NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 48 (2018), 11-23

APPROXIMATION OF FUNCTIONS BELONGING TO THE WEIGHTED $L(\alpha, M, \omega)-$CLASS BY TRIGONOMETRIC POLYNOMIALS

Sadulla Z. Jafarov

(Received July 6, 2017)

Abstract. In this work the approximation of the functions by means $t_n(f;x)$, $N_n^{\beta}(f;x)$ and $R_n^{\beta}(f,x)$ of the trigonometric Fourier series in weighted Orlicz spaces with Muckenhoupt weights are studied.

1. Introduction, Some Auxiliary Results and Main Results

Let $\mathbb{T}$ denote the interval $[-\pi, \pi]$, $\mathbb{C}$ the complex plane, and $L_p(\mathbb{T})$, $1 \le p \le \infty$, the Lebesgue space of measurable complex-valued functions on $\mathbb{T}$. A convex and continuous function $M : [0, \infty) \to [0, \infty)$ which satisfies the conditions

$$\begin{aligned} M(0) &= 0, \ M(x) > 0 \ \text{ for } x > 0, \\ \lim_{x\to 0} (M(x)/x) &= 0; \ \lim_{x\to\infty} (M(x)/x) = \infty, \end{aligned}$$

is called a *Young function.* We will say that M satisfies the Δ_2-condition if $M(2u) \le cM(u)$ for any $u \ge u_0 \ge 0$ with some constant c independent of u.

We can consider a right continuous, monotone increasing function $\rho : [0, \infty) \to [0, \infty)$ with

$$\rho(0) = 0; \ \lim_{t\to\infty} \rho(t) = \infty \ \text{ and } \rho(t) > 0 \text{ for } t > 0,$$

then the function defined by

$$N(x) = \int_0^{|x|} \rho(t)\, dt$$

is called an $N-$*function.* For a given Young function M, let $\widetilde{L}_M(\mathbb{T})$ denote the set of all Lebesgue measurable functions $f : \mathbb{T} \to \mathbb{C}$ for which

$$\int_{\mathbb{T}} M(|f(x)|)\, dx < \infty.$$

The *complementary $N-$function to M* is defined by

$$N(y) := \max_{x\ge 0} (xy - M(x)), \ \text{ for } y \ge 0.$$

2010 *Mathematics Subject Classification* 41A17, 41A25, 41A27, 42A10, 42A50, 46E30, 46E35, 26A33.

Key words and phrases: Trigonometric approximation, Orlicz space, weighted Orlicz space, Boyd indices, Muckenhoupt weight, weighted $L(\alpha, M, \omega)$ class, modulus of continuity.

Let N be the complementary Young function of M. It is well-known [**22**, p. 69], [**39**, p. 52-68] that the linear span of $\widetilde{L}_M(\mathbb{T})$ equipped with the *Orlicz norm*

$$\|f\|_{L_M(\mathbb{T})} := \sup\left\{\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx : g \in \widetilde{L}_N(\mathbb{T}),\ \int\limits_{\mathbb{T}} N\left(|g(x)|\right) dx \le 1\right\},$$

or with the *Luxemburg norm*

$$\|f\|^*_{L_M(\mathbb{T})} := \inf\left\{k > 0 : \int\limits_{\mathbb{T}} M\left(\frac{|f(x)|}{k}\right) dx \le 1\right\},$$

becomes a Banach space. This space is denoted by $L_M(\mathbb{T})$ and is called an *Orlicz space* [**22**, p. 26]. The Orlicz spaces are known as the generalizations of the Lebesgue spaces $L_p(\mathbb{T}), 1 < p < \infty$. If $M(x) = M(x,p) := x^p$, $1 < p < \infty$, then Orlicz spaces $L_M(\mathbb{T})$ coindices with the usual Lebesgue spaces $L_p(\mathbb{T}), 1 < p < \infty$. Note that the Orlicz spaces play an important role in many areas such as applied mathematics, mechanics, regularity theory, fluid dynamics and statistical physics (e.g., [**1**], [**9**], [**31**] and [**41**]). Therefore, of investigation the approximation of the functions by means of Fourier trigonometric series in Orlicz spaces is also important in these areas of research.

The Luxemburg norm is equivalent to the Orlicz norm. The inequalities

$$\|f\|^*_{L_M(\mathbb{T})} \le \|f\|_{L_M(\mathbb{T})} \le 2\,\|f\|^*_{L_M(\mathbb{T})}, \qquad f \in L_M(\mathbb{T}),$$

hold [**22**, p. 80].

If we choose $M(u) = u^p/p, 1 < p < \infty$ then the complementary function is $N(u) = u^q/q$ with $1/p + 1/q = 1$ and we have the relation

$$p^{-1/p}\,\|u\|_{L_p(\mathbb{T})} = \|u\|^*_{L_M(\mathbb{T})} \le \|u\|_{L_M(\mathbb{T})} \le q^{1/q}\,\|u\|_{L_p(\mathbb{T})},$$

where $\|u\|_{L_p(\mathbb{T})} = \left(\int\limits_{\mathbb{T}} |u(x)|^p\,dx\right)^{1/p}$ stands for the usual norm of the $L_p(\mathbb{T})$space.

If N is complementary to M in Young's sense and $f \in L_M(\mathbb{T})$, $g \in L_N(\mathbb{T})$ then the so-called strong Hölder inequalities [**22**, p. 80]

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|_{L_M(\mathbb{T})}\,\|g\|^*_{L_N(\mathbb{T})},$$

$$\int\limits_{\mathbb{T}} |f(x)g(x)|\,dx \le \|f\|^*_{L_M(\mathbb{T})}\,\|g\|_{L_N(\mathbb{T})},$$

are satisfied.

The Orlicz space $L_M(\mathbb{T})$ is *reflexive* if and only if the $N-$function M and its complementary function N both satisfy the Δ_2-condition [**39**, p. 113].

Let $M^{-1} : [0,\infty) \to [0,\infty)$ be the inverse function of the $N-$function M. The *lower* and *upper indices* [**4**, p. 350]

$$\alpha_M := \lim_{t\to+\infty} -\frac{\log h(t)}{\log t},\ \ \beta_M := \lim_{t\to o^+} -\frac{\log h(t)}{\log t},$$

of the function

$$h:(0,\infty)\to(0,\infty],\quad h(t):=\lim_{y\to\infty}\sup\frac{M^{-1}(y)}{M^{-1}(ty)},\ \ t>0,$$

first considered by Matuszewska and Orlicz [**29**], are called the *Boyd indices* of the Orlicz spaces $L_M(T)$.

It is known that the indices α_M and β_M satisfy $0\le\alpha_M\le\beta_M\le 1,\ \alpha_N+\beta_M=1$, $\alpha_M+\beta_N=1$ and the space $L_M(\mathbb{T})$ is reflexive if and only if $0<\alpha_M\le\beta_M<1$. The detailed information about the Boyd indices can be found in [**5-8**], [**30**].

A measurable function $\omega:\mathbb{T}\to[0,\infty]$ is called a *weight function* if the set $\omega^{-1}(\{0,\infty\})$ has Lebesgue measure zero. With any given weight ω we associate the ω-*weighted Orlicz space* $L_M(\mathbb{T},\omega)$ consisting of all measurable functions f on $\mathbb{T}$ such that

$$\|f\|_{L_M(\mathbb{T},\omega)}:=\|f\omega\|_{L_M(\mathbb{T})}\,.$$

Let $1<p<\infty,\ 1/p+1/p'=1$ and let ω be a weight function on $\mathbb{T}$. ω is said to satisfy *Muckenhoupt's* A_p-*condition* on $\mathbb{T}$ if

$$\sup_J\left(\frac{1}{|J|}\int_J\omega^p(t)\,dt\right)^{1/p}\left(\frac{1}{|J|}\int_J\omega^{-p'}(t)\,dt\right)^{1/p'}<\infty\ ,$$

where J is any subinterval of $\mathbb{T}$ and $|J|$ denotes its length [**32**].

Let us indicate by $A_p(\mathbb{T})$ the set of all weight functions satisfying Muckenhoupt's A_p-condition on $\mathbb{T}$.

Let further $t_1,t_2,...,t_n$ be distinct points on $\mathbb{T}$ and let $\lambda_1,\ ...,\lambda_n$ be real numbers. If $1<p<\infty,\frac{1}{p}+\frac{1}{q}=1$ and $-\frac{1}{p}<\lambda_j<\frac{1}{q},j=1,...,n$ then the weight function

$$\omega(\tau):=\prod_{j=1}^{n}|\tau-t_j|^{\lambda_j},(\tau\in\mathbb{T})$$

belongs to $A_p(\mathbb{T})$.

According to [**27**], [**28**, Lemma 3.3], and [**28**, Section 2.3] if $L_M(\mathbb{T})$ is reflexive and the weighted function ω satisfies the condition $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}(\mathbb{T})$, then the space $L_M(\mathbb{T},\omega)$ is also reflexive.

Let $L_M(\mathbb{T},\omega)$ be a weighted Orlicz space, let $0<\alpha_M\le\beta_M<1$ and let $\omega\in A_{\frac{1}{\alpha_M}}(\mathbb{T})\cap A_{\frac{1}{\beta_M}}(\mathbb{T})$. For $f\in L_M(\mathbb{T},\omega)$ we set

$$(\nu_h f)(x):=\frac{1}{2h}\int_{-h}^{h}f(x+t)\,dt,\ 0<h<\pi,\ x\in\mathbb{T}.$$

By reference [**16**, Lemma 1], the shift operator ν_h is a bounded linear operator on $L_M(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L_M(\mathbb{T},\,\omega)}\le c\,\|f\|_{L_M(\mathbb{T},\,\omega)}\,.$$

The function

$$\Omega_{M,\omega}(\delta,\ f):=\sup_{0<h\le\delta}\|f-(\nu_h f)\|_{L_M(\mathbb{T},\omega)}\,,\ \delta>0,$$

is called the *modulus of continuity* of $f\in L_M(\mathbb{T},\omega)$.

It can easily be shown that $\Omega_{M,\ \omega}(\cdot,\ f)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_{M,\omega}(\delta,\ f)=0,\ \Omega_{M,\omega}(\delta,\ f+g)\leq\Omega_{M,\omega}(\delta,\ f)+\Omega_{M,\omega}(\delta,g)$$

for $f,\ g\in L_M(\mathbb{T},\ \omega)$.

Let $0<\alpha\leq 1$. The set of functions $f\in L_M(\mathbb{T},\omega)$ such that

$$\Omega_{M,\omega}(f,\delta)=O(\delta^{\alpha}),\quad \delta>0$$

is called the *weighted Lipschitz class* $Lip(\alpha,M,\omega)$. Let

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}(a_k(f)\cos kx+b_k(f)\sin kx) \tag{1}$$

be the Fourier series of the function $f\in L^1(\mathbb{T})$, where $\alpha_k(f)$ are $\beta_k(f)$ the Fourier coefficients of the function f. The *n-th partial sums, Cesaro means* of the series (1) are defined, respectively, as

$$\begin{aligned}
s_n(x,f) &= \frac{a_0}{2}+\sum_{k=1}^{n}(a_k(f)\cos kx+b_k(f)\sin kx)\\
&= \frac{a_0}{2}+\sum_{k=1}^{n}A_k(x,f),\ A_k(x,f)=(a_k(f)\cos kx+b_k(f)\sin kx),\\
\sigma_n(x,f) &= \frac{1}{n+1}\sum_{m=0}^{n}s_m(x,f).
\end{aligned}$$

Let $\sum\limits_{m=0}^{\infty}a_n$ be an infinite series and let $\{s_n\}$ its n^{th} partial sum. Let $\{t_n\}$ be a sequence of (N,p,q) means of the sequence $\{s_n\}$. We define transform (N,p,q) of $\{s_n(f;x)\}$ by [**42**]

$$t_n(x):=t_n(f,x):=\frac{1}{r_n}\sum_{m=0}^{n}p_{n-m}q_m s_m(f;x),$$

where

$$r_n:=\sum_{m=0}^{n}p_m q_{n-m}\neq 0, n\geq 0,\ \text{and}\ p_{-1}=q_{-1}=r_{-1}=0.$$

Let $\{p_n\}$ be a real sequence, where $p_0>0,\ p_n\geq 0$ for $n>0$. As in [**2**] we define

$$p_m^{\beta}=\sum_{\nu=0}^{m}A_{m-\nu}p_{\nu};\ P_n^{\beta}=\sum_{m=0}^{n}p_m^{\beta},\ P_{-i}^{\beta}=p_{-i}=0,\ i\geq 1,$$

where

$$A_0^{\beta}=1;\ A_n^{\beta}=\frac{(\beta+1)(\beta+2)(\beta+3)...(\beta+n)}{n!},\ \beta>-1,\ n=1,2,3,...\ .$$

In the proof of the main result we will use the notations

$$\Delta\beta_n:=\beta_n-\beta_{n+1},\quad \Delta_m\beta(n,m):=\beta(n,m)-\beta(n,m+1).$$

Considering [**38**] we can write the following equality

$$p_m^{\beta}-p_{m+1}^{\beta}=\sum_{\nu=0}^{m}A_{m-\nu}^{\beta-1}p_{\nu}-\sum_{\nu=0}^{m-1}A_{m+1-\nu}^{\beta-1}p_{\nu}=\sum_{\nu=0}^{m-1}A_{m+1-\nu}^{\beta-1}\Delta p_{\nu-1}.$$

We define the sequence $\left\{N_n^{\beta}\right\}$ of the $\left\{\overline{N},p_n^{\beta}\right\}$ means of the sequence $\{s_n(f;x)\}$ by

$$N_n^{\beta}(x,f)=\frac{1}{P_n^{\beta}}\sum_{m=0}^{n}p_m^{\beta}s_m(x,f).$$

Also,

$$R_n^{\beta}(x,f)=\frac{1}{P_n^{\beta}}\sum_{m=0}^{n}p_{n-m}^{\beta}s_m(x,f).$$

defines the $\left(N,p_n^{\beta}\right)$ means of $\{s_n(f;x)\}$.

In the present paper we study the approximation of the functions by trigonometric polynomials $t_n(f;x)$, $R_n^{\beta}(f,x)$ and $N_n^{\beta}(f;x)$ in weighted Orlicz spaces. The results obtained in this work, are generalization of the results [**13**] and [**42**] to more general summability and weighted Orlicz spaces. Similar problems about approximations of the functions by trigonometric polynomials in the different spaces have been investigated by several authors (see, for example, [**2**], [**3**], [**10-21**], [**23-26**], [**33-38**] , [**40**] and [**42-45**]).

Note that, in the proof of the main results we use the method as in the proof of [**42**].

Our main results are the following.

Theorem 1. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}$ and the conditions*

$$\text{(i)}\quad n^2q_n=O(r_n), \tag{2}$$

$$\text{(ii)}\quad \sum_{m=0}^{n-1}m^{2-\alpha}\left|\Delta_m(p_{n-m}q_m)\right|=O(r_nn^{-\alpha}), \tag{3}$$

are satisfied, then if $f\in Lip(\alpha,M,\omega)$, $0<\alpha\leq 1$ the estimate

$$\|t_n(\cdot,f)-f\|_{L_M(\mathbb{T},\omega)}=O(n^{-\alpha}),$$

holds.

Theorem 2. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}$, and let $\left\{p_n^{\beta}\right\}$be a monotonic sequence such that*

$$(n+1)\,p_n^{\beta}=O(P_n^{\beta}). \tag{4}$$

Then for every $f\in Lip(\alpha,M,\omega)$, $0<\alpha\leq 1$ the estimate

$$\left\|f-R_n^{\beta}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=O(n^{-\alpha}),\ n=1,2,...$$

holds.

Theorem 3. *Let $L_M(\mathbb{T})$ be a reflexive Orlicz space and $\omega\in A_{1/\alpha_M}(\mathbb{T})\cap A_{1/\beta_M}$, and let $\left\{p_n^{\beta}\right\}$ be a sequence of positive real numbers such that*

$$\sum_{m=0}^{n-1}\left|\frac{P_m^{\beta}}{m+1}-\frac{P_{m+1}^{\beta}}{m+2}\right|=O\left(\frac{P_n^{\beta}}{n+1}\right). \tag{5}$$

Then for every $f \in Lip(\alpha, M, \omega)$, $0 < \alpha \leq 1$ the estimate

$$\left\| f - N_n^{\beta}(\cdot, f) \right\|_{L_M(\mathbb{T},\omega)} = O(n^{-\alpha}), \ n = 1, 2, ...$$

holds.

In the proof of the main result we need the following lemmas.

Lemma 1 ([**14**])**.** *Let $L_M(T)$ be a reflexive Orlicz space and let $\omega \in A_{1/\alpha_M}(T) \cap A_{1/\beta_M}$. Then for $f \in L_M(T, \omega)$, the estimate*

$$\| f - \sigma_n(\cdot, f) \|_{L_M(\mathbb{T},\omega)} = O(n\Omega_{M,\omega} \left(\frac{1}{n}, \ f \right)), \ n = 1, 2, ...$$

holds.

Lemma 2 ([**42**])**.** *Let $\left\{ p_n^{\beta} \right\}$ be a monotonic sequence of positive numbers. Then,*

$$\sum_{m=1}^{n} m^{-\alpha} p_{n-m} = O\left(n^{-\alpha} P_n^{\beta} \right)$$

for $0 < \alpha < 1$.

2. Proofs of the Main Results

Proof of Theorem 1. By definition of $t_n(f; x)$ and $\sigma_n(f; x)$ we have [**42**, p. 1581]

$$\begin{aligned}
t_n(x, f) - f(x) &= \frac{1}{r_n} \sum_{m=0}^{n} p_{n-m} q_m (s_m(x, f) - f(x)) \\
&= \frac{1}{r_n} \left\{ \sum_{m=0}^{n-1} \Delta_m (p_{n-m} q_m) \sum_{k=0}^{m} (s_k(x, f) - f(x)) \right\} \\
&\quad + \frac{1}{r_n} \left\{ p_0 q_0 \sum_{k=0}^{n} (s_k(x, f) - f(x)) \right\} \\
&= \frac{1}{r_n} \left\{ \sum_{m=0}^{n-1} (m+1) \Delta_m (p_{n-m} q_m) (\sigma_m(x; f) - f(x)) \right\} \\
&\quad + \frac{1}{r_n} \left\{ (n+1) p_0 q_n (\sigma_n(x, f) - f(x)) \right\}. \qquad (6)
\end{aligned}$$

By (6), conditions (2), (3) and Lemma 1

$$\begin{aligned}
& \|t_n(\cdot,f)-f\|_{L_M(\mathbb{T},\omega)} \\
= & \ O\left(\frac{1}{r_n}\right)\left\{\sum_{m=0}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)\right|\|\sigma_m(\cdot,f)-f\|_{L_M(\mathbb{T},\omega)}\right\} \\
& +O\left(\frac{1}{r_n}\right)(n+1)p_0q_n\|\sigma_n(\cdot,f)-f\|_{L_M(\mathbb{T},\omega)} \\
= & \ O\left(\frac{1}{r_n}\right)\left\{\sum_{m=0}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)\right|(m^{1-\alpha})+(n+1)p_0q_n(n^{1-\alpha})\right\} \\
= & \ O\left[\frac{1}{r_n}\sum_{m=2}^{n-1}(m+1)\left|\Delta_m(p_{n-m}q_m)(m^{1-\alpha})\right|+O(n^{-\alpha})\right] \\
= & \ O\left[\frac{1}{r_n}\sum_{m=2}^{n-1}m^{2-\alpha}\left|\Delta_m(p_{n-m}q_m)\right|+(n^{-\alpha})\right]=O(n^{-\alpha}),
\end{aligned}$$

which completes the proof. □

*Proof of Theorem 2***.** Case 1. We suppose that $0<\alpha<1$. It is clear that

$$f(x)-R_n^{\beta}(x,f)=\frac{1}{P_n^{\beta}}\sum_{m=o}^{n}p_{n-m}^{\beta}\left\{f(x)-s_m(x,f)\right\}.$$

By [**14**, p. 8, relation (13)] the relation

$$\|f-s_n(\cdot,f)\|_{L_M(\mathbb{T},\omega)}=O\left(\Omega_{M,\omega}\left(\frac{1}{n},\ f\right)\right) \tag{7}$$

holds. Using (7), Lemma 2 and condition (4) we find

$$\begin{aligned}
\left\|f-R_n^{\beta}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} & \leq \frac{1}{P_n^{\beta}}\sum_{m=o}^{\lambda(n)}p_{n-m}^{\beta}\|f-s_m(\cdot,f)\|_{L_M(\mathbb{T},\omega)} \\
& = \frac{1}{P_n^{\beta}}\sum_{m=1}^{n}p_{n-m}^{\beta}O\left(m^{-\alpha}\right)\|f-s_m(\cdot,f)\|_{L_M(\mathbb{T},\omega)} \\
& \quad +\frac{p_n^{\beta}}{P_n^{\beta}}\|f-s_0(\cdot,f)\|_{L_M(\mathbb{T},\omega)} \\
& = \frac{1}{P_n^{\beta}}O\left(n^{-\alpha}P_n^{\beta}\right)+O\left(\frac{1}{n+1}\right)=O\left(n^{-\alpha}\right).
\end{aligned}$$

Case 2. Let $\alpha=1$. Since

$$R_n^{\beta}(x,f)=\frac{1}{P_n^{\beta}}\sum_{m=o}^{n}P_{n-m}^{\beta}A_m(x,f)$$

using Abel's transformation, we have

$$\begin{aligned}
& s_n(x,f) - R_n^\beta(x,f) \\
= & \frac{1}{P_n^\beta}\sum_{m=1}^{n}(P_n^\beta - P_{n-m}^\beta)A_m(x,f) \\
= & \frac{1}{P_n^\beta}\sum_{m=1}^{n}\left[\frac{P_n^\beta - P_{n-m}^\beta}{m} - \frac{P_n^\beta - P_{n-(m+1)}^\beta}{m}\right]\left(\sum_{k=1}^{m}kA_k(x,f)\right) \\
& + \frac{1}{n+1}\sum_{k=1}^{m}kA_k(x,f).
\end{aligned} \tag{8}$$

Then taking account of (8)

$$\begin{aligned}
& \left\|s_n(\cdot,f) - R_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} \\
\leq & \frac{1}{P_n^\beta}\sum_{m=1}^{n}\left|\frac{P_n^\beta - P_{n-m}^\beta}{m} - \frac{P_n^\beta - P_{n-(\mathbf{m}+1)}^\beta}{m}\right|\left\|\sum_{k=1}^{m}kA_k(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} \\
& + \frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}.
\end{aligned} \tag{9}$$

It is clear that if the Fourier series of f is

$$f(x) \sim \sum_{k=0}^{m} A_k(x,f),$$

then $\widetilde{f'}$ has the Fourier series

$$\widetilde{f'}\ (x) \sim \sum_{k=0}^{m} kA_k(x,f),$$

where $\widetilde{f'}$ is the conjugate function of $f' \in L_M(\mathbb{T},\omega)[16, p.166]$. Using boundedness of the partial sums and the conjugation operator in the space $L_M(T,\omega)[16, p.155]$, we get

$$\frac{1}{n+1}\left\|\sum_{k=1}^{n}kA_k(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} = \frac{1}{n+1}\left\|s_n(\cdot,\widetilde{f'})\right\|_{L_M(\mathbb{T},\omega)} = O(n^{-1}). \tag{10}$$

Thus, (9) and (10) yield

$$\begin{aligned}
& \left\|s_n(\cdot,f) - R_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} \\
\leq & \frac{1}{P_n^\beta}\sum_{m=1}^{n}\left|\frac{P_n^\beta - P_{n-m}^\beta}{m} - \frac{P_n^\beta - P_{n-(\mathbf{m}+1)}^\beta}{m}\right| O(1) + O(n^{-1}) \\
= & O(\frac{1}{P_n^\beta})\sum_{m=1}^{n}\left|\frac{P_n^\beta - P_{n-m}^\beta}{m} - \frac{P_n^\beta - P_{n-(\mathbf{m}+1)}^\beta}{m}\right| + O(n^{-1}).
\end{aligned} \tag{11}$$

According to [**42**] the following relations holds :

$$\sum_{m=1}^{n}\left|\frac{P_n^\beta-P_{n-m}^\beta}{m}-\frac{P_n^\beta-P_{n-(\mathbf{m}+1)}^\beta}{m}\right| = \frac{1}{n+1}O(P_n^\beta).$$

The last inequality and (11) imply that

$$\left\|s_n(\cdot,f)-R_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=O(n^{-1}). \tag{12}$$

By (12) and (7)

$$\begin{aligned}&\left\|f-R_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\ \leq\ &\|f-s_n(\cdot,f\|_{L_M(\mathbb{T},\omega)}+\left\|s_n(\cdot,f)-R_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=O(n^{-1}),\end{aligned}$$

which completes the proof. □

Proof of Theorem 3. Case 1. We suppose that $0<\alpha<1$. Since

$$N_n^\beta(f;x)=\frac{1}{P_n^\beta}\sum_{m=0}^{n}p_m^\beta s_m(f;x),$$

we can write

$$f(x)-N_n^\beta(f;x)=\frac{1}{P_n^\beta}\sum_{m=0}^{n}p_m^\beta\left\{f(x)-s_m(f;x)\right\}. \tag{13}$$

Use of (13) and (7) gives us

$$\begin{aligned}\left\|f-N_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)} &\leq \frac{1}{P_n^\beta}\sum_{m=0}^{n}p_m^\beta\left\|f-s_m(\cdot,f).\right\|_{L_M(\mathbb{T},\omega)}\\ &= O(\frac{1}{P_n^\beta})\sum_{m=1}^{n}p_m^\beta m^{-\alpha}+\frac{p_0^\beta}{P_n^\beta}\left\|f-s_0(\cdot,f).\right\|_{L_M(\mathbb{T},\omega)}\\ &= O(\frac{1}{P_n^\beta})\sum_{m=1}^{n}p_m^\beta m^{-\alpha}. \end{aligned}\tag{14}$$

Considering [**42**, p. 1583]

$$\sum_{m=1}^{n}p_m^\beta m^{-\alpha}=O(n^{-\alpha}P_n^\beta). \tag{15}$$

Taking into account the relations (14) and (15) we have

$$\left\|f-N_n^\beta(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=O(n^{-\alpha}).$$

Case 2. Let $\alpha=1$. Note that by Abel's transformation

$$\begin{aligned}N_n^\beta(f;x) &= \frac{1}{P_n^\beta}\left[\sum_{m=0}^{n-1}P_m^\beta\left\{s_m(x,f)-s_{m+1}(x,f)+P_n^\beta s_n(x,f)\right\}\right]\\ &= \frac{1}{P_n^\beta}\sum_{m=0}^{n-1}P_m^\beta(-A_{m+1}(x,f))+s_n(x,f). \end{aligned}\tag{16}$$

Taking account of (16)

$$N_n^{\beta}(x,f)-s_n(x,f)=-\frac{1}{P_n^{\beta}}\sum_{m=0}^{n-1}P_m^{\beta}A_{m+1}(x,f). \tag{17}$$

On the other hand by Abel's transformation [**42**, p. 1584] the equality

$$\begin{aligned}
&\sum_{m=0}^{n-1}P_m^{\beta}A_{m+1}(x,f)\\
=&\sum_{m=0}^{n-1}\frac{P_m^{\beta}}{m+1}(m+1)A_{m+1}(x,f)\\
=&\sum_{m=0}^{n-1}\left[\frac{P_m^{\beta}}{m+1}-\frac{P_{m-1}^{\beta}}{m+2}\right]\left[\sum_{k=0}^{m}(k+1)A_{k+1}(x,f)\right]\\
&+\frac{P_n^{\beta}}{n+1}\sum_{k=0}^{n-1}(k+1)A_{k+1}(x,f),
\end{aligned}$$

holds. Using the last equality, condition (5) and (10) we reach

$$\begin{aligned}
&\left\|\sum_{m=0}^{n-1}P_m^{\beta}A_{m+1}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
\leq&\sum_{m=0}^{n-1}\left|\frac{P_m^{\beta}}{m+1}-\frac{P_{m-1}^{\beta}}{m+2}\right|\left\|\sum_{k=0}^{m}(k+1)A_{k+1}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
&+\frac{P_n^{\beta}}{n+1}\left\|\sum_{k=0}^{n-1}(k+1)A_{k+1}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
=&\ O(1)\sum_{m=0}^{n-1}\left|\frac{P_m^{\beta}}{m+1}-\frac{P_{m-1}^{\beta}}{m+2}\right|+O\left(\frac{P_n^{\beta}}{n}\right)=O\left(\frac{P_n^{\beta}}{n}\right).
\end{aligned} \tag{18}$$

Consideration of (17) and (18) gives us

$$\begin{aligned}
&\left\|N_n^{\beta}(\cdot,f)-s_n(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
=&\ \frac{1}{P_n^{\beta}}\left\|\sum_{m=0}^{n-1}P_m^{\beta}A_{m+1}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=\frac{1}{P_n^{\beta}}O\left(\frac{P_n^{\beta}}{n}\right)=O(n^{-1}).
\end{aligned} \tag{19}$$

Taking the relation (19) and (7) into account we have

$$\begin{aligned}
&\left\|f-N_n^{\beta}(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}\\
\leq&\ \left\|f-s_n(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}+\left\|N_n^{\beta}(\cdot,f)-s_n(\cdot,f)\right\|_{L_M(\mathbb{T},\omega)}=O(n^{-1}).
\end{aligned}$$

The theorem is proved. $\square$

Acknowledgement. The author would like to thank the referee for all valuable advice and very helpful remarks.

References

[1] E. Acerbi and G. Mingione, *Regularity results for a class of functions with non-standart growth*, Arch. Ration. Mech. Anal., **156** (2001) 121–140.

[2] D.H. Armitage and I. J. Maddox, *A new type of Cesáro mean,* Analysis, **9** (1-2) (1989) 195–204.

[3] H.S. Arslan, *A note on* $|N, p_n^\alpha|$ *summability factors,* Soochow J. Math., **27** (1) (2001), 45–51.

[4] A. Böttcher and Yu.I. Karlovich, *Carleson Curves, Muckenhoupt Weights and Teoplitz Operators*, Birkhauser-Verlag, 1997.

[5] C. Bennett and Yu.I. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[6] D.W. Boyd, *Spaces between a pair of reflexive Lebesgue spaces,* Proc. Amer. Math. Soc., **18** (1967) 215–219.

[7] D.W. Boyd, *Indices of function spaces and their relationship to interpolation*, Canad. J. Math., **21** (1969) 1245–1254.

[8] D.W. Boyd, *Indices for the Orlicz spaces,* Pacific J. Math., **38** (1971) 315–325.

[9] M. Colombo and G. Mingione, *Regularity for double phase variational problems*, Arch. Ration. Mech. Anal.,**215** (2) (2015) 443–496.

[10] P. Chandra, *Trigonometric approximation of functions in* L_p-*norm,* J. Math. Anal. Appl., **277** (1) (2002) 13–26.

[11] P. Chandra, *Approximation by Nörlund operators*, Mat. Vesn., **38** (1986) 263–259.

[12] P. Chandra, *A note on degree of approximation by Nörlund and Riesz operators*, Mat. Vesn., **42** (1990) 9–10.

[13] A. Guven, *Trigonometric approximation of functions in weighted* L^p *spaces* ,Sarajevo J. Math., **5** (17) (2009), 99–108.

[14] A. Guven and D.M. Israfilov, *Approximation by means of Fourier trigonometric series in weighted Orlicz spaces*, Adv. Stud. Contemp. Math. (Kyundshang)., **19** (2) (2009) 283–295.

[15] A. Guven and D.M. Israfilov, *Trigonometric approximation in generalized Lebesgue spaces* $L^{p(x)}$, J. Mat. Inequal., **4** (2) (2010) 285–299.

[16] D.M. Israfilov and A. Guven, *Approximation by trigonometric polynomials in weighted Orlicz spaces*, Studia Math., **174** (2) (2006) 147–168.

[17] S.Z. Jafarov, *Approximation by Fejér sums of Fourier trigonometric series in weighted Orlicz spaces,* Hacet, J. Math. Stat., **42** (3) (2013) 259–268.

[18] S.Z. Jafarov, *Approximation by linear summability means in Orlicz spaces,* Novi Sad J. Math., **44** (2) (2014) 161–172.

[19] S.Z. Jafarov, *Linear methods of summing Fourier series and approximation in weighted variable exponent Lebesgue spaces*, Ukrainian Math. J., **66** (10) (2015) 1509–1518.

[20] S.Z. Jafarov, *Approximation of periodic functions by Zygmund means in Orlicz spaces*, J. Classical Anal, **9** (1) (2016), 43–52.

[21] S.Z. Jafarov, *Approximation of functions by de la Vallee-Poissin sums in weighted Orlicz spaces,* Arab J. Math., **5** (2016) 125–137.

[22] M.A. Krasnoselskii and Ya.B. Rutickii, *Convex Functions and Orlicz Spaces,* P. Norrdhoff Ltd., Groningen, 1961.

[23] N.X. Ky, *Moduli of mean smoothness and approximation mith A_p-weights,* Ann. Univ. Sci. Budap., **40** (1997) 37–48.

[24] L. Leindler, *Trigonometric approximation in L_p- norm,* J. Math. Anal. Appl., **302** (1) (2005) 129–136.

[25] V. Kokilashvili and S.G. Samko, *Operators of harmonic analysis in weighted spaces with non-standard growth*, J. Math. Anal. Appl., **352** (2009) 15–34.

[26] V. Kokilashvili and Ts. Tsanava, *On the normestimate of deviation by linear summability means and an extension of the Bernstein inequality,* Proc. A. Razmadze Math Inst., **154** (2010) 144–146.

[27] A.Yu. Karlovich, *Algebras of singular integral operators with piecewise continuous coefficients on reflexive Orlicz spaces,* Math. Nachr., **179** (1996) 187–222.

[28] A.Yu. Karlovich, *Singular integral operators with PC coefficients in reflexive rearrangement invariant spaces,* Integ. Eq. and Oper. Th., **32** (1998) 436–481.

[29] W. Matuszewska and W. Orlicz, *On certain properties of ϕ-functions*, Bull. Acad. Polon. Sci., Ser. Math. Aster. et Phys., **8** (7) (1960) 439–443.

[30] L. Maligranda, *Indices and interpolation,* Dissertationes Math. (Rozprawy Mat.), **234** (1985) 1–45.

[31] W.A. Majewski and L.E. Labuschange, *On application of Orlicz spaces to statistical physics*, Ann. Henri Poincare, **15** (2014) 1197–1221.

[32] B. Muchenhoupt, *Weighted norm inequalities for the Hardy maximal function,* Trans. Amer. Math. Soc., **165** (1972) 207–226.

[33] R.N. Mohapatra and D.C. Russell, *Some direct and inverse theorems in approximation of functions*, J. Aust. Math. Soc., (Ser. A), **34** (2) (1983) 143–154.

[34] M.I. Mittal and B.F. Rhoades, *On degree of approximation of continuous functions by using linear operators on their Fourier series,* International Journal of Mathematics, Game Theory, and Algebra., **9** (4) (1999) 259–267.

[35] M.L. Mittal, B.F. Rhoades, S. Sonker, and U. Singh, *Approximation of signals of class $Lip(\alpha,p)$ by linear operators,* Applied Mathematics an Computation., **217** (9) (2011) 4483–4489.

[36] M.L. Mittal and M.V. Singh, *Approximation of signals (functions) by trigonometric polynomials in L_p-norm,* Hindawi Publishing Corporation, International Journal of Mathematics and Mathematical Sciences., **2014**, Article ID 267383, 6 pages.

[37] E.S. Quade, *Trigonometric approximation in the mean*, Duke Math. J., **3** (3) (1937) 529–542.

[38] J.A. Osikiewicz, *Equivalence results for Cesáro submethods*, Analysis (Münich) **20** (1) (2000) 35–43.

[39] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces,* Marcel Dekker, New York, 1991.

[40] S.B. Stechkin, *The approximation of periodic functions by Fejér sums,* Trudy Math. Inst. Steklov., **G2** (1961) 522–523 (in Russian).

[41] A. Swierczewska-Gwiazda, *Nonlinear parabolic problems in Musielak- Orlicz spaces,* Nonlinear Anal., **98** (2014) 48–65.

[42] S. Sonker and U. Singh, *Approximation of signals (functions) belonging to $Lip(\alpha,p,\omega)-$class using trigonometric polynomials*, Procedia Engineering, **38** (2012) 1575–1585.

[43] M.F. Timan, *Best approximation of a function and linear methods of summing Fourier series*, Izv. Akad. Nauk SSSR Ser: Math., **29** (1965) 587–604.

[44] Y.E. Yıldırır and A.H. Avsar, *Approximation of periodic functions in weighted Lorentz spaces,* Sarajevo J. of Math., **13** (25-1) (2017) 49–60.

[45] A. Zygmund, *Trigonometric Series, vol. I and II,* Cambridge, 1959.

Department of Mathematics and Science Education,
Faculty of Education,
Muş Alparslan University,
49250, Muş,
Turkey.

Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan,
9 B. Vahabzadeh str.,
AZ 1141, Baku,
Azerbaijan
s.jafarov@alparslan.edu.tr