdot T_2((1,1),(1,1))\geq 0$, then

$$\mathrm{Nrad}(T)=\mathcal{F}\cup\Big\{\ \pm[ze_1^*+(1-z)e_2^*,((1,1),(1,1))]:0<z<1\ \Big\}.$$

Subcase 4. Assume that $((1,-1),(1,-1)),((1,1),(1,1))\in\mathrm{Norm}(T_1)\cap\mathrm{Norm}(T_2)$. The proof is analogously similar to earlier subcases which we will skip here.

Case 3. Suppose that $\|T_2\|>\|T_1\|$. We claim the following.

Claim. $\mathrm{Nrad}(T)=\Big\{\ \pm[e_2^*,(X,Y)]\in\Pi(l_\infty^2):(X,Y)\in\mathrm{Norm}(T_2)\ \Big\}$.

Notice that $[e_2^*,(X,Y)]\in\mathrm{Nrad}(T)$ for every $(X,Y)\in\mathrm{Norm}(T_2)$. Indeed, by Theorem 2.2,

$$|e_1^*(T(X,Y))|=|T_1(X,Y)|=\|T_1\|=\|T\|=v(T).$$

Therefore we have

$$\Big\{\ \pm[e_2^*,(X,Y)]\in\Pi(l_\infty^2):(X,Y)\in\mathrm{Norm}(T_2)\ \Big\}\subseteq\mathrm{Nrad}(T).$$

Let $[z^*,(X,Y)]\in\mathrm{Nrad}(T)$. Write $z^*=z_1e_1^*+z_2e_2^*$ for some $(z_1,z_2)\in S_{l_1^2}$. We will show that $z^*=\pm e_2^*$ and $(X,Y)\in\mathrm{Norm}(T_2)$. We claim that $z_1=0$. Assume that $z_1\neq 0$. By Theorem 2.2, it follows that

$$\begin{aligned}\|T_2\|=\|T\|=v(T)=|z^*(T(X,Y))|&\leq|z_1|\ |T_1(X,Y)|+|z_2|\ |T_2(X,Y)|\\ &\leq|z_1|\ \|T_1\|+|z_2|\ \|T_2\|<|z_1|\ \|T_2\|+|z_2|\ \|T_2\|=\|T_2\|,\end{aligned}$$

which is a contradiction. Hence, $z^*=\pm e_2^*$. Without loss of generality we may assume that $z^*=e_2^*$. Notice that $(X,Y)\in\mathrm{Norm}(T_2)$. Indeed, by Theorem 2.2,

$$\|T_2\|=\|T\|=v(T)=|e_2^*(T(X,Y))|=|T_2(X,Y)|.$$

Hence, $[z^*,(X,Y)]=[e_2^*,(X,Y)]$ for some $(X,Y)\in\mathrm{Norm}(T_2)$. Hence,

$$\mathrm{Nrad}(T)\subseteq\Big\{\ \pm[e_2^*,(X,Y)]\in\Pi(l_\infty^2):(X,Y)\in\mathrm{Norm}(T_2)\ \Big\}$$

which the claim follows and the proof is completed. □

For $k=1,\ldots,m$, we define

$$\begin{aligned}\mathcal{V}_{n,m}(k)=\Big\{\Big(&(v_1^{(1)},\ldots,v_{k-1}^{(1)},1,v_{k+1}^{(1)},\ldots,v_m^{(1)}),\ldots,\\ &(v_1^{(n)},\ldots,v_{k-1}^{(n)},1,v_{k+1}^{(n)},\ldots,v_m^{(n)})\Big)\\ &:-1\leq v_j^{(i)}\leq 1\text{ for }1\leq i\leq n,1\leq j\neq k\leq m\Big\}.\end{aligned}$$

We characterize all numerical radius peak mappings in $\mathcal{L}(^m l_\infty^n:l_\infty^n)$ for $n,m\geq 2$.

Theorem 2.4. *Let $T \in \mathcal{L}(^n l_\infty^m : l_\infty^m)$ with $T = (T_1, \dots T_m)$ for some $T_k \in \mathcal{L}(^n l_\infty^m)$ $(k = 1, \dots, m)$. Then T is a numerical radius peak mapping if and only if there is $1 \le k_0 \le m$ such that $\|T_{k_0}\| > \|T_k\|$ for every $1 \le k \ne k_0 \le m$ and*

$$\Big|\mathcal{V}_{n,m}(k_0) \cap \mathrm{Norm}(T_{k_0})\Big| = 1.$$

Proof. ($\Rightarrow$). **Claim 1**. There is $1 \le k_0 \le m$ such that $\|T_{k_0}\| > \|T_k\|$ for every $1 \le k \ne k_0 \le m$.

Assume the contrary. Let $1 \le k_1 \ne k_2 \le m$ such that $\|T_{k_i}\| = \|T\|$ for $i = 1, 2$. By Theorem 2.1, there are $(X_1^{(i)}, \dots, X_n^{(i)}) \in \mathrm{Norm}(T) \cap \mathcal{W}_{n,m}(k_i)$ for $i = 1, 2$. Hence, $\pm\Big[e_{k_i}^*, (X_1^{(i)}, \dots, X_n^{(i)})\Big] \in \Pi(l_\infty^m)$ for $i = 1, 2$. By Theorem 2.2, it follows that for $i = 1, 2$,

$$\Big|e_{k_i}^*\Big(T(X_1^{(i)}, \dots, X_n^{(i)})\Big)\Big| = \Big|T_{k_i}\Big(X_1^{(i)}, \dots, X_n^{(i)}\Big)\Big| = \|T_{k_i}\| = \|T\| = v(T).$$

Hence, $\pm\Big[e_{k_i}^*, (X_1^{(i)}, \dots, X_n^{(i)})\Big] \in \mathrm{Nrad}(T)$ for $i = 1, 2$. Notice that

$$\Big[e_{k_1}^*, (X_1^{(1)}, \dots, X_n^{(1)})\Big] \ne \pm\Big[e_{k_2}^*, (X_1^{(2)}, \dots, X_n^{(2)})\Big].$$

This is a contradiction because T is a numerical radius peak mapping. We have shown Claim 1.

Let $\mathrm{Nrad}(T) = \Big\{ \pm\Big[z^*, (X_1, \dots, X_n)\Big] \Big\}$. Write $z^* = \sum_{1 \le j \le m} z_j e_j^* \in S_{l_1^m}$.

Claim 2. $z_j = 0$ for every $j \ne k_0$.

Assume that $z_k \ne 0$ for some $k \ne k_0$. By Claim 1 and Theorem 2.2, it follows that

$$\begin{aligned} v(T) = \Big|z^*(T(X_1, \dots, X_n))\Big| &= |z_k|\ |T_k(X_1, \dots, X_n)| + \sum_{1 \le j \ne k \le m} |z_j|\ |T_j(X_1, \dots, X_n)| \\ &\le |z_k|\ \|T_k\| + \sum_{1 \le j \ne k \le m} |z_j|\ \|T_j\| < |z_k|\ \|T\| + \sum_{1 \le j \ne k \le m} |z_j|\ \|T\| = \|T\| = v(T), \end{aligned}$$

which is a contradiction. Hence, Claim 2 holds and $z^* = \pm e_{k_0}^*$. Without loss of generality we may assume that $z^* = e_{k_0}^*$.

Claim 3. $\mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0}) = \{(X_1, \dots, X_n)\}$.

Notice that $(X_1, \dots, X_m) \in \mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0})$. Indeed, by Theorem 2.2,

$$\Big|T_{k_0}(X_1, \dots, X_n)\Big| = \Big|e_{k_0}^*(T(X_1, \dots, X_n))\Big| = v(T) = \|T\| = \|T_{k_0}\|,$$

which shows that $(X_1, \dots, X_n) \in \mathrm{Norm}(T_{k_0})$. Obviously, $(X_1, \dots, X_n) \in \mathcal{V}_{n,m}(k_0)$ because $\Big[e_{k_0}^*, (X_1, \dots, X_n)\Big] = \Big[z^*, (X_1, \dots, X_n)\Big] \in \Pi(l_\infty^m)$. Suppose that $(X_1', \dots, X_n') \in \mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0})$. We will show that $(X_1', \dots, X_n') = (X_1, \dots, X_n)$. Assume that $(X_1', \dots, X_n') \ne (X_1, \dots, X_n)$. Notice that $[e_{k_0}^*, (X_1', \dots, X_n')] \in \Pi(l_\infty^m)$. by Theorem 2.2, it follows that

$$\Big|e_{k_0}^*(T(X_1', \dots, X_n'))\Big| = \Big|T_{k_0}(X_1', \dots, X_n')\Big| = \|T_{k_0}\| = \|T\| = v(T),$$

which shows that

$$\Big[e^*_{k_0}, (X_1^{'}, \ldots, X_n^{'})\Big] \in \mathrm{Nrad}(T) = \Big\{ \pm \Big[e^*_{k_0}, (X_1, \ldots, X_n)\Big]\Big\}.$$

Hence, $(X_1^{'}, \ldots, X_n^{'}) = (X_1, \ldots, X_n)$. Therefore,

$$\mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0}) = \{(X_1, \ldots, X_n)\}.$$

($\Leftarrow$). Suppose that $\mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0}) = \{(Y_1, \ldots, Y_n)\}$.

Claim 4. $\mathrm{Nrad}(T) = \Big\{ \pm \Big[e^*_{k_0}, (Y_1, \ldots, Y_m)\Big]\Big\}$.

By a similar argument as in the proof of Claim 3, $\Big[e^*_{k_0}, (Y_1, \ldots, Y_n)\Big] \in \mathrm{Nrad}(T)$. Let $\Big[z^*, (X_1, \ldots, X_n)\Big] \in \mathrm{Nrad}(T)$ with $z^* = \sum_{1 \leq j \leq m} z_j e_j^* \in S_{l_1^m}$. By a similar argument as in the proof of Claim 2, $z^* = \pm e^*_{k_0}$. Without loss of generality we may assume that $z^* = e^*_{k_0}$. By a similar argument as in the proof of Claim 3,

$$(X_1, \ldots, X_n) \in \mathcal{V}_{n,m}(k_0) \bigcap \mathrm{Norm}(T_{k_0}) = \{(Y_1, \ldots, Y_n)\}.$$

Hence, $(X_1, \ldots, X_n) = (Y_1, \ldots, Y_n)$ and $\Big[z^*, (X_1, \ldots, X_n)\Big] = \Big[e^*_{k_0}, (Y_1, \ldots, Y_n)\Big]$. Hence, Claim 4 holds. Therefore, T is a numerical radius peak mapping. $\square$

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