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EXACT VALUE OF INTEGRALS INVOLVING PRODUCT OF SINE OR COSINE FUNCTION

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Abstract. By considering the number of all choices of signs + and − such that $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 \cdots \pm \alpha_n = 0$ and the number of sign − appeared therein, this paper can give the exact value of $\int_0^{2\pi} \prod_{k=1}^{n} \sin(\alpha_k x)dx$. In addition, without using the Fourier transformation technique, we can also find the exact value of $\int_0^\infty \frac{(\cos\alpha x - \cos\beta x)^p}{x^q}dx$. These two integrals are motivated by the work of Andrican and Bragdasar in 2021, Andria and Tomescu in 2002, and Borwein and Borwein in 2001, respectively.

1. Introduction

Our work is inspired by the integrals

$$\int_0^{2\pi} \prod_{k=1}^{n} \cos(\alpha_k x)dx \tag{1.1}$$

and

$$\int_0^{\infty} \prod_{k=1}^{n} \frac{\sin(\alpha_k x)}{\alpha_k x}dx, \tag{1.2}$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ are positive numbers.

By considering the coefficient of a certain polynomial expansion [**1**] and [**2**] gave a relation between (1.1) and an integer sequence $S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ where $S(\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n)$ is the number of ways of choosing + and − signs such that $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 \cdots \pm \alpha_n = 0$. For (1.2), Borwein and Borwein [**3**] used the Fourier transform techniques to find its value. Surprisingly, the calculation involved the summation of $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ with either + or − sign.

In this paper, by manipulating the idea of [**1**] and [**2**], we can calculate the value of

$$\int_0^{2\pi} \prod_{k=1}^{n} \sin(\alpha_k x)dx. \tag{1.3}$$

However, our calculation also depends on the number of − sign appeared in $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 \pm \cdots \pm \alpha_k = 0$ where $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_k$ are positive integers. Without using the knowledge of Fourier transformation, we also find the value of an infinite integral involving the product of a difference of cosine, namely

$$\int_0^{\infty} \frac{(\cos\alpha x - \cos\beta x)^p}{x^q}dx, \tag{1.4}$$

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where α, β are real numbers and p, q are positive integers. Note that (1.4) in the case that $p = q = 2$ was appeared in [**5**].

The presentation of our manuscript is as follows. In Section 2, we give some preliminaries needed in this work. Then, (1.3) is calculated in Section 3. In Section 4, we consider the case that $p \geq q$ and q is even and give the exact value of (1.4). Finally, conclusion and discussion about some open problems are given in Section 5.

2. Preliminaries

Definition 2.1. Let $n \in \mathbb{N}$, $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n > 0$ and $(\alpha_1 : \alpha_n) = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n)$. Define the set of sum expressions as follows.

- $R(\alpha_1 : \alpha_n) := \{\sum_{k=1}^{n} \gamma_k \alpha_k : (\gamma_1, \gamma_2, \gamma_3, ..., \gamma_n) \in \{-1, 1\}^n\}$,
- $R^+(\alpha_1 : \alpha_n) := \left\{\alpha_1 + \sum_{k=2}^{n} \gamma_k \alpha_k : (\gamma_2, \gamma_3, \gamma_4, ..., \gamma_n) \in \{-1, 1\}^{n-1}\right\}$ and
- $R^-(\alpha_1 : \alpha_n) := \left\{-\alpha_1 + \sum_{k=2}^{n} \gamma_k \alpha_k : (\gamma_2, \gamma_3, \gamma_4, ..., \gamma_n) \in \{-1, 1\}^{n-1}\right\}$.

That is, R^+ consists of all the expressions in R with $\gamma_1 = 1$, while R^- consists of all the expressions in R with $\gamma_1 = -1$. Let σ and s be functions from $R(\alpha_1 : \alpha_n)$ to the set of real numbers given by

$$\sigma(r) := \text{the number of } - \text{ sign(s) appear in } r.$$

and

$$s(r) := \text{the sum of } r \text{ or the exact value of } r,$$

for all $r \in R(\alpha_1 : \alpha_n)$. For example, $R(1, 2) = \{1 + 2, 1 - 2, -1 + 2, -1 - 2\}$. Then, $\sigma(1 + 2) = 0, \sigma(-1 - 2) = 2, \sigma(1 - 2) = \sigma(-1 + 2) = 1$, $s(1 + 2) = 3, s(-1 - 2) = -3, s(1 - 2) = -1$, and $s(-1 + 2) = 1$. It is easy to see that

$$R^+(\alpha_1 : \alpha_n) \cup R^-(\alpha_1 : \alpha_n) = R(\alpha_1 : \alpha_n) \text{ and } R^+(\alpha_1 : \alpha_n) \cap R^-(\alpha_1 : \alpha_n) = \emptyset.$$

In addition, if $r \in R^+(\alpha_1 : \alpha_n)$, then we use notation $-r := -\alpha_1 - \sum_{k=2}^{n} \gamma_k \alpha_k$. Thus, $r \in R^+(\alpha_1 : \alpha_n)$ if and only if $-r \in R^-(\alpha_1 : \alpha_n)$.

From Definition 2.1, we can have the following remarks.

Remark 2.2. Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n, \alpha_{n+1} > 0$ and $r \in R(\alpha_1 : \alpha_n)$. Then, $r + \alpha_{n+1} \in R(\alpha_1 : \alpha_{n+1})$, $\sigma(r + \alpha_{n+1}) = \sigma(r)$ and $\sigma(r - \alpha_{n+1}) = \sigma(r) + 1$.

Remark 2.3. $\sigma(-r) = n - \sigma(r)$ and $s(-r) = -s(r)$ for all $r \in R^+(\alpha_1 : \alpha_n)$.

Next, we prove important identities that will be used in this work.

Lemma 2.4. *Let $n \in \mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n > 0$. Then,*

$$\prod_{k=1}^{n} (x^{\alpha_k} - x^{-\alpha_k}) = \sum_{r \in R(\alpha_1 : \alpha_n)} (-1)^{\sigma(r)} x^{s(r)}.$$

Proof. We use the induction on n. For $n = 1$, it is easy to see that

$$x^{\alpha_1} - x^{-\alpha_1} = (-1)^{\sigma(\alpha_1)} x^{s(\alpha_1)} + (-1)^{\sigma(-\alpha_1)} x^{s(-\alpha_1)}.$$

Next, let $m \in \mathbb{N}$ such that

$$\prod_{k=1}^{m} (x^{\alpha_k} - x^{-\alpha_k}) = \sum_{r \in R(\alpha_1 : \alpha_m)} (-1)^{\sigma(r)} x^{s(r)}.$$

Then,

$$\begin{aligned}
&\prod_{k=1}^{m+1}(x^{\alpha_k}-x^{-\alpha_k})\\
&=\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{\sigma(r)}x^{s(r)}x^{\alpha_{m+1}}-\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{\sigma(r)}x^{s(r)}x^{-\alpha_{m+1}}\\
&=\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{\sigma(r)}x^{s(r)+\alpha_{m+1}}+\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{1+\sigma(r)}x^{s(r)-\alpha_{m-1}}\\
&=\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{\sigma(r+\alpha_{m+1})}x^{s(r+\alpha_{m+1})}+\sum_{r\in R(\alpha_1:\alpha_m)}(-1)^{\sigma(r-\alpha_{m+1})}x^{s(r-\alpha_{m+1})}\\
&=\sum_{r'\in R(\alpha_1:\alpha_{m+1})}(-1)^{\sigma(r')}x^{s(r')}.
\end{aligned}$$

By mathematical induction, this statement holds for $n \in \mathbb{N}$. □

We can easily rewrite Lemma 2.4 in the following form.

Corollary 2.5. *Let* $n \in \mathbb{N}$ *and* $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n > 0$. *Then,*

$$\prod_{k=1}^{n}(x^{\alpha_k}-x^{-\alpha_k})=\sum_{r\in R^+(\alpha_1:\alpha_n)}(-1)^{\sigma(r)}(x^{s(r)}+(-1)^n x^{-s(r)}).$$

Proof. By Remarks 2.2, 2.3, Lemma 2.4 and $(-1)^{-\sigma(r)} = (-1)^{\sigma(r)}$ for $r \in R^+(\alpha_1 : \alpha_n)$, we have

$$\begin{aligned}
\prod_{k=1}^{n}(x^{\alpha_k}-x^{-\alpha_k}) &= \sum_{r\in R^+(\alpha_1:\alpha_n)}(-1)^{\sigma(r)}x^{s(r)}+\sum_{r\in R^-(\alpha_1:\alpha_n)}(-1)^{\sigma(r)}x^{s(r)}\\
&=\sum_{r\in R^+(\alpha_1:\alpha_n)}[(-1)^{\sigma(r)}x^{s(r)}+(-1)^{\sigma(-r)}x^{s(-r)}]\\
&=\sum_{r\in R^+(\alpha_1:\alpha_n)}(-1)^{\sigma(r)}(x^{s(r)}+(-1)^{n-\sigma(r)}x^{-s(r)})\\
&=\sum_{r\in R^+(\alpha_1:\alpha_n)}(-1)^{\sigma(r)}(x^{s(r)}+(-1)^{n}x^{-s(r)}).
\end{aligned}$$

□

3. Product of Sine Formula and Its Definite Integral

In this section, before finding the exact value of (1.3), we first give the formula for the product of sine.

Lemma 3.1. *(Generalized Product of Sine Formula) Let $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n > 0$. Then,*

$$\prod_{k=1}^{n} \sin(\alpha_k x) = \begin{cases} \dfrac{1}{2^{n-1}} \displaystyle\sum_{r\in R^+(\alpha_1:\alpha_n)} (-1)^{\sigma(r)} \cos(s(r)x) & ; n \equiv 0 \pmod 4, \\ \dfrac{1}{2^{n-1}} \displaystyle\sum_{r\in R^+(\alpha_1:\alpha_n)} (-1)^{\sigma(r)} \sin(s(r)x) & ; n \equiv 1 \pmod 4, \\ -\dfrac{1}{2^{n-1}} \displaystyle\sum_{r\in R^+(\alpha_1:\alpha_n)} (-1)^{\sigma(r)} \cos(s(r)x) & ; n \equiv 2 \pmod 4, \\ -\dfrac{1}{2^{n-1}} \displaystyle\sum_{r\in R^+(\alpha_1:\alpha_n)} (-1)^{\sigma(r)} \sin(s(r)x) & ; n \equiv 3 \pmod 4. \end{cases}$$

Proof. We show the cases when $n \equiv 0 \pmod 4$ and $n \equiv 1 \pmod 4$. For the remaining cases, we left for the reader. By using the Euler's identity, De Moivre's Formula and Corollary 2.5, we have

$$\begin{aligned} \prod_{k=1}^{n} \sin(\alpha_k x) &= \frac{1}{(2i)^n} \prod_{k=1}^{n} (e^{i\alpha_k x} - e^{-i\alpha_k x}) \\ &= \frac{1}{(2i)^n} \sum_{r\in R^+(\alpha_1:\alpha_n)} (-1)^{\sigma(r)} (e^{is(r)x} + (-1)^n e^{-is(r)x}). \end{aligned}$$

If $n = 4m$ for $m \in \mathbb{N} \cup \{0\}$, then

$$\begin{aligned} \prod_{k=1}^{4m} \sin(\alpha_k x) &= \frac{1}{2^{4m}} \sum_{r\in R^+(\alpha_1:\alpha_{4m})} (-1)^{\sigma(r)} (e^{is(r)x} + e^{-is(r)x}) \\ &= \frac{1}{2^{4m-1}} \sum_{r\in R^+(\alpha_1:\alpha_{4m})} (-1)^{\sigma(r)} \cos(s(r)x). \end{aligned}$$

If $n = 4m + 1$ for $m \in \mathbb{N} \cup \{0\}$, then

$$\begin{aligned} \prod_{k=1}^{4m+1} \sin(\alpha_k x) &= \frac{1}{2^{4m+1} i} \sum_{r\in R^+(\alpha_1:\alpha_{4m+1})} (-1)^{\sigma(r)} (e^{is(r)} - e^{-is(r)}) \\ &= \frac{1}{2^{4m}} \sum_{r\in R^+(\alpha_1:\alpha_{4m+1})} (-1)^{\sigma(r)} \sin(s(r)x). \end{aligned}$$

□

From Lemma 3.1, we obtain a well-known identity for the power of sine and the product of cosine as follows.

Corollary 3.2. *Let $n \in \mathbb{N}$. Then,*

$$\sin^n x = \begin{cases} \dfrac{(-1)^{\frac{n(n+1)}{2}+1}}{2^{n-1}} \displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sin((n-2k)x) & ; n \text{ is odd}, \\ \dfrac{(-1)^{\frac{n(n+1)}{2}}}{2^{n-1}} \displaystyle\sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \cos((n-2k)x) & ; n \text{ is even} \end{cases}$$

and

$$\cos^n x = \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} \binom{n-1}{k} \cos((n-2k)x).$$

Proof. By applying Lemma 3.1 with $\alpha_i = 1$ for $i \in \{1, 2, 3, \dots, n\}$, it is easy to see that for $r \in R^+(1:1)$, if $\sigma(r) = k$, then $s(r) = n - 2k$ and it repeats $\binom{n-1}{k}$ times for $k \in \{0, 1, 2, \dots, n-1\}$. Hence,

$$\sin^n x = \begin{cases} \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 0 \pmod 4, \\ \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sin((n-2k)x) & ; n \equiv 1 \pmod 4, \\ -\frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 2 \pmod 4, \\ -\frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \sin((n-2k)x) & ; n \equiv 3 \pmod 4 \end{cases} \tag{3.1}$$

and we are done. Next, we substitutes x by $\frac{\pi}{2} - x$ in (3.1). Note that

$$\sin\left(\frac{p\pi}{2} - x\right) = (-1)^{\frac{p-1}{2}} \cos x, \quad \text{and} \quad \cos(q\pi - x) = (-1)^q \cos x$$

for $p, q \in \mathbb{Z}$ and p is odd. Hence,

$$\cos^n x = \begin{cases} \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k (-1)^{\frac{n-2k}{2}} \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 0 \pmod 4, \\ \frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k (-1)^{\frac{n-2k-1}{2}} \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 1 \pmod 4, \\ -\frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k (-1)^{\frac{n-2k}{2}} \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 2 \pmod 4, \\ -\frac{1}{2^{n-1}} \sum_{k=0}^{n-1} (-1)^k (-1)^{\frac{n-2k-1}{2}} \binom{n-1}{k} \cos((n-2k)x) & ; n \equiv 3 \pmod 4 \end{cases}$$

and we are done. □

Remark 3.3. Actually, we can obtain $\cos^n x$ in an easier way. By the Euler's identity and the binomial theorem, we have

$$\cos^n x = \left(\frac{e^{ix} + e^{-ix}}{2}\right)^n = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{i(n-2k)x}$$

and

$$\cos^n x = \cos^n(-x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} e^{-i(n-2k)x}.$$

Hence, we obtain

$$\cos^n x = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{e^{i(n-2k)x} + e^{-i(n-k)x}}{2}\right) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \cos((n-2k)x). \tag{3.2}$$

Note that

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\cos((n-2k)x) &= \sum_{k=0}^{n-1}\binom{n-1}{k}\cos((n-2k)x)+\sum_{k=1}^{n}\binom{n-1}{k-1}\cos((n-2k)x)\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\cos((n-2k)x)\\ &\quad + \sum_{n-k=1}^{n}\binom{n-1}{n-k-1}\cos((n-2(n-k))x)\\ &= \sum_{k=0}^{n-1}\binom{n-1}{k}\cos((n-2k)x)+\sum_{m=0}^{n-1}\binom{n-1}{m}\cos((n-2m)x)\\ &= 2\sum_{k=0}^{n-1}\binom{n-1}{k}\cos((n-2k)x).\end{aligned}$$

That is

$$\cos^n x = \frac{1}{2^{n-1}}\sum_{k=0}^{n-1}\binom{n-1}{k}\cos((n-2k)x). \tag{3.3}$$

For $\sin^n x$, we can obtain the same formula as shown in Corollary 3.2 by substituting $\frac{\pi}{2}-x$ into x in (3.3).

According to [**2**], Andrica and Tomescu showed that if $\alpha_k \in \mathbb{N}$ for $k \in \{1,2,3,\ldots,n\}$, then

$$S(\alpha_1 : \alpha_n) = \frac{2^{n-1}}{\pi}\int_0^{2\pi}\prod_{k=1}^{n}\cos(\alpha_k x)dx,$$

where $S(\alpha_1 : \alpha_n)$ is the number of all choices of $+$ and $-$ such that $\pm\alpha_1 \pm\cdots\pm \alpha_k = 0$. Thus, this formula motivated our definition, indeed, we define two numbers as follows. $S_E(\alpha_1 : \alpha_n)$ is the number of all choices of $+$ and $-$ such that $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 \cdots \pm \alpha_n = 0$ and $\sigma(r)$ is even. On the other hand, $S_O(\alpha_1 : \alpha_n)$ is the number of all choices of $+$ and $-$ such that $\pm\alpha_1 \pm \alpha_2 \pm \alpha_3 \cdots \pm \alpha_n = 0$ and $\sigma(r)$ is odd. Hence, we obtain the following result.

Theorem 3.4. *Let $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n \in \mathbb{N}$. Then,*

$$\int_0^{2\pi}\prod_{k=1}^{n}\sin(\alpha_k x)dx = \begin{cases}\frac{\pi}{2^{n-1}}(S_E(\alpha_1 : \alpha_n) - S_O(\alpha_1 : \alpha_n)) & ; n \equiv 0 \pmod 4,\\ \frac{\pi}{2^{n-1}}(S_O(\alpha_1 : \alpha_n) - S_E(\alpha_1 : \alpha_n)) & ; n \equiv 2 \pmod 4,\\ 0 & ;\text{ otherwise.}\end{cases}$$

Proof. This follows from integrating Lemma 3.1 on $[0, 2\pi]$. Note that $s(r) \in \mathbb{Z}$ for $r \in R^+(\alpha_1 : \alpha_n)$. We have

$$\int_0^{2\pi}\cos(s(r)x)dx = \begin{cases}2\pi & ; s(r) = 0\\ 0 & ; s(r) \neq 0\end{cases} \quad\text{and}\quad \int_0^{2\pi}\sin(s(r)x) = 0.$$

For the cases $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, we have

$$\int_0^{2\pi}\prod_{k=1}^{n}\sin(\alpha_k x)dx = 0.$$

For the cases $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, we obtain

$$\int_0^{2\pi} \prod_{k=1}^{n} \sin(\alpha_k x)dx = \begin{cases} \frac{\pi}{2^{n-2}} \sum\limits_{\substack{r\in R^+(\alpha_1:\alpha_n)\\ s(r)=0}} (-1)^{\sigma(r)} & ; n \equiv 0 \pmod 4, \\ -\frac{\pi}{2^{n-2}} \sum\limits_{\substack{r\in R^+(\alpha_1:\alpha_n)\\ s(r)=0}} (-1)^{\sigma(r)} & ; n \equiv 2 \pmod 4. \end{cases}$$

Since

$$\begin{aligned} \sum_{\substack{r\in R^+(\alpha_1:\alpha_n)\\ s(r)=0}} (-1)^{\sigma(r)} &= \sum_{\substack{r\in R^+(\alpha_1:\alpha_n)\\ \sigma(r) \text{ is even and } s(r)=0}} 1 - \sum_{\substack{r\in R^+(\alpha_1:\alpha_n)\\ \sigma(r) \text{ is odd and } s(r)=0}} 1 \\ &= \frac{1}{2}(S_E(\alpha_1 : \alpha_n) - S_O(\alpha_1 : \alpha_n)), \end{aligned}$$

the proof is completed. □

4. Power of Difference of Cosine Integral

First of all, it is well known that

$$\int_0^{\infty} \frac{\cos\alpha x - \cos\beta x}{x} = \ln|\beta| - \ln|\alpha| \quad \text{and} \quad \int_0^{\infty} \frac{1-\cos\gamma x}{x^2}dx = \frac{\pi}{2}|\gamma|,$$

where $\alpha, \beta \neq 0$ and $\gamma \in \mathbb{R}$ as shown in [**4**]. In this section, we find the exact value of

$$\int_0^{\infty} \frac{(\cos\alpha x - \cos\beta x)^p}{x^q},$$

where $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathbb{N}$ such that $p \geq q$ and q is even by using the technique shown in [**3**] regardless of the Fourier transformation. First, we need the following identity to help us tackle a problem.

Lemma 4.1. *Let $\alpha, \beta \in \mathbb{R}$ and $m, n \in \mathbb{N} \cup \{0\}$. Then,*

$$\cos^m\alpha \cos^n\beta = \frac{1}{2^{m+n-1}} \sum_{k=0}^{m-1} \sum_{l=0}^{n} \binom{m-1}{k} \binom{n}{l} \cos((m-2k)\alpha + (n-2l)\beta).$$

Proof. By the cosine part of Corollary 3.2 and (3.2) we have

$$\begin{aligned} \cos^m\alpha \cos^n\beta &= \frac{1}{2^{m+n-1}} \sum_{k=0}^{m-1} \sum_{l=0}^{n} \binom{m-1}{k} \binom{n}{l} \cos((m-2k)\alpha) \cos((n-2l)\beta) \\ &= \frac{1}{2^{m+n}} \sum_{k=0}^{m-1} \binom{m-1}{k} F(k), \end{aligned}$$

where

$$
\begin{aligned}
F(k) &= \sum_{l=0}^{n} \binom{n}{l} \cos((m-2k)\alpha + (n-2l)\beta) + \sum_{l=0}^{n} \binom{n}{l} \cos((m-2k)\alpha - (n-2l)\beta) \\
&= \sum_{l=0}^{n} \binom{n}{l} \cos((m-2k)\alpha + (n-2l)\beta) \\
&\qquad + \sum_{n-l=0}^{n} \binom{n}{n-l} \cos((m-2k)\alpha - (n-2(n-l))\beta) \\
&= \sum_{l=0}^{n} \binom{n}{l} \cos((m-2k)\alpha + (n-2l)\beta) \\
&\qquad + \sum_{l=0}^{n} \binom{n}{n-l} \cos((m-2k)\alpha + (n-2l)\beta) \\
&= 2\sum_{l=0}^{n} \binom{n}{l} \cos((m-2k)\alpha + (n-2l)\beta).
\end{aligned}
$$

Thus, the lemma is proved. □

Next, we clarify an important lemma that was used without proof in [**3**].

Lemma 4.2. *Let f be a smooth function on $\mathbb{R}$ with $x = 0$ is its zero of order k for some $k \geq 2$ and there exists an integer $l \leq k$ such that $\int_0^\infty \frac{f^{(l-1)}(x)}{x}dx$ exists, and $f^{(m)}$ is bounded on $\mathbb{R}$ for $0 \leq m \leq l-2$ where $f^{(n)}$ is the n^{th} derivative of f. Then*

$$\int_0^\infty \frac{f(x)}{x^l}dx = \frac{1}{(l-1)!}\int_0^\infty \frac{f^{(l-1)}(x)}{x}dx.$$

Proof. For $2 \leq l \leq k$, by using the integration by part with indefinite integral $\int \frac{f(x)}{x^l}dx$, we get

$$\int \frac{f(x)}{x^l}dx = -\frac{1}{(l-1)}\frac{f(x)}{x^{l-1}} - \cdots - \frac{1}{(l-1)!}\frac{f^{(l-2)}(x)}{x} + \frac{1}{(l-1)!}\int \frac{f^{(l-1)}(x)}{x}dx.$$

Since f has a zero order k at $x = 0$ and $l \leq k$, we have

$$f^{(m)}(x) = 0 \quad \text{and} \quad \lim_{x\to 0^+} \frac{f(x)}{x^{l-1}} = 0$$

for $0 \leq m \leq l-1$. By using the L'Hopital's rule, we obtain $\lim_{x\to 0^+} \frac{f^{(m)}(x)}{x^{l-m-1}} = 0$ for $0 \leq m \leq l-2$. Moreover, since $f^{(m)}$ is bounded on $\mathbb{R}$ for $0 \leq m \leq l-2$, by using the squeeze theorem, we get that $\lim_{x\to\infty} \frac{f^{(m)}(x)}{x^{l-m-1}} = 0$ for $0 \leq m \leq l-2$.

Hence,

$$\int_0^\infty \frac{f(x)}{x^l}dx = \frac{1}{(l-1)!}\int_0^\infty \frac{f^{(l-1)}(x)}{x}dx.$$

□

Before we prove our result, we introduce a well-known improper integral as shown in [**4**]

$$\int_0^\infty \frac{\sin rx}{x}dx = \begin{cases} \operatorname{sgn}(r)\dfrac{\pi}{2} & ; r \neq 0, \\ 0 & ; r = 0, \end{cases},$$

for $r \in \mathbb{R}$, where sgn is a signum function defined by :

$$\operatorname{sgn}(t) = \begin{cases} 1 & ; t > 0 \\ 0 & ; t = 0 \\ -1 & ; t < 0 \end{cases},$$

for $t \in \mathbb{R}$. Now, we give and prove our result.

Theorem 4.3. *Let $\alpha, \beta \in \mathbb{R}$ and $p, q \in \mathbb{N}$ such that $p \geq q$ and q is even. Then,*

$$\int_0^\infty \frac{(\cos\alpha x - \cos\beta x)^p}{x^q}dx$$
$$= \frac{\pi}{2^p(q-1)!}\sum_{k=0}^{p}\sum_{l=0}^{p-k-1}\sum_{m=0}^{k}\Bigg((-1)^{k+\frac{q}{2}}\binom{p}{k}\binom{p-k-1}{l}\binom{k}{m}\times$$
$$|(p-k-2l)\alpha + (k-2m)\beta|^{q-1}\Bigg).$$

Proof. By the binomial theorem and Lemma 4.1, we obtain

$$(\cos\alpha x - \cos\beta x)^p \tag{4.1}$$
$$= \frac{1}{2^{p-1}}\sum_{k=0}^{p}\sum_{l=0}^{p-k-1}\sum_{m=0}^{k}(-1)^k\binom{p}{k}\binom{p-k-1}{l}\binom{k}{m}\cos((p-k-2l)\alpha x + (k-2m)\beta x).$$

Hence,

$$\int_0^\infty \frac{(\cos\alpha x - \cos\beta x)^p}{x^q} = \frac{1}{2^{p-1}}\int_0^\infty \frac{G(x)}{x^q}dx,$$

where $G(x) = \sum_{k=0}^{p}\sum_{l=0}^{p-k-1}\sum_{m=0}^{k}(-1)^k\binom{p}{k}\binom{p-k-1}{l}\binom{k}{m}\cos((p-k-2l)\alpha x + (k-2m)\beta x)$. Note that $G(x)$ is a smooth function on $\mathbb{R}$ with a zero of order $2p$ at $x = 0$. Since q is even, we have

$$G^{(q-1)}(x) = \sum_{k=0}^{p}\sum_{l=0}^{p-k-1}\sum_{m=0}^{k}\Bigg((-1)^{k+\frac{q}{2}}\binom{p}{k}\binom{p-k-1}{l}\binom{k}{m}\times$$
$$((p-k-2l)\alpha + (k-2m)\beta)^{q-1}\sin((p-k-2l)\alpha x + (k-2m)\beta x)\Bigg).$$

By Lemma 4.2, we have

$$\int_0^\infty \frac{(\cos\alpha x - \cos\beta x)^p}{x^q} dx$$

$$= \frac{1}{2^{p-1}(q-1)!} \sum_{k=0}^{p} \sum_{l=0}^{p-k-1} \sum_{m=0}^{k} \Bigg((-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{p-k-1}{l}$$

$$\times \binom{k}{m} ((p-k-2l)\alpha + (k-2m)\beta)^{q-1} \int_0^\infty \frac{\sin((p-k-2l)\alpha x + (k-2m)\beta x)}{x} dx \Bigg)$$

$$= \frac{\pi}{2^p(q-1)!} \sum_{k=0}^{p} \sum_{l=0}^{p-k-1} \sum_{m=0}^{k} \Bigg((-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{p-k-1}{l} \binom{k}{m} \times$$

$$\frac{((p-k-2l)\alpha + (k-2m)\beta)^q}{|(p-k-2l)\alpha + (k-2m)\beta)|} \Bigg)$$

$$= \frac{\pi}{2^p(q-1)!} \sum_{k=0}^{p} \sum_{l=0}^{p-k-1} \sum_{m=0}^{k} \Bigg((-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{p-k-1}{l} \binom{k}{m} \times$$

$$|(p-k-2l)\alpha + (k-2m)\beta|^{q-1} \Bigg).$$

□

If we consider $(\alpha, \beta, p, q) = (0, \beta, p, q)$, then we have the following result.

Corollary 4.4. *Let $\beta \in \mathbb{R}$ and $p, q \in \mathbb{N}$ such that $p \geq q$ and q is even. Then,*

$$\int_0^\infty \frac{(1-\cos\beta x)^p}{x^q} dx = \frac{\pi|\beta|^{q-1}}{(q-1)!} \sum_{k=0}^{p} \sum_{m=0}^{k} 2^{-k-1} (-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{k}{m} |k-2m|^{q-1}.$$

Proof. By plug in $\alpha = 0$ in theorem 4.3, we have

$$\int_0^\infty \frac{(1-\cos\beta x)^p}{x^q} dx$$

$$= \frac{\pi}{2^p(q-1)!} \sum_{k=0}^{p} \sum_{l=0}^{p-k-1} \sum_{m=0}^{k} (-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{p-k-1}{l} \binom{k}{m} |(k-2m)\beta|^{q-1}$$

$$= \frac{\pi}{2^p(q-1)!} \sum_{k=0}^{p} \sum_{m=0}^{k} \Bigg((-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{k}{m} |(k-2m)\beta|^{q-1} \sum_{l=0}^{p-k-1} \binom{p-k-1}{l} \Bigg)$$

$$= \frac{\pi}{2^p(q-1)!} \sum_{k=0}^{p} \sum_{m=0}^{k} 2^{p-k-1} (-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{k}{m} |(k-2m)\beta|^{q-1}$$

$$= \frac{\pi|\beta|^{q-1}}{(q-1)!} \sum_{k=0}^{p} \sum_{m=0}^{k} 2^{-k-1} (-1)^{k+\frac{q}{2}} \binom{p}{k} \binom{k}{m} |(k-2m)|^{q-1}.$$

□

5. Conclusion and Discussion

We present the exact values of (1.3) and (1.4). However, the exact value of (1.4) is not found in the case that q is odd. Thus, our future study is try to solve this remaining case. In addition, we may consider (1.4) with arbitrary arguments of cosine, namely $\int_0^\infty \frac{\prod_{k=1}^{p}(\cos(\alpha_k x) - \cos(\beta_k x))}{x^q} dx$.

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