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AMENDMENT TO "LINDELÖF WITH RESPECT TO AN IDEAL" [NEW ZEALAND J. MATH. 42, 115-120, 2012]

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Abstract. We give a counterexample in this amendment to show that there is an error in consideration of the statement "*if* $f : X \to Y$ *and* $\mathbf{J}$ *is an ideal on* Y*, then* $f^{-1}(\mathbf{J}) = \{f^{-1}(J) : J \in \mathbf{J}\}$ *is an ideal on* X" by Hamlett in his paper "Lindelöf with respect to an ideal" [New Zealand J. Math. 42, 115-120, 2012]. We also modify it here in a new way and henceforth put forward correctly all the results that were based on the said statement derived therein.

1. Clarification and Amendment

We use here notation and terminology from [**1**]. In [**1**], Page 117, Section 4, Line 3, the following statement has been considered by Hamlett:

"If $f : X \to Y$ and $\mathbf{J}$ is an ideal on Y, then $f^{-1}(\mathbf{J}) = \{f^{-1}(J) : J \in \mathbf{J}\}$ is an ideal on X."

In this amendment, we give a counterexample to clarify the above statement.

Example 1.1. Consider the map $f : \mathbb{Z} \to \mathbb{N} \cup \{0\}$ as $x \mapsto |x|$. Here, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all integers and the set of all positive integers, respectively, and $|\cdot|$ is the modulus function. Consider the subset O of all odd positive integers, and take $\mathbf{J} = \wp(O)$, the power set of O. Then $\mathbf{J}$ is an ideal on $\mathbb{N} \cup \{0\}$. Now, $\{1\} \in \mathbf{J}$ implies $f^{-1}(\{1\}) = \{-1, +1\} \in f^{-1}(\mathbf{J})$. Though $\{-1\} \subseteq \{-1, +1\}$, $\{-1\} \notin f^{-1}(\mathbf{J})$. Thus, $f^{-1}(\mathbf{J})$ is not an ideal on $\mathbb{Z}$.

In view of the above example, we can say that the statement "if $f : X \to Y$ and $\mathbf{J}$ is an ideal on Y, then $f^{-1}(\mathbf{J}) = \{f^{-1}(J) : J \in \mathbf{J}\}$ is an ideal on X" is not true. Here, we give a modification of this statement in the following theorem:

Theorem 1.2. *Let* $f : X \to Y$ *be a map, and* $\mathbf{J}$ *an ideal on* Y*. Define*

$$f^{\leftarrow}(\mathbf{J}) = \{A : A \subseteq f^{-1}(J) \in f^{-1}(\mathbf{J})\}.$$

Then $f^{\leftarrow}(\mathbf{J})$ *is an ideal on* X*. Moreover,* $f^{\leftarrow}(\mathbf{J})$ *contains* $f^{-1}(\mathbf{J})$*.*

Proof. (i). Since $\varnothing \in \mathbf{J}$, $\varnothing = f^{-1}(\varnothing) \in f^{-1}(\mathbf{J})$, and hence $\varnothing \in f^{\leftarrow}(\mathbf{J})$.

(ii) Let $A \subseteq B$ and $B \in f^{\leftarrow}(\mathbf{J})$. Then there exists $J \in \mathbf{J}$ such that $B \subseteq f^{-1}(J)$. Clearly, $A \subseteq f^{-1}(J)$, and hence $A \in f^{\leftarrow}(\mathbf{J})$.

(iii) Let A, $B \in f^{\leftarrow}(\mathbf{J})$. Then there exist J_1, $J_2 \in \mathbf{J}$ such that $A \subseteq f^{-1}(J_1)$ and $B \subseteq f^{-1}(J_2)$. Now, $A \cup B \subseteq f^{-1}(J_1) \cup f^{-1}(J_2) = f^{-1}(J_1 \cup J_2)$. Since $J_1 \cup J_2 \in \mathbf{J}$, $A \cup B \in f^{\leftarrow}(\mathbf{J})$.

Hence $f^{\leftarrow}(\mathbf{J})$ is an ideal on X.

From the definition of $f^{\leftarrow}(\mathbf{J})$, it is clear that $f^{-1}(\mathbf{J})$ is contained in $f^{\leftarrow}(\mathbf{J})$. □

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In his paper [**1**], Hamlett derived the results Theorem 4.2, Lemma 4.3, Theorem 4.4, Lemma 4.5 and Theorem 4.7, where f was a surjection map. It is very clear that the map considered in Example 1.1 is a surjection but $f^{-1}(\mathbf{J})$ is, still now, not an ideal on X. Thus, the statements considered in Theorem 4.2, Lemma 4.3, Theorem 4.4, Lemma 4.5 and Theorem 4.7 of [**1**] become meaningless for arbitrary ideal $\mathbf{J}$ on Y. The results will be valid whenever $f^{-1}(\mathbf{J})$ becomes an ideal on X in its first appearance.

In this amendment, we modify the results one by one with the help of $f^{\leftarrow}(\mathbf{J})$.

Theorem 1.3 (**Modification of Theorem 4.2,** [**1**])**.** *Let $f : X \to (Y, \sigma, \mathbf{J})$ be a surjection onto a $\mathbf{J}$-Lindelöf space. If $f^{-1}(\sigma)$ is the weak topology induced by f and σ, then $(X, f^{-1}(\sigma))$ is $f^{\leftarrow}(\mathbf{J})$-Lindelöf space.*

Before entering to the proof of Theorem 1.3, we give the explicit meaning of '*the weak topology $f^{-1}(\sigma)$ induced by f and σ*' here. Let (Y, σ) be a topological space, X a non-empty set, and $f : X \to (Y, \sigma)$ a map. Then $f^{-1}(\sigma) = \{f^{-1}(U) : U \in \sigma\}$ gives a topology (one can easily verify this) on X. This topology is called *the weak topology induced by f and σ*.

Proof of Theorem 1.3 : Let $\mathbf{U} = \{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ be an open cover of X, where each $V_\alpha \in \sigma$. Then $X = \bigcup_{\alpha\in\Delta} f^{-1}(V_\alpha)$. Now, f being surjective, we have $Y = f(X) = f(\bigcup_{\alpha\in\Delta} f^{-1}(V_\alpha)) = \bigcup_{\alpha\in\Delta} f(f^{-1}(V_\alpha)) = \bigcup_{\alpha\in\Delta} V_\alpha$ yielding that $\mathbf{V} = \{V_\alpha : \alpha \in \Delta\}$ is an open cover of Y. Since Y is $\mathbf{J}$-Lindelöf, so there is a countable subcollection $\mathbf{V}_0 = \{V_i : i \in \mathbb{N}\}$ of $\mathbf{V}$ and a $J \in \mathbf{J}$ such that $Y = (\bigcup_{i\in\mathbb{N}} V_i) \cup J$. Therefore, $X = f^{-1}(Y) = f^{-1}((\bigcup_{i\in\mathbb{N}} V_i) \cup J) = f^{-1}(\bigcup_{i\in\mathbb{N}} V_i) \cup f^{-1}(J) = (\bigcup_{i\in\mathbb{N}} f^{-1}(V_i)) \cup f^{-1}(J)$. Take $\mathbf{U}_0 = \{f^{-1}(V_i) : i \in \mathbb{N}\}$. Then $\mathbf{U}_0$ is a countable subcollection of $\mathbf{U}$. On the other side, $f^{-1}(J) \in f^{\leftarrow}(\mathbf{J})$. Therefore, $(X, f^{-1}(\sigma))$ is $f^{\leftarrow}(\mathbf{J})$-Lindelöf.

Lemma 1.4 (**Modification of Lemma 4.3,** [**1**])**.** *If $f : X \to (Y, \sigma, \mathbf{J})$ is a surjection and $\mathbf{J}$ is σ-codense, then $f^{\leftarrow}(\mathbf{J})$ is $f^{-1}(\sigma)$-codense.*

Proof. Assume on the contrary, that $f^{\leftarrow}(\mathbf{J})$ is not $f^{-1}(\sigma)$-codense. Then there is an $A \in (f^{\leftarrow}(\mathbf{J}) \cap f^{-1}(\sigma)) \setminus \{\varnothing\}$. Now, choose $J \in \mathbf{J} \setminus \{\varnothing\}$ and $V \in \sigma \setminus \{\varnothing\}$ such that $A \subseteq f^{-1}(J)$ and $A = f^{-1}(V)$. Now, f being surjective, we have $V = f(f^{-1}(V)) = f(A) \subseteq f(f^{-1}(J)) = J$ implying that $V \in \mathbf{J} \setminus \{\varnothing\}$. Therefore, $V \in (\mathbf{J} \setminus \{\varnothing\}) \cap (\sigma \setminus \{\varnothing\})$ witnesses that $\mathbf{J}$ is not σ-codense, a contradiction. □

Besides the problem of $f^{-1}(\mathbf{J})$ to be an ideal, after a deep observation to the proof of Theorem 4.4 of [**1**], we reach the decision that the conditions 'continuous' and 'openness' of f are redundant. A correction is presented in the following theorem:

Theorem 1.5 (**Modification of Theorem 4.4,** [**1**])**.** *Let $f : (X, \tau) \to (Y, \sigma, \mathbf{J})$ be a closed surjection and has compact fibers. If (Y, σ) is $\mathbf{J}$-Lindelöf, then (X, τ) is $f^{\leftarrow}(\mathbf{J})$-Lindelöf.*

Proof. To show that (X, τ) is $f^{\leftarrow}(\mathbf{J})$-Lindelöf, let $\mathbf{U} = \{U_\alpha : \alpha \in \Delta\}$ be an open cover of X. For each $y \in Y$, $\mathbf{U}$ is then also an open cover of $f^{-1}(y)$, a compact subset of X. Therefore, for each $y \in Y$, there is a finite subcollection $\mathbf{U}_y = \{U^y_{\alpha_i} : i = 1, 2, \dots, n_y\}$ of $\mathbf{U}$ such that $f^{-1}(y) \subseteq \bigcup_{i=1}^{n_y} U^y_{\alpha_i}$. Take $U_y = \bigcup_{i=1}^{n_y} U^y_{\alpha_i}$. Then U_y is open in X and $f^{-1}(y) \subseteq U_y$. Each $f(X \setminus U_y)$ is closed in Y, and note that $y \notin f(X \setminus U_y)$ (if $y \in f(X \setminus U_y)$, then there exists $x_* \in X \setminus U_y$ such that $f(x_*) = y$ implying that $x_* \in f^{-1}(y) \subseteq U_y$, a contradiction). Take $V_y = Y \setminus f(X \setminus U_y)$. Then V_y is open in Y, $y \in V_y$ and $f^{-1}(V_y) \subseteq U_y$ (since $p \notin U_y$ implies $f(p) \in f(X \setminus U_y)$ and hence $f(p) \notin V_y$ gives $p \notin f^{-1}(V_y)$). The collection $\mathbf{V} = \{V_y : y \in Y\}$ is then an open cover of Y. So there is a countable subcollection $\mathbf{V}_0 = \{V_{y_j} : j \in \mathbb{N}\}$ and a $J \in \mathbf{J}$ such that $Y = (\bigcup_{j\in\mathbb{N}} V_{y_j}) \cup J$. Now, let $x \in X$ is arbitrary. Let $f(x) = y_* \in Y$. Then $y_* \in V_{y_j} \cup J$ for some y_j. Now, $x \in f^{-1}(y_*) \subseteq f^{-1}(V_{y_j}) \cup f^{-1}(J) \subseteq U_{y_j} \cup f^{-1}(J) = (\bigcup_{i=1}^{n_{y_j}} U^{y_j}_{\alpha_i}) \cup f^{-1}(J) \subseteq (\bigcup_{j\in\mathbb{N}} \bigcup_{i=1}^{n_{y_j}} U^{y_j}_{\alpha_i}) \cup f^{-1}(J)$. Take $\mathbf{U}_0 = \{U^{y_j}_{\alpha_i} : i = 1, 2, \dots, n_{y_j}$ and $j = 1, 2, \dots\}$. Then $\mathbf{U}_0$ is a countable subcollection of $\mathbf{U}$ such that $X = (\bigcup_{j\in\mathbb{N}} \bigcup_{i=1}^{n_{y_j}} U^{y_j}_{\alpha_i}) \cup f^{-1}(J)$, where $f^{-1}(J) \in f^{\leftarrow}(\mathbf{J})$. This completes the proof. □

Lemma 1.6 (**Modification of Lemma 4.5,** [**1**])**.** *If $f : (X, \tau) \to (Y, \sigma, \mathbf{J})$ is an open surjection and $\mathbf{J}$ is σ-codense, then $f^{\leftarrow}(\mathbf{J})$ is τ-codense.*

Proof. Assume on the contrary, that $f^{\leftarrow}(\mathbf{J})$ is not τ-codense. Then there are an $A \in \tau \setminus \{\varnothing\}$ and a $J \in \mathbf{J}$ such that $A \subseteq f^{-1}(J)$. Now, $f(A) \subseteq f(f^{-1}(J)) = J \in \mathbf{J}$ gives $f(A) \in \mathbf{J}$. Also, $f(A) \in \sigma$. Therefore, $\varnothing \neq f(A) \in \mathbf{J} \cap \sigma$, and hence $\mathbf{J}$ is not τ-codense. □

Theorem 1.7 (**Modification of Theorem 4.7,** [**1**])**.** *If $(X, \tau, \mathbf{I})$ is $\mathbf{I}$-Lindelöf and (Y, σ) is compact, then $(X \times Y, \eta)$ is $p^{\leftarrow}(\mathbf{I})$-Lindelöf, where η is the usual product topology and $p : X \times Y \to X$ is the projection map onto X defined by $p(x, y) = x$.*

Proof. Follows from Theorem 1.5. □

References

[1] T. R. Hamlett, *Lindelöf with respect to an ideal*, New Zealand J. Math., **42** (2012), 115-120.

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