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ON A THEOREM OF COOPER

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Abstract. The classical result of Cooper states that every pure strongly continuous semigroup of isometries $\{V_t\}_{t\geq 0}$ on a Hilbert space is unitarily equivalent to the shift semigroup on $L^2([0,\infty))$ with some multiplicity. The purpose of this note is to record a proof which has an algebraic flavour. The proof is based on the groupoid approach to semigroups of isometries initiated in [**8**]. We also indicate how our proof can be adapted to the Hilbert module setting and gives another proof of the main result of [**3**].

1. Introduction

Two classical results in the theory of semigroups of isometries are Wold decomposition for a single isometry and its generalisation, by Cooper ([**5**]), to the continuous case. Let $\{S_t\}_{t\geq 0}$ be the shift semigroup on $L^2([0,\infty))$ defined by

$$S_t(\xi)(s) := \begin{cases} \xi(s-t) & \text{if} \quad s \geq t, \\ 0 & \text{if} \quad s < t. \end{cases} \tag{1.1}$$

for $\xi \in L^2([0,\infty))$.

Cooper's result asserts that if $\{V_t\}_{t\geq 0}$ is a strongly continuous semigroup of isometries on a Hilbert space $\mathcal{H}$ which is pure, i.e. $V_t^* \to 0$ strongly as $t \to \infty$, then up to unitary equivalence, $\mathcal{H} = L^2([0,\infty)) \otimes \mathcal{L}$ for some Hilbert space $\mathcal{L}$ and $V_t = S_t \otimes 1$. The traditional proofs of Cooper's result available in the literature ([**5**], [**15**] and [**7**]) make essential use of unbounded operators. A proof relying only on bounded operators can be found in [**16**]. Another such proof can be found in [**3**], where Cooper's result was generalised to the Hilbert C^*-module setting. The purpose of this note is to add another such proof, based on the groupoid approach to semigroup C^*-algebras initiated by Muhly and Renault in [**8**]. Also, the proof adapts well to the Hilbert module setting and gives another proof of the main result of [**3**].

We hope to convince the reader that Cooper's result is yet another reflection of the fundamental fact regarding the representation theory of the C^*-algebra of compact operators that asserts that any non-degenerate representation of $\mathcal{K}(\mathcal{H})$ is unitarily equivalent to the 'natural standard representation' of $\mathcal{K}(\mathcal{H})$ on $\mathcal{H}$, with some multiplicity. This fact, not difficult to prove, has many powerful implications. Two well known corollaries, when viewed from the 'correct algebraic perspective', are the following.

(i) A finite group has only finitely many irreducible representations, up to unitary equivalence.

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(ii) Stone-von Neumann theorem that asserts the uniqueness of the irreducible family of two 1-parameter unitary groups that obey the Weyl commutation relation.

We wish to demonstrate that Cooper's result is a third such corollary, when the result is cast in operator algebraic terms. Moreover, we wish to convey that the algebraic reason for the validity of Cooper's result in the Hilbert module setting is the fact that $\mathcal{K}(L^2([0,\infty)))$ and $\mathbb{C}$ are Morita equivalent.

Next, we explain the strategy that we pursue. We imitate the operator algebraic proof, which is folklore, of the Wold decomposition for a single pure isometry. The proof for a single isometry proceeds as follows. Let $\mathcal{T}$ be the universal C^*-algebra generated by a single isometry v. Set $p := 1 - vv^*$. Then, the ideal generated by p in $\mathcal{T}$ is isomorphic to the algebra of compact operators $\mathcal{K} := \mathcal{K}(\ell^2(\mathbb{N}))$.

By the universal property of $\mathcal{T}$, isometries on Hilbert spaces are in bijective correspondence with unital representations of $\mathcal{T}$. Moreover, pure isometries correspond to representations π of $\mathcal{T}$ for which the restriction $\pi|_{\mathcal{K}}$ is non-degenerate. Cleary, if $\pi|_{\mathcal{K}}$ is non-degenerate, then the restriction $\pi|_{\mathcal{K}}$ determines π. Applying the representation theory of $\mathcal{K}$, it follows immediately that if V is a pure isometry on a Hilbert space $\mathcal{H}$, then $\mathcal{H} = \ell^2(\mathbb{N}) \otimes \mathcal{L}$ for some Hilbert space $\mathcal{L}$ and $V = S \otimes 1$, where S is the standard shift on $\ell^2(\mathbb{N})$.

The Wold decomposition for a generic isometry as a direct sum of a shift with multiplicity and a unitary can be proved, as indicated above, by making use of the following fundamental short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{T}) \longrightarrow 0.$$

The above proof can be imitated in the continuous case if we have the analog of the Toeplitz algebra in the continuous case. Such a universal algebra[1] that encodes semigroups of isometries, indexed by an Ore semigroup P, with commuting range projections, is provided by Theorem 7.4 of [**13**] which is the main theorem of [**13**]. Thanks to the total order structure on $[0,\infty)$, it follows that any 1-parameter semigroup of isometries has commuting range projections. Thus, we can apply Theorem 7.4 of [**13**]. We take Theorem 7.4 of [**13**] as our starting point and work exactly as in the discrete setup. To keep this paper as much self-contained as possible, we include details concerning the main points of Theorem 7.4 of [**13**] in the special case when the semigroup $P = [0,\infty)$. We have repeated some details from [**13**].

The idea of exploiting groupoids to study C^*-algebras associated to semigroups was due to Muhly and Renault ([**8**]), where the Wiener-Hopf algebra associated to a cone was shown to be isomorphic to the C^*-algebra of the Wiener-Hopf groupoid. We use the Wiener-Hopf groupoid, considered in [**8**], in the base case when the cone is 1-dimensional. It is probable that the proof presented in this paper is already known to many experts. Nevertheless, the author is unable to locate a reference and believes that the proof is worth recording. A few reasons for recording this proof are given at the end of this paper.

[1] It is well known that such a universal algebra is nothing but the Wiener-Hopf algebra. However, the proofs found in the literature, for example [**16**], always derives the universal picture of the Wiener-Hopf algebra from Cooper's result. The proof given in [**13**] does not a priori assume Cooper's result.

Since we will talk about Hilbert C^*-modules at the end of this article, our convention is that inner products on Hilbert spaces and Hilbert modules are linear in the second variable.

2. Proof of Cooper's Theorem

We make essential use of the Wiener-Hopf groupoid first considered in [**8**]. We refer the reader to Section 2 of [**8**] for the basics on groupoids and their C^*-algebras. In this paper, we make use of only one groupoid and its associated *-algebra (it is not even necessary to complete it) whose multiplication and adjoint formula we recall first.

Let $[0,\infty]$ be the one point compactification of $[0,\infty)$. Let

$$\mathcal{G} := \{(x,t) \in [0,\infty] \times \mathbb{R} : x + t \geq 0\}.$$

We use the standard convention that $\infty + t = \infty$ for $t \in \mathbb{R}$. Then, $\mathcal{G}$ is a locally compact Hausdorff groupoid with multiplication and inversion given by

$$\begin{aligned}(x,t)(y,s) &:= (x, t+s) \text{ if } x+t = y \\ (x,t)^{-1} &:= (x+t, -t).\end{aligned}$$

The groupoid $\mathcal{G}$ is called *the Wiener-Hopf groupoid* associated to the semigroup $[0,\infty)$. The unit space of $\mathcal{G}$ can be identified with $[0,\infty]$. Let r and s be the range and source maps respectively. Then, $r(x,t) = x$ and $s(x,t) = x+t$. For $x \in [0,\infty]$, observe that the fibre

$$\mathcal{G}^{(x)} := r^{-1}(x) \cong [-x,\infty) \cap \mathbb{R}.$$

For $x \in [0,\infty]$, let $\lambda^{(x)}$ be the restriction of the Lebesgue measure to the set $\mathcal{G}^{(x)} = [-x,\infty) \cap \mathbb{R}$. Then, $\lambda := \{\lambda^{(x)}\}_{x \in [0,\infty]}$ is a Haar system for $\mathcal{G}$. Denote the associated *-algebra, w.r.t λ, of $\mathcal{G}$ by $C_c(\mathcal{G})$.

To be more precise, denote the space of continuous complex valued functions on $\mathcal{G}$ with compact support by $C_c(\mathcal{G})$. Then, $C_c(\mathcal{G})$ is a *-algebra with multiplication and involution given by

$$\begin{aligned} f * g(x,t) : &= \int f(x,r) g(x+r, t-r) 1_{[-x,\infty)}(r) dr, \\ f^*(x,t) : &= \overline{f(x+t,-t)}. \end{aligned}$$

Let us fix notation that we will use throughout. For $f \in C_c(\mathbb{R})$, define $\widetilde{f} \in C([0,\infty])$ by

$$\widetilde{f}(x) := \int_{-\infty}^{x} f(t) dt.$$

For $\phi \in C([0,\infty])$ and $f \in C_c(\mathbb{R})$, let $\phi \otimes f \in C_c(\mathcal{G})$ be defined by

$$\phi \otimes f(x,t) := \phi(x) f(t).$$

Fix a strongly continuous semigroup of isometries $V = \{V_t\}_{t \geq 0}$ on a Hilbert space $\mathcal{H}$. The semigroup of isometries V will be fixed until further mention. It is not difficult to show that $[0,\infty) \ni t \to V_t^* \in B(\mathcal{H})$ is strongly continuous. For a proof, see Proposition 3.4 of [**3**]. For $t \geq 0$, let E_t be the range projection of V_t.

For $t \in \mathbb{R}$, set

$$W_t := \begin{cases} V_t & \text{if} \quad t \geq 0, \\ V_{-t}^* & \text{if} \quad t < 0. \end{cases} \tag{2.2}$$

For $t \in \mathbb{R}$, the operator W_t is a partial isometry whose range projection we denote by E_t. Note that if $s \geq t$, then $E_s \leq E_t$ (for a proof, see Example 7.7 of [**13**]). This implies in particular that $\{E_t : t \in \mathbb{R}\}$ is a commuting family of projections. Moreover the maps $\mathbb{R} \ni t \to W_t \in B(\mathcal{H})$ and $\mathbb{R} \ni t \to E_t \in B(\mathcal{H})$ are strongly continuous. For a proof of this assertion, we refer the reader to Proposition 3.4 of [**13**].

The main result of [**13**], i.e. Theorem 7.4 in the one parameter situation, asserts the following.

Theorem 2.1. *Keep the foregoing notation.*

(1) *There exists a unique unital* **-homomorphism* $\pi : C([0,\infty]) \to B(\mathcal{H})$ *such that*

$$\pi(\widetilde{f}) = \int f(t) E_t dt$$

for $f \in C_c(\mathbb{R})$.

(2) *There exists a unique* **-homomorphism* $\lambda : C_c(\mathcal{G}) \to B(\mathcal{H})$ *such that*

$$\lambda(\phi \otimes f) = \pi(\phi) \int f(t) W_{-t} dt$$

for $\phi \in C([0,\infty])$ *and* $f \in C_c(\mathbb{R})$*. Moreover, the map* λ *is continuous when* $C_c(\mathcal{G})$ *is given the inductive limit topology and* $B(\mathcal{H})$ *is given the norm topology.*

Proof of uniqueness: Since $\{\widetilde{f} : f \in C_c(\mathbb{R})\}$ generates $C([0,\infty])$, it is clear that the homomorphism π is unique. To show that λ is unique, it suffices to show that $\{\phi \otimes f : \phi \in C([0,\infty]), f \in C_c(\mathbb{R})\}$ is total in $C_c(\mathcal{G})$, when $C_c(\mathcal{G})$ is given the inductive limit topology.

Let $\mathcal{S}$ be the linear span of $\{\phi \otimes f : \phi \in C([0,\infty]), f \in C_c(\mathbb{R})\}$. Let $G \in C_c(\mathcal{G})$ be given. We claim that there exists a compact subset K of $\mathbb{R}$ such that for every $n \geq 1$, there exists $G_n \in \mathcal{S}$ such that

(1) the support of G_n is contained in $[0,\infty] \times K$, and
(2) for $(x,t) \in \mathcal{G}$, $|G_n(x,t) - G(x,t)| \leq \frac{1}{n}$.

By Tietze extension theorem, we can extend G to a continuous function $\widetilde{G}$ defined on $[0,\infty] \times \mathbb{R}$. Multiplying $\widetilde{G}$ by a compactly supported function that equals 1 on $supp(G)$, we can assume that $\widetilde{G}$ is compactly supported. Choose $M > 0$ such that $supp(\widetilde{G})$ is contained in $[0,\infty] \times [-M, M]$.

Let $n \geq 1$ be given. By the uniform continuity of $\widetilde{G}$, there exists $\delta_n > 0$ such that if $|t - s| < \delta_n$, then

$$|\widetilde{G}(x,t) - \widetilde{G}(x,s)| \leq \frac{1}{n}$$

for all $x \in [0,\infty]$.

Cover $\mathbb{R}$ by open intervals $\{I_k\}_{k=1}^{\infty}$ of length atmost δ_n such that the following holds. There exists N such that

(1) $I_1, I_2, \cdots, I_N$ cover $[-M, M]$ and $\displaystyle\bigcup_{k=1}^{N} I_k$ is contained in $(-M-1, M+1)$.

(2) For $k \geq N+1$, the interval I_k and $[-M, M]$ are disjoint.
Let $\{f_k\}_{k=1}^{\infty}$ be a partition of unity subordinate to the cover $\{I_k\}_{k=1}^{\infty}$. For each k, pick a point $t_k \in I_k$. Note that $\widetilde{G}(x, t_k) = 0$ if $k \geq N+1$.

Define $G_n : \mathcal{G} \to \mathbb{C}$ by

$$G_n(x,t) := \sum_{k=1}^{N} \widetilde{G}(x, t_k) f_k(t).$$

Then, $G_n \in \mathcal{S}$ and $supp(G_n) \subset [0, \infty] \times [-M-1, M+1]$.

Let $(x,t) \in \mathcal{G}$ be given. Calculate as follows to observe that

$$\begin{aligned}
|G(x,t) - G_n(x,t)| &= \Big|\widetilde{G}(x,t) - \sum_{k=1}^{N} \widetilde{G}(x,t_k) f_k(t)\Big| \\
&= \Big|\sum_{k=1}^{\infty} f_k(t)\widetilde{G}(x,t) - \sum_{k=1}^{\infty} \widetilde{G}(x,t_k) f_k(t)\Big| \\
&\leq \sum_{k=1}^{\infty} f_k(t)|\widetilde{G}(x,t) - \widetilde{G}(x,t_k)| \\
&\leq \frac{1}{n}\sum_{k=1}^{\infty} f_k(t) \\
&\leq \frac{1}{n}.
\end{aligned}$$

This proves the claim. Therefore, $\mathcal{S}$ is dense in $C_c(\mathcal{G})$ when $C_c(\mathcal{G})$ is given the inductive limit topology. This completes the proof of uniqueness.

Remark 2.2. *The validity of Theorem 2.1 is well known for a long time. If we assume Cooper's result, then Theorem 3.7 of* [**8**], *for the case $P = [0, \infty)$, gives the desired result. However, the proof of Theorem 2.1 given in* [**13**] *does not a priori assume Cooper's theorem.*

The homomorphisms π and λ of Theorem 2.1 depend on the isometric representation V. If we wish to stress their dependence on V, we denote π and λ by π_V and λ_V respectively.

To keep the paper self-contained (as much as possible), we include the main details of the proof of Theorem 2.1. We repeat some details from [**13**]. Let

$$\mathcal{D} := C^*\Big\{\int f(t)E_t dt : f \in C_c(\mathbb{R})\Big\} \subset B(\mathcal{H}).$$

Then, $\mathcal{D}$ is a unital commutative C^*-subalgebra of $B(\mathcal{H})$. We first determine the spectrum $\widehat{\mathcal{D}}$.

Lemma 2.3. *Let $\chi \in \widehat{\mathcal{D}}$ be given. Then, there exists a unique $x =: x_\chi \in [0, \infty]$ such that, for every $f \in C_c(\mathbb{R})$,*

$$\chi\Big(\int f(t)E_t dt\Big) = \int_{-\infty}^{x} f(t)dt.$$

Proof. Observe that the map
$$C_c(\mathbb{R}) \ni f \to \chi\Big(\int f(t)E_t dt\Big) \in \mathbb{C}$$
extends to a bounded linear functional on $L^1(\mathbb{R})$. Thus, there exists $\phi \in L^\infty(\mathbb{R})$ such that
$$\chi\Big(\int f(t)E_t dt\Big) = \int f(t)\phi(t)dt \tag{2.3}$$
for every $f \in C_c(\mathbb{R})$. Since $E_t = 1$ if $t \leq 0$, it follows that for every $f \in C_c(\mathbb{R})$ with $supp(f) \subset (-\infty, 0)$, we have
$$\int f(t)\phi(t)dt = \chi\Big(\int f(t)E_t dt\Big) = \int_{-\infty}^{0} f(t)dt.$$
Therefore, $\phi(t) = 1$ for almost all $t \in (-\infty, 0]$.

Define
$$A := \{a \in \mathbb{R} : \phi(t) = 1 \text{ for almost all } t \in (-\infty, a]\}.$$
We have shown that $0 \in A$. Set $x := \sup(A)$. Clearly, $\phi(t) = 1$ for almost all $t \in (-\infty, x]$. If $x = \infty$, then $\phi = 1$ a.e. and we are done. Suppose that $x < \infty$.

We claim that $\phi(t) = 0$ for almost all $t \in (x, \infty)$. Suppose not. Then, there exist real numbers a, b with $b > a > x$ and a function $g \in C_c(\mathbb{R})$ with $supp(g) \subset (a, b)$ and $\int_a^b g(s)\phi(s)ds = 1$. Let $f \in C_c(\mathbb{R})$ be any function such that $supp(f) \subset (-\infty, a)$. Note that $E_t E_s = E_s$ whenever $t < a < s$. Therefore,
$$\begin{aligned}\Big(\int f(t)E_t dt\Big)\Big(\int g(s)E_s ds\Big) &= \Big(\int_{-\infty}^{a} f(t)E_t dt\Big)\Big(\int_a^b g(s)E_s ds\Big) \\ &= \Big(\int_{(t,s)\in(-\infty,a)\times(a,b)} f(t)g(s)E_tE_s dtds\Big) \\ &= \Big(\int_{-\infty}^{a} f(t)dt\Big)\int_{-\infty}^{\infty} g(s)E_s ds.\end{aligned}$$
Applying χ to the above equation and appealing to the fact that χ is a character, we deduce that
$$\begin{aligned}\int_{-\infty}^{a} f(t)dt &= \Big(\int_{-\infty}^{a} f(t)dt\Big)\Big(\int_a^b g(s)\phi(s)ds\Big) \\ &= \Big(\int_{-\infty}^{a} f(t)dt\Big)\chi\Big(\int g(s)E_s ds\Big) \\ &= \chi\Big(\Big(\int_{-\infty}^{a} f(t)dt\Big)\Big(\int g(s)E_s ds\Big)\Big) \\ &= \chi\Big(\Big(\int f(t)E_t dt\Big)\Big(\int g(s)E_s ds\Big)\Big) \\ &= \chi\Big(\int f(t)E_t dt\Big)\chi\Big(\int g(s)E_s ds\Big) \\ &= \int_{-\infty}^{a} f(t)\phi(t)dt \int_a^b g(s)\phi(s)ds \\ &= \int_{-\infty}^{a} f(t)\phi(t)dt.\end{aligned}$$

Since f is arbitrary, we can conclude that $\phi(t) = 1$ for almost all $t \in (-\infty, a]$. This implies that $A \ni a > x = \sup(A)$, which is a contradiction. Hence the claim is true. Therefore, $\phi = 1_{(-\infty,x]}$. Hence the proof. $\square$

As $\left\{ \displaystyle\int f(t)E_t dt : f \in C_c(\mathbb{R}) \right\}$ generates $\mathcal{D}$, it follows that the map, which we denote by T,

$$\widehat{\mathcal{D}} \ni \chi \to x_\chi \in [0, \infty]$$

is 1-1. Clearly, the map $\chi \to x_\chi$ is continuous.

We are now in a position to define the unital homomorphism π of Theorem 2.1. For $\phi \in C([0, \infty])$, set

$$\pi(\phi) := G(\phi \circ T).$$

Here, $G : C(\widehat{\mathcal{D}}) \to \mathcal{D} \subset B(\mathcal{H})$ is the Gelfand transformation.

Let $\chi \in \widehat{D}$ and $f \in C_c(\mathbb{R})$ be given. Then, by Lemma 2.3, we have

$$\chi\Big(\int f(t)E_t dt \Big) = \int_0^{x_\chi} f(t)dt = \widetilde{f}(x_\chi) = (\widetilde{f} \circ T)(\chi).$$

The above equality implies that for $f \in C_c(\mathbb{R})$, $G(\widetilde{f} \circ T) = \int f(t)E_t dt$. Hence, for $f \in C_c(\mathbb{R})$,

$$\pi(\widetilde{f}) = \int f(t)E_t dt.$$

The proof of Statement (1) of Theorem 2.1 is now complete.

Let $B([0, \infty])$ be the algebra of complex valued bounded Borel functions on $[0, \infty]$. Denote the extension of the homomorphism $\pi : C([0, \infty]) \to B(\mathcal{H})$, obtained via the measurable functional calculus, to $B([0, \infty])$ by π itself. Note that π is the unique extension that satisfies DCT in the following sense.

Let (f_n) be a sequence in $B([0, \infty])$ and $f \in B([0, \infty])$ be given. Suppose $f_n \to f$ pointwise and there exists $M > 0$ such that $|f_n| \leq M$. Then, $\pi(f_n) \to \pi(f)$ strongly.

For $f \in C_c(\mathbb{R})$, let $R_s(f) \in C_c(\mathbb{R})$ be defined by $R_s(f)(t) := f(t + s)$. For $\phi \in B([0, \infty])$ and $s \in \mathbb{R}$, let $R_s(\phi) \in B([0, \infty])$ be defined by

$$R_s(\phi)(x) := \begin{cases} \phi(x + s) & \text{if } x + s \geq 0, \\ \\ 0 & \text{if } x + s < 0. \end{cases} \tag{2.4}$$

Proposition 2.4. *With the above notation, we have the following covariance relation. Let $s \in \mathbb{R}$ be given. For $\phi \in B([0, \infty])$,*

$$W_s^* \pi(\phi) W_s = \pi(R_s(\phi)).$$

Proof. We split into cases.

Case 1: Suppose $s \geq 0$. Observe that $\{E_t\}_{t \in \mathbb{R}}$ and $\{\pi(\phi) : \phi \in C([0, \infty])\}$ commute. Consequently, $\{E_t : t \in \mathbb{R}\}$ and $\{\pi(\phi) : \phi \in B([0, \infty])\}$ commute. Therefore, the map, denoted π_1,

$$B([0, \infty]) \ni \phi \to V_s^* \pi(\phi) V_s \in B(\mathcal{H})$$

is a unital *-homomorphism. Moreover, π_1 satisfies DCT. Also, the homomorphism $\phi \to \pi(R_s(\phi))$, denoted π_2, satisfies DCT.

Thus, it suffices to check that $V_s^*\pi(\phi)V_s = \pi(R_s(\phi))$ for $\phi \in C([0,\infty])$. However, $\{\widetilde{f} : f \in C_c(\mathbb{R})\}$ generates $C([0,\infty])$. Thus, it is enough to verify the equality

$$V_s^*\pi(\widetilde{f})V_s = \pi(R_s(\widetilde{f})).$$

Let $f \in C_c(\mathbb{R})$ be given. Note that

$$V_s^*\Big(\int f(t)E_t dt\Big)V_s = \int f(t)V_s^*E_tV_s dt = \int f(t)E_{t-s}dt = \int f(t+s)E_t dt = \pi\big(\widetilde{R_s(f)}\big).$$

A similar computation shows that $R_s(\widetilde{f}) = \widetilde{R_s(f)}$. Therefore, $V_s^*\pi(\widetilde{f})V_s = \pi(R_s(\widetilde{f}))$. The proof in Case 1 is complete.

Case 2: Suppose $s = -t < 0$. We claim that $V_tV_t^* = \pi(1_{[t,\infty]})$. Choose a sequence $f_n \in C_c(\mathbb{R})$ such that $f_n \geq 0$, $supp(f_n) \subset (t - \frac{1}{n}, t]$ and $\int f_n(r)dr = 1$. Then, $0 \leq \widetilde{f_n} \leq 1$ and $\widetilde{f_n} \to 1_{[t,\infty]}$. Thus, in the strong operator topology,

$$\int f_n(r)E_r dr = \pi(\widetilde{f_n}) \to \pi(1_{[t,\infty]}).$$

However, as the map $\mathbb{R} \ni r \to E_r \in B(\mathcal{H})$ is strongly continuous and (f_n) is a sequence of 'bump functions' supported around t, it is clear that $\int f_n(r)E_r dr \to E_t$. This proves the claim.

Let $\phi \in B([0,\infty])$ be given. Calculate as follows, by using Case 1, to observe that

$$\begin{aligned}
V_t\pi(\phi)V_t^* &= V_tV_t^*\pi(R_{-t}(\phi))V_tV_t^* \\
&= E_t\pi(R_{-t}(\phi))E_t \\
&= \pi(1_{[t,\infty]})\pi(R_{-t}(\phi))\pi(1_{[t,\infty]}) \\
&= \pi(1_{[t,\infty]}R_{-t}(\phi)) \\
&= \pi(R_{-t}(\phi)).
\end{aligned}$$

The proof is complete. $\square$

We are all set to define the homomorphism $\lambda : C_c(\mathcal{G}) \to B(\mathcal{H})$ of Theorem 2.1. Let $\phi \in C_c(\mathcal{G})$ be given. For $t \in \mathbb{R}$, let $\phi_t \in B([0,\infty])$ be defined by

$$\phi_t(x) := \begin{cases} \phi((x,t)) & \text{if } x+t \geq 0, \\ \\ 0 & \text{if } x+t < 0. \end{cases}$$

It is not difficult to check that the map $\mathbb{R} \ni t \to \pi(\phi_t)W_t^* \in B(\mathcal{H})$ is compactly supported and is measurable. This is true for linear combinations of functions of the form $\phi \otimes f$ where $\phi \in C([0,\infty])$ and $f \in C_c(\mathbb{R})$ and when $C_c(\mathcal{G})$ is given the inductive limit topology, the set $\{\phi \otimes f : \phi \in C([0,\infty]), f \in C_c(\mathbb{R})\}$ is total in $C_c(\mathcal{G})$. Define

$$\lambda(\phi) := \int \pi(\phi_t)W_t^* dt.$$

It is routine to check using Proposition 2.4 that λ is the desired homomorphism. We refer the reader to Theorem 6.7 of [**13**] where the verification is carried out in complete detail. This completes the proof of Theorem 2.1.

Let us fix more notation. Let $\mathcal{U}$ be the open subgroupoid of $\mathcal{G}$ defined by

$$\mathcal{U} := \{(x,t) \in \mathcal{G} : x \in [0,\infty)\}.$$

Note that $C_c(\mathcal{U})$ is a *-ideal of $C_c(\mathcal{G})$. For $f, g \in C_c([0,\infty))$, define $\widetilde{\theta}_{f,g} \in C_c(\mathcal{U})$ by

$$\widetilde{\theta}_{f,g}(x,t) := \overline{f(x)}g(x+t).$$

Then, $\{\widetilde{\theta}_{f,g} : f, g \in C_c([0,\infty))\}$ forms a set of 'matrix units', i.e.

(i) the map $C_c([0,\infty)) \times C_c([0,\infty)) \ni (f,g) \to \widetilde{\theta}_{f,g} \in C_c(\mathcal{U})$ is linear in the second variable and conjugate linear in the first variable, and

(ii) for $f_1, f_2, g_1, g_2 \in C_c([0,\infty))$,

$$\widetilde{\theta}_{f_1,g_1} * \widetilde{\theta}_{f_2,g_2} = \langle f_2 | g_1 \rangle_{L^2([0,\infty))} \widetilde{\theta}_{f_1,g_2}.$$

(iii) for $f, g \in C_c([0,\infty))$, $\widetilde{\theta}^*_{f,g} = \widetilde{\theta}_{g,f}$.

For $f, g \in C_c([0,\infty))$, let $\theta_{f,g}$ be the rank one operator on $L^2([0,\infty))$ defined by

$$\theta_{f,g}(\xi) = f\langle g|\xi\rangle.$$

Proposition 2.5. *Suppose that $V = \{V_t\}_{t\geq 0}$ is pure, i.e. $V_t^* \to 0$ strongly as $t \to \infty$. Then, the homomorphism λ, of Theorem 2.1, restricted to $C_c(\mathcal{U})$ is non-degenerate.*

Proof. Note that $V_s^* \to 0$ strongly as $s \to \infty$ is equivalent to the assertion that $E_s^\perp \to 1$ as $s \to \infty$.

Choose a sequence $f_n \in C_c(\mathbb{R})$ such that $f_n \geq 0$, $\int f_n(t)dt = 1$ and *supp*(f_n) is contained in $(n, n+1)$. Set $\phi_n = 1 - \widetilde{f}_n$. Then, $\phi_n \in C_c([0,\infty))$. Also, $0 \leq \phi_n \leq 1$. Therefore, $\pi(\phi_n)$ has norm atmost 1. Let (g_n) be a sequence in $C_c(\mathbb{R})$ such that $g_n \geq 0$, $\int g_n(t)dt = 1$ and *supp*$(g_n) \subset (-\frac{1}{n}, \frac{1}{n})$. Note that $\phi_n \otimes g_n \in C_c(\mathcal{U})$. Observe that

$$||\lambda(\phi_n \otimes g_n)|| \leq ||\pi(\phi_n)|| \Big|\Big| \int g_n(t) W_t^* dt \Big|\Big| \leq ||\pi(\phi_n)|| \int g_n(t)dt \leq 1.$$

We complete the proof by showing that $\lambda(\phi_n \otimes g_n) = \pi(\phi_n)W_n \to 1$ strongly. Here, $W_n := \int g_n(t)W_t^* dt$. Using the fact that the map $\mathbb{R} \ni t \to W_t \in B(\mathcal{H})$ is strongly continuous, it is easy to verify that $W_n \to 1$ strongly. Thus, it suffices to show that $\pi(\phi_n) \to 1$ strongly. Since $\pi(\phi_n)$ has norm atmost one and the set $\{E_s^\perp \xi : s > 0, \xi \in \mathcal{H}\}$ is total in $\mathcal{H}$, it suffices to verify that for each $s > 0$ and $\xi \in \mathcal{H}$, $\pi(\phi_n)E_s^\perp \xi \to E_s^\perp \xi$.

Let $s > 0$ and $\xi \in \mathcal{H}$ be given. Observe that, for large n, $\pi(\phi_n)E_s^\perp = E_s^\perp$, equivalently $\pi(\widetilde{f_n})E_s^\perp = 0$. This is because, $E_tE_s^\perp = 0$ if $t > s$. Therefore, if $n > s$, then

$$\pi(\widetilde{f}_n)E_s^\perp = \int_n^{n+1} f_n(t)E_tE_s^\perp dt = 0.$$

Therefore, $\pi(\phi_n)E_s^\perp \xi \to E_s^\perp \xi$ for every $s > 0$ and $\xi \in \mathcal{H}$. Hence the proof. $\square$

Theorem 2.6 (Cooper)**.** *Let $V := \{V_t\}_{t\geq 0}$ be a strongly continuous semigroup of isometries on a Hilbert space $\mathcal{H}$. Suppose that V is pure, i.e. $V_t^* \to 0$ strongly as $t \to \infty$. Then, up to a unitary equivalence, $\mathcal{H} = L^2([0,\infty)) \otimes \mathcal{L}$ for some Hibert space $\mathcal{L}$ and for $t \geq 0$, $V_t = S_t \otimes 1$.*

Proof. Denote the homomorphism λ of Theorem 2.1 associated to V by λ_V. This is to stress the dependence of λ on V. For $f,g \in C_c([0,\infty))$, let $\overline{\theta}_{f,g} := \lambda_V(\widetilde{\theta}_{f,g})$. Observe that $\{\overline{\theta}_{f,g} : f,g \in C_c([0,\infty))\}$ form a system of 'matrix units' in $B(\mathcal{H})$. Moreover, $C_c([0,\infty))$ is a dense subspace of $L^2([0,\infty))$. Since $\mathcal{K} := \mathcal{K}(L^2([0,\infty)))$ is the universal C^*-algebra generated by such a system of matrix units, it follows that there exists a unique *-homomorphism $\overline{\lambda}_V : \mathcal{K}(L^2([0,\infty))) \to B(\mathcal{H})$ such that

$$\overline{\lambda}_V(\theta_{f,g}) = \lambda_V(\widetilde{\theta}_{f,g})$$

for $f,g \in C_c([0,\infty))$.

Since, $\{\widetilde{\theta}_{f,g} : f,g \in C_c([0,\infty))\}$ is total in $C_c(\mathcal{U})$ and λ_V is continuous, when $C_c(\mathcal{U})$ is given the inductive limit topology, it follows that $\lambda_V(C_c(\mathcal{U})) \subset \overline{\lambda}_V(\mathcal{K})$. This, together with Proposition 2.5, imply that the representation $\overline{\lambda}_V$ is non-degenerate. It is well known that any non-degenerate representation of $\mathcal{K}$ is a multiple of the representation of $\mathcal{K}$ on $L^2([0,\infty))$ given by the inclusion $\mathcal{K} \subset B(L^2([0,\infty)))$. Thus, we can assume that, up to a unitary equivalence, $\mathcal{H} = L^2([0,\infty)) \otimes \mathcal{L}$ for some Hilbert space $\mathcal{L}$ and we can assume that $\overline{\lambda}_V$ is of the form

$$\overline{\lambda}_V(T) = T \otimes 1$$

for $T \in \mathcal{K} = \mathcal{K}(L^2([0,\infty)))$. Therefore, for $f,g \in C_c([0,\infty))$,

$$\lambda_V(\widetilde{\theta}_{f,g}) = \theta_{f,g} \otimes 1. \tag{2.5}$$

Let $S := \{S_t\}_{t \geq 0}$ be the usual shift semigroup on $L^2([0,\infty))$. Set $\widetilde{S}_t := S_t \otimes 1$. Then, $\widetilde{S} := \{\widetilde{S}_t\}_{t\geq 0}$ is a pure strongly continuous semigroup of isometries on the Hilbert space $\mathcal{H} = L^2([0,\infty)) \otimes \mathcal{L}$.

By a routine direct computation, which we omit, we can deduce that for every $f,g \in C_c([0,\infty))$,

$$\lambda_{\widetilde{S}}(\widetilde{\theta}_{f,g}) = \theta_{f,g} \otimes 1. \tag{2.6}$$

From Equation 2.5 and Equation 2.6 and the fact that $\{\widetilde{\theta}_{f,g} : f,g \in C_c([0,\infty))\}$ is total in $C_c(\mathcal{U})$, we see that λ_V and $\lambda_{\widetilde{S}}$ agree on $C_c(\mathcal{U})$. However, $C_c(\mathcal{U})$ is an ideal in $C_c(\mathcal{G})$ and it acts non-degenerately on $\mathcal{H}$ through $\lambda_V = \lambda_{\widetilde{S}}$. Therefore, $\lambda_V = \lambda_{\widetilde{S}}$ on $C_c(\mathcal{G})$.

Consequently, for every $f \in C_c((0,\infty))$,

$$\int f(t)V_t^* dt = \lambda_V(1 \otimes f) = \lambda_{\widetilde{S}}(1 \otimes f) = \int f(t)\widetilde{S}_t^* dt.$$

As the above equality holds for every $f \in C_c((0,\infty))$ and the maps $t \to V_t^*$ and $t \to \widetilde{S}_t^*$ are strongly continuous, we deduce that for every $t > 0$, $V_t = S_t \otimes 1$. Hence the proof. □

We end this article by showing how our proof of Theorem 2.6 can be adapted to the Hilbert module setting. We start by collecting some definitions concerning vector valued integrals.

(a) Let E be a Banach space and let $f : \mathbb{R} \to E$ be a compactly supported continuous function. Then, there exists a unique vector denoted $\int f(t)dt \in E$ such that for $\rho \in E^*$,

$$\rho\Big(\int f(t)dt\Big) = \int \rho(f(t))dt.$$

We can prove the existence of $\int f(t)dt$ by making use of the Krein-Smulian theorem ([**4**]).

(b) Let B be a C^*-algebra and let E be a Hilbert B-module. Denote the algebra of adjointable operators on E by $\mathcal{L}_B(E)$. For $x \in E$, define a seminorm $|| \, ||_x$ on $\mathcal{L}_B(E)$ by setting

$$||T||_x := ||Tx|| + ||T^*x||.$$

The topology on $\mathcal{L}_B(E)$ generated by the family of seminorms $\{|| \, ||_x : x \in E\}$ is called the *-strong topology.

Let $f : \mathbb{R} \to \mathcal{L}_B(E)$ be continuous, when $\mathcal{L}_B(E)$ is given the *-strong topology. This means that for every $x \in E$, the maps

$$\mathbb{R} \ni t \to f(t)x \in E \ ; \text{ and } \mathbb{R} \ni t \to f(t)^*x \in E$$

are continuous. Assume that f is compactly supported. Define an operator, denoted $\int f(t)dt$, on E by setting

$$\Big(\int f(t)dt\Big)x := \int f(t)xdt.$$

Then, $\int f(t)dt$ is adjointable and it is the unique adjointable operator on E such that for $x, y \in E$,

$$\Big\langle \Big(\int f(t)dt\Big)x|y\Big\rangle = \int \langle f(t)x|y\rangle dt.$$

Let B be a C^*-algebra which will be fixed for the rest of this section. Let E be a Hilbert B-module. Suppose $V := \{V_t\}_{t\geq 0}$ is a strongly continuous semigroup of adjointable isometries on E. Then, the map $\mathbb{R} \ni t \to V_t \in \mathcal{L}_B(E)$ is *-strong continuous. Define, for $t \in \mathbb{R}$, the partial isometries W_t and the range projections E_t as in the Hilbert space setting.

Assume for the moment that we have proved Theorem 2.1 in the setting of Hilbert modules, i.e. with E in place of $\mathcal{H}$ and with $B(\mathcal{H})$ replaced with $\mathcal{L}_B(E)$. The proof of Proposition 2.5 works well and establishes the analogous result with $\mathcal{H}$ replaced with E. Similarly, the proof of Theorem 2.6 adapts well to establish the following analogous result proved in [**3**].

Theorem 2.7 ([**3**]). *Let $V := \{V_t\}_{t\geq 0}$ be a strongly continuous semigroup of adjointable isometries on a Hilbert B-module E. Suppose that V is pure, i.e. $V_t^* \to 0$ strongly as $t \to \infty$. Then, up to a unitary equivalence, $E = L^2([0,\infty)) \otimes F$ for some Hibert B-module F and for $t \geq 0$, $V_t = S_t \otimes 1$.*

Remark 2.8. *In the statement of Theorem 2.7, the tensor product $L^2([0,\infty)) \otimes F$ is the external tensor product. While adapting the proof of Theorem 2.6, we need to know the non-degenerate representations of $\mathcal{K} := \mathcal{K}(L^2([0,\infty)))$ on Hilbert B-modules, which we do know, as the Hilbert space $L^2([0,\infty))$ is an imprimitivity module implementing the Morita equivalence between $\mathcal{K}$ and $\mathbb{C}$. Therefore, thanks to Rieffel (*[**10**]*), if $\pi : \mathcal{K} \to \mathcal{L}_B(E)$ is non-degenerate, then up to unitary equivalence, $E = L^2([0,\infty)) \otimes F$ for some Hilbert module F and π is of the form*

$$\pi(T) = T \otimes 1$$

for $T \in \mathcal{K}$.

Next, we explain why Theorem 2.1 is valid in the Hilbert module setting. For simplicity, we assume that the C^*-algebra B is separable. Suppose E is a Hilbert B-module. Suppose $V := \{V_t\}_{t\geq 0}$ is a strongly continuous semigroup of isometries in $\mathcal{L}_B(E)$. Define the partial isometries $\{W_t\}_{t\in\mathbb{R}}$ as in the Hilbert space setting, i.e.

$$W_t := \begin{cases} V_t & \text{if} \quad t \geq 0, \\ V_{-t}^* & \text{if} \quad t < 0. \end{cases} \tag{2.7}$$

For $t \in \mathbb{R}$, we denote the range projection of W_t by E_t.

Theorem 2.9. *Keep the foregoing notation.*

(1) *There exists a unique unital* **-homomorphism* $\pi : C([0,\infty]) \to \mathcal{L}_B(E)$ *such that*
$$\pi(\widetilde{f}) = \int f(t)E_t dt$$
for $f \in C_c(\mathbb{R})$.

(2) *There exists a unique* **-homomorphism* $\lambda : C_c(\mathcal{G}) \to \mathcal{L}_B(E)$ *such that*
$$\lambda(\phi \otimes f) = \pi(\phi) \int f(t)W_{-t} dt$$
for $\phi \in C([0,\infty])$ *and* $f \in C_c(\mathbb{R})$. *Moreover, the map* λ *is continuous when* $C_c(\mathcal{G})$ *is given the inductive limit topology and* $\mathcal{L}_B(E)$ *is given the norm topology.*

Proof. Fix a faithful state ω on B. Define a $\mathbb{C}$-valued inner product on E by setting
$$\langle x|y\rangle_\omega := \omega(\langle x|y\rangle).$$
The completion of E w.r.t. the inner product $\langle\ |\ \rangle_\omega$ is denoted by E_ω. An element $x \in E$ when viewed in E_ω will be denoted $\overline{x}$. It is not difficult to show that for $T \in \mathcal{L}_B(E)$, there exists a unique bounded operator $\overline{T}$ on E_ω such that
$$\overline{T}(\overline{x}) = \overline{Tx}$$
for $x \in E$. Moreover, the map $\mathcal{L}_B(E) \ni T \to \overline{T}$, denoted ρ, is a faithful unital representation.

Note that $\overline{V} := \{\overline{V}_t\}_{t\geq 0}$ is a strongly continuous semigroup of isometries on the Hilbert space E_ω. Let $\overline{\pi}$ and $\overline{\lambda}$ be the homomorphisms of Theorem 2.1 associated to $\overline{V} := \{\overline{V}_t\}_{t\geq 0}$.

Observe that for $f \in C_c(\mathbb{R})$,
$$\rho\Big(\int f(t)E_t dt\Big) = \int f(t)\overline{E}_t dt = \overline{\pi}(\widetilde{f}).$$
Since $\{\widetilde{f} : f \in C_c(\mathbb{R})\}$ generates $C([0,\infty])$, it follows that the range of $\overline{\pi}$ is contained in $\rho\big(\mathcal{L}_B(E)\big)$. Since ρ is faithful, there exists a unique unital *-homomorphism $\pi : C([0,\infty]) \to \mathcal{L}_B(E)$ such that $\rho \circ \pi = \overline{\pi}$.

Similarly, for $f \in C_c(\mathbb{R})$,
$$\rho\Big(\int f(t)W_{-t} dt\Big) = \int f(t)\overline{W}_{-t} dt.$$
Therefore, for $\phi \in C([0,\infty])$ and $f \in C_c(\mathbb{R})$, $\overline{\lambda}(\phi\otimes f) \in \rho\big(\mathcal{L}_B(E)\big)$. As the collection $\{\phi\otimes f : \phi \in C([0,\infty]), f \in C_c(\mathbb{R})\}$ is total in $C_c(\mathcal{G})$ and $\overline{\lambda}$ is continuous when $C_c(\mathcal{G})$

is given the inductive limit topology and when $B(E_\omega)$ is given the norm topology, it follows that the range of $\overline{\lambda}$ is contained in $\rho\big(\mathcal{L}_B(E)\big)$. Consequently, there exists a unique *-homomorphism $\lambda : C_c(\mathcal{G}) \to \mathcal{L}_B(E)$ such that $\rho \circ \lambda = \overline{\lambda}$.

It is clear that π and λ are the desired homomorphisms which completes the proof. □

3. Concluding Remarks

We end this article with a few remarks especially about the history of the groupoid approach to semigroups of isometries.

(1) Analysing semigroups of isometries from a groupoid perspective was initiated in the seminal paper [**8**], where Muhly and Renault, with great success, demonstrated the use of groupoids in understanding the Wiener-Hopf algebra associated to a closed convex cone P in $\mathbb{R}^d$ for $d \geq 1$. The groupoid that we have used in this paper is the groupoid in [**8**] used for the base case $P = [0, \infty)$.
In the 1-parameter situation, the Wiener-Hopf algebra $\mathcal{W}([0, \infty))$ is the C^*-algebra generated by the Wiener-Hopf operators $\{W_f : f \in C_c(\mathbb{R})\}$. The Wiener-Hopf operator with symbol f, where $f \in C_c(\mathbb{R})$, is the operator on $L^2([0, \infty))$ defined by the equation
$$W_f\xi(s) := \int_0^\infty f(s-t)\xi(t)dt.$$
The main result of [**8**], in the base case $P = [0, \infty)$, is that the Wiener-Hopf algebra $\mathcal{W}([0, \infty)) = C^*(\{W_f : f \in C_c(\mathbb{R})\}) \cong C^*(\mathcal{G})$, where $\mathcal{G}$ is the Wiener-Hopf groupoid associated to the semigroup $[0, \infty)$.
Note that, in general, $W_f W_g \neq W_{f*g}$, where $f * g$ is the usual convolution. However, the remarkable insight due to Muhly and Renault is that the formula
$$W_f W_g = W_{f*g}$$
is valid if we interpret the symbols f and g not as functions on $\mathbb{R}$, but as functions on the Wiener-Hopf groupoid $\mathcal{G}$. All the credit for the approach presented in this paper must therefore go to Muhly and Renault for this great insight.
As already mentioned in the introduction, it is probable that the approach presented in this article is already known to many experts. Neverthless, the author could not find it recorded in the literature and the author believes it is worth recording not only for its own intrinsic interest, but also for reasons that are mentionned below.

(2) The subtleties involving the Wiener-Hopf groupoid associated to a cone in the higher dimensional case, especially the subtle points around the Haar system of the Wiener-Hopf groupoid, was noted and clarified by Nica in [**9**]. The groupoid approach to Wiener-Hopf operators was later generalised to Lie semigroups by Hilgert and Neeb in [**6**].
The Wiener-Hopf groupoid encodes only the left regular representation of a semigroup. What about other isometric representations? It soon became clear that there should be an universal groupoid encoding all isometric representations of a semigroup, or at least those with commuting range projections. This was carried out in the discrete setting for the semigroup $\mathbb{N}^k$ by Salas in

[**11**] and by the author for a topological Ore semigroup with mild assumptions in [**13**]. Thus, for example, if P is a closed convex cone in $\mathbb{R}^d$, then a certain universal groupoid $\mathcal{G}_u$ was constructed in [**13**] whose C^*-algebra encodes all the isometric representations of P with commuting range projections. If $P = [0,\infty)$, then the groupoid $\mathcal{G}_u$ is the Wiener-Hopf groupoid.
This also provides a conceptual reason why it is hard to think of a 'complete classification' of semigroups of isometries if we replace $\mathbb{R}_+$ by a higher dimensional cone, even if we restrict attention to those isometric representations that have commuting range projections. For, if P is a pointed spanning cone in $\mathbb{R}^d$, where $d \geq 2$, then it is not difficult to prove that the C^*-algebra $C^*(\mathcal{G}_u)$ is not of type I. However, in the 1-dimensional case, $C^*(\mathcal{G}_u)$ is of type I. This, the author believes, is one algebraic reason why we lack 'a good classification' of semigroups of isometries in the higher dimensional case and why we have complete classification in the 1-parameter case.

(3) A third point why the author believes it is worth recording the groupoid proof of Cooper's result is the following. Cooper's result is a fundamental result that gets exploited heavily in the 1-parameter theory of E_0-semigroups. Arveson's remarkable efforts that resulted in the classification and the characterisation of 1-parameter CCR flows ([**2**]) rest deeply on Cooper's theorem which provides a 'good coordinatization' of a 1-parameter semigroup of isometries. Such a coordinatization is not available in the multiparameter case which makes the theory of multiparameter E_0-semigroups both interesting and complicated.
In the recent works on multiparameter CCR flows done, in collaboration, by the author ([**1**], [**12**],[**14**]), the universal groupoid $\mathcal{G}_u$ (or the associated transformation groupoid) was exploited heavily to show that "There are enough multiparameter CCR flows exhibiting behaviour different from (and similar to) that of the 1-parameter case". The key is that $C^*(\mathcal{G}_u)$ is not of type I, and hence, there are enough semigroups of isometries exhibiting 'wild' behaviour. Having the groupoid proof of Cooper's result recorded and available in the literature will help those working in the theory of E_0-semigroups appreciate that groupoids provide a natural framework to analyse the semigroups of isometries appearing in the theory. This is another reason why this proof is recorded.

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