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ON LEVELS OF FAST ESCAPING SETS AND SPIDER'S WEB OF TRANSCENDENTAL ENTIRE FUNCTIONS

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Abstract. Let f be a transcendental entire function and let $I(f)$ be the points which escape to infinity under iteration. Bergweiler and Hinkkanen introduced the fast escaping sets $A(f)$ and subsequently, Rippon and Stallard introduced 'Levels' of fast escaping sets $A_R^L(f)$. These sets under some restriction have the properties of "infinite spider's web" structure. Here we give some topological properties of the infinite spider's web and show some of the transcendental entire functions whose levels of the fast escaping sets have infinite spider's web structure.

1. Introduction

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a transcendental entire function. For $n \in \mathbb{N}$, f^n denotes the n^{th} iteration of f. Thus $f^n(z) = f(f^{n-1})(z)$, where $f^0(z) = z$ and $n = 1, 2, \ldots$
A family of functions $\mathfrak{F}$ is said to be a normal family, if every infinite sequence in the family has a subsequence which converges locally uniformly. The Fatou set $F(f)$ is defined to be the set of points $z \in \mathbb{C}$, such that$(f^n)_{n\in\mathbb{N}}$ forms a normal family in some neighbourhood of z. The complement of $F(f)$ denoted by $J(f)$ is called the Julia set. Clearly the Fatou set is open while the Julia set is closed. Also for a transcendental entire function it is known that the Julia set is unbounded. Baker [**1**] proved that the Julia set coincides with the closure of repulsive periodic points. For an introduction to the other properties of these sets one can refer, for instance, [**2**], [**7**].

For a transcendental entire function f, Eremenko [**4**] defined the escaping set as:

$$I(f) = \{z \in \mathbb{C} : f^n(z) \longrightarrow \infty \text{ as } n \longrightarrow \infty\} \tag{1.1}$$

and proved that

$$I(f) \cap J(f) \neq \emptyset, \partial I(f) = J(f)$$

and

$$\text{all the components of } \overline{I(f)} \text{ are unbounded.}$$

He further conjectured that all the components of $I(f)$ are unbounded. This led to a rich development of the field. Some partial results in the confirmation of this conjecture have been obtained, see for instance, [**9**], [**10**] and [**11**].

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The rate at which the points of $I(f)$ escape to infinity also plays an important role. Bergweiler and Hinkkanen in [**3**] defined the fast escaping set, denoted by $A(f)$ as,

$$A(f) = \{z \in I(f) : \text{ there exists } L \in \mathbb{N} \text{ such that } |f^{n+L}(z)| \geq M^n(R,f), \text{ for n} \in \mathbb{N}\}, \tag{1.2}$$

where

$$M(r,f) = \max_{|z|=r} |f(z)|, r > 0$$

and $M^n(r,f)$ is the n^{th} iterate of $M(r,f)$ with respect to variable r, $R > 0$ is such that $M(r,f) > r$ for $r \geq R$, so that $M^n(r,f) \to \infty$ as $n \to \infty$.
Clearly $A(f) \subset I(f)$. It was shown in [**3**] that $A(f) \neq \emptyset$ and also that, $\partial A(f) = J(f)$.

An alternative to $A(f)$, which has a geometric flavour was defined by Rippon and Stallard [**10**]. They defined:

$$B(f) = \{z : \text{ there exists } L \in \mathbb{N} \text{ such that } f^{n+L}(z) \notin \mathcal{T}(f^n(D)), \text{ for } n \in \mathbb{N}\} \tag{1.3}$$

where D is any open disc meeting $J(f)$ and $\mathcal{T}(D)$ denotes the union of domain D with all of it's bounded complementary components.
They showed that $B(f)$ is independent of D, completely invariant, $B(f^p) = B(f)$ for $p \in \mathbb{N}$ and finally $B(f) = A(f)$. They further showed that all the components of $A(f)$ are unbounded and so $I(f)$ has at least one unbounded component.

In [**11**], Rippon and Stallard defined subsets of $A(f)$ called levels of $A(f)$, defined below. This lead to simplification of proofs of several earlier results and new insight into the properties of $A(f)$. See for instance [**8**], [**11**].

Definition 1.1. *Let f be a transcendental entire function and $R > 0$ be such that $M(r,f) > r$ for $r \geq R$. Let $L \in \mathbb{Z}$, then the L^{th} level (with respect to variable R) is defined to be*

$$A_R^L(f) = \{z : |f^n(z)| \geq M^{n+L}(R,f), n \in \mathbb{N}, \ \ n + L \geq 0\}. \tag{1.4}$$

Clearly

$$A(f) = \cup_{L \in \mathbb{N}} A_R^{-L}(f) \tag{1.5}$$

and thus $A_R^{-L}(f) \subset A_R^{-(L+1)}(f)$ for $L \in \mathbb{N}$. They denote the 0^{th} level $A_R^0(f)$ by $A_R(f)$ thus

$$A_R(f) = \{z : |f^n(z)| \geq M^n(R,f)\}.$$

For such $A_R(f)$ Rippon and Stallard [**11**] proved the following:

Theorem A. Let f be a transcendental entire function and $n \in \mathbb{N}$. If $R > 0$ is sufficiently large then

$$A_R(f) \subset A_R(f^n) \ \subset A_{R/2}(f).$$

Here we supplement the above result by proving the next theorem, which deals with the levels of higher order. We observe that the above result cannot be generalized to each Level, i.e., the set relation

$$A_R^L(f) \subset A_R^L(f^n) \ \subset A_{R/2}^L(f)$$

need not hold true for all $n \in \mathbb{N}$. However we can prove the following:

Theorem 1.1. *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L, n \in \mathbb{N}$ then*
$$A_R^{nL}(f) \subset A_R^n(f^L) \subset A_{R/2}^{nL}(f)$$
and also
$$A_R^{nL}(f) \subset A_R^L(f^n) \subset A_{R/2}^{nL}(f).$$
An interesting observation of the above theorem is that, if p is composite number having two different factorizations, say $p = p_1.q_1 = p_2.q_2$, then
$$A_R^p(f) \subset A_R^{p_1}(f^{q_1}) \subset A_{R/2}^p(f)$$
and also
$$A_R^p(f) \subset A_R^{p_2}(f^{q_2}) \subset A_{R/2}^p(f).$$
Note also that $A_R^{p_1}(f^{q_1})$ need not equal $A_R^{p_2}(f^{q_2})$.

Rippon and Stallard [**11**] first observed that $A(f), A_R(f)$ have interesting and intricate structure and called it as infinite spider's web, which is defined below.

Definition 1.2. *A set E is an (infinite) spider's web if E is connected and there exists a sequence of bounded simply connected domains $(G_n)_{n\in\mathbb{N}}$ with $G_n \subset G_{n+1}$ for $n \in \mathbb{N}$, $\partial G_n \subset E$, for $n \in \mathbb{N}$ and $\cup_{n\in N} G_n = \mathbb{C}$.*

Several examples of functions having $A_R(f)$ as spider's web have been given. Sixsmith [**13**] gave examples of transcendental entire functions for which spider's web formation is there. Some work on spider's web has also been done by Helena and Peter [**5**] and Osborne[**8**]. In their paper [**11**], Rippon and Stallard showed that for sufficiently large R, if $A_R(f)^c$ has a bounded component then each of $A_R(f), A(f), I(f)$ is a spider's web. Also if f is a multiply connected Fatou component, then for sufficiently large R, $A_R(f), A(f), I(f)$ are all spider's web. It is thus natural to look for relations between spider's web and levels of fast escaping sets, which we consider here. Also we study the topological properties of spider's web and show certain composite transcendental entire function h having the structure of $A_R(h)$ as spider's web.

We begin with a small but important observation that a bounded set can not be a spider's web. For suppose S is a bounded set and it also forms a spider's web, then there exits a positive constant K such that $|z| \leq K$, for all $z \in S$, and there exits a sequence of bounded simply connected domain G_n such that $G_n \subset G_{n+1}$ for $n \in \mathbb{N}, \partial G_n \subset S$ and $\cup_{n\in\mathbb{N}} G_n = \mathbb{C}$. Since $\cup_{n\in\mathbb{N}} G_n = \mathbb{C}$ and G_n are bounded, there exists $G_t, t \in \mathbb{N}$ whose boundary contains z_0 such that $|z_0| > K$, thus boundary of G_t will not be a part of S, which is contradiction to $\partial G_n \subset S$ for $n \in \mathbb{N}$.

Theorem 1.2. *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L, n \in \mathbb{N}$ then $A_R^{nL}(f)$ is a spider's web if and only if $A_R^n(f^L)$ is a spider's web.*

The next Theorem deals with the union of spider's web.

Theorem 1.3. *Let $S_i,\ i = 1, 2, 3, \ldots, n$ be spider's web, then $\cup_{i=1}^n S_i$ is also a spider's web.*

Note that intersection of two spider's web need not form spider's web. Also continuous image of a spider's web need not form spider's web, for let $f : \mathbb{C} \to \mathbb{C}$ be a non constant continuous map defined by

$$\begin{aligned} f(0) &= 0, \\ f(re^{i\theta}) &= re^{i\theta} \ \text{ for } 0 < r \leq 1, \ \ 0 < \theta \leq 2\pi, \\ f(re^{i\theta}) &= e^{i\theta} \quad \text{for } r > 1, \ \ 0 < \theta \leq 2\pi. \end{aligned}$$

Then clearly f is a continuous map and if S be any spider's web, then $f(S)$ is not a spider's web being bounded. However if we take f to be continuous open surjective map which maps bounded domain to bounded domain, then we have following theorem. In particular if f is a transcendental entire function without any exceptional value Picard, and without any asymptotic values, then also the image of a spider's web will be a spider's web.

Theorem 1.4. *If $f : \mathbb{C} \longrightarrow \mathbb{C}$ be open continuous and a surjective map. Further let f map every bounded domain to bounded domain. If S be a spider's web, then $f(S)$ is also a spider's web.*

Corollary 1.1. *Let f be a transcendental entire function with no finite asymptotic value and no exceptional value Picard. If S is a spider's web then so is f(S).*

2. Proofs of Theorems on Levels of Fast Escaping Sets and Spider's Web

For the proof of Theorem 1.1 we need following lemmas.

Lemma 2.1. *For any $L, K \in \mathbb{N}$ and sufficiently large R,*

$$M^{KL}(R, f) \geq M^K(R, f^L). \tag{2.1}$$

The proof is an immediate consequence of Maximum modulus principle and the following lemma of Rippon and Stallard [**11**].

Lemma 2.2. *Let f be a transcendental entire function and let $D = \{z : |z| < R\}$. If $R > 0$ be sufficiently large then for $n \in \mathbb{N}$*

$$\{z : |z| \leq M^n(R/2, f)\} \subset \{z : |z| \leq M(R, f^n)\} \subset \{z : |z| \leq M^n(R, f)\}. \tag{2.2}$$

Proof of Theorem 1.1. Let $R > 0$ be such that $M(r, f) > r$, for $r \geq R, L, n \in \mathbb{N}$. If $z \in A_R^{nL}(f)$, then

$$|f^m(z)| \geq M^{m+nL}(R, f), \text{ for } m \in \mathbb{N}.$$

So $|(f^L)^m(z)| = |f^{mL}(z)| \geq M^{mL+nL}(R, f) = M^{(m+n)L}(R, f) \geq M^{m+n}(R, f^L)$, for $m \in \mathbb{N}$ by Lemma 2.1. Hence $z \in A_R^n(f^L)$, and so

$$A_R^{nL}(f) \subset A_R^n(f^L). \tag{2.3}$$

Next let $z \in A_R^n(f^L)$, then

$$|(f^L)^m(z)| \geq M^{m+n}(R, f^L) \geq M^{(m+n)L}(R/2, f), \ \text{ for } \ m \in \mathbb{N}$$

by Lemma 2.2. So we must have

$$|f^m(z)| \geq M^{m+nL}(R/2, f), \ \text{ for } \ m \in \mathbb{N}.$$

Hence $z \in A^{nL}_{R/2}(f)$. Thus

$$A^n_R(f^L) \subset A^{nL}_{R/2}(f). \tag{2.4}$$

From (2.3) and (2.4) we have

$$A^{nL}_R(f) \subset A^n_R(f^L) \subset A^{nL}_{R/2}(f).$$

The second set relation follows on similar lines. □

For proving Theorem 1.2 we need following lemmas:

Lemma 2.3. [**11**] *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L \in \mathbb{Z}$.*

(a) If G is a bounded component of $A^L_R(f)^c$ then $\partial G \subset A^L_R(f)$ and f^n is a proper map of G on to a bounded component of $A^{n+L}_R(f)^c$ for each $n \in \mathbb{N}$.
(b) If $A^L_R(f)^c$ has a bounded component, then $A^L_R(f)$ is a spider's web and hence every component of $A^L_R(f)^c$ is bounded.
(c) $A_R(f)$ is a spider's web if and only if $A^L_R(f)$ is a spider's web.
(d) Let $R^{'} > R$ then $A_R(f)$ is a spider's web if and only if $A_{R'}(f)$ is a spider's web.

As a consequence of Lemma 2.3 we have the following Lemma:

Lemma 2.4. *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \geq R$ and $L \in \mathbb{Z}$.*

(a) If $A^L_R(f)^c$ has a bounded component then, $A^m_R(f)$ is a spider's web, for all $m \geq L + 1$.
(b) Let $R^{'} > R$ then $A^L_R(f)$ is a spider's web if and only if $A^L_{R'}(f)$ is a spider's web.

Proof. (a) If $A^L_R(f)^c$ has a bounded component then so has $A^{L+1}_R(f)^c$, by Lemma 2.3(a). So $A^{L+1}_R(f)$ is spider's web, by Lemma (2.3)(b). In a similiar way we can prove that $A^{L+2}_R(f), A^{L+3}_R(f), \dots$ all will be spider's web. In general $A^m_R(f)$, for all $m \geq L + 1$ will be spider's web.
(b) Proof is immediate from 2.3 (c) and (d). □

Proof of Theorem 1.2. Firstly suppose that $A^{nL}_R(f)$ is a spider's web. It follows from Lemma 2.3(b) that each component of $A^{nL}_R(f)^c$ is bounded.
We know by Theorem 1.1 that $A^{nL}_R(f) \subset A^n_R(f^L)$. So each component of $A^n_R(f^L)^c$ is bounded. Thus by Lemma 2.3 (b), $A^n_R(f^L)$ is a spider's web. Conversely let us suppose that $A^n_R(f^L)$ is a spider's web. If $R^{'} > 2R$ for R sufficiently large, then we have by Theorem 1.1, $A^n_{R'}(f^L) \subset A^{nL}_{R'/2}(f)$. Now from Lemma 2.4 (b), $A^n_{R'}(f^L)$ is a spider's web. Hence by Lemma 2.3 (b), it follows that every component of $A^n_{R'}(f^L)^c$ is bounded. So every component of $A^{nL}_{R'/2}(f)^c$ is bounded. So by Lemma 2.3(b), $A^{nL}_{R'/2}(f)$ is a spider's web. Hence by Lemma 2.4 (b), $A^{nL}_R(f)$ is a spider's web. □

Before we prove next theorem we shall introduce a new notation. If E is a spider's web, then by Definition 1.2 there exist sequence of bounded simply connected domains G_n with $G_n \subset G_{n+1}$, for $n \in \mathbb{N}, \partial G_n \subset E$, for $n \in \mathbb{N}$ and $\cup_{n \in N} G_n = \mathbb{C}$. It is quite possible that there might exist more than one sequence of such domains say $(H_m)_{m\in\mathbb{N}}$ satisfying the Definition 1.2. In order to distinguish the spider's web with corresponding domains we shall use the notation $(E, G_n)_{n\in\mathbb{N}}$ and $(E, H_m)_{m\in\mathbb{N}}$ respectively.

Proof of Theorem 1.3. Using induction it is sufficient to prove the theorem for two spider's webs. Let $(S, G_n)_{n\in\mathbb{N}}$ and $(T, H_m)_{m\in\mathbb{N}}$, be two spider's webs.
Now consider G_1. Then there exits some H_m such that $G_1 \bigcap H_m \neq \emptyset$. This is possible since $\cup_{m\in\mathbb{N}} H_m = \mathbb{C}$. Let k be the smallest positive integer such that $G_1 \bigcap H_k \neq \emptyset$.
Let $\mathcal{T}(D)$ denotes the union of domain D with all it's bounded complementary components, and define $K_n = \mathcal{T}(G_n \cup H_{k+(n-1)})$, $n = 1, 2, \ldots$
Then clearly $\cup_{n=1}^{\infty} K_n \supset \cup_{n=1}^{\infty} G_n = \mathbb{C}$, and for $n = 1, 2, \ldots,$ G_n and $H_{k+(n-1)}$ are simply connected and $G_n \cap H_{k+(n-1)} \supset G_1 \cap H_k \neq \emptyset$, so $\mathcal{T}(G_n \cup H_{k+(n-1)})$ is simply connected. Thus K_n is simply connected as well as bounded domain being union of two bounded domains. Further
$K_{n+1} = \mathcal{T}(G_{n+1} \cup H_{k+n}) \supset \mathcal{T}(G_n \cup H_{k+(n-1)}) = K_n$ and
$\partial K_n = \partial\mathcal{T}((G_n \cup H_{k+(n-1)})) \subset \partial(G_n) \cup \partial(H_{k+(n-1)}) \subset S \cup T$.
Thus $(S \cup T, K_n)_{n\in\mathbb{N}}$ is also a spider's web. □

For proving the Theorem 1.4 we shall need the following lemma.

Lemma 2.5. *(a) If A_1 and A_2 are domains such that $A_1 \subset A_2$, then*

$$\mathcal{T}(A_1) \subset \mathcal{T}(A_2)$$

and hence

$$\mathcal{T}(A_1) \cup \mathcal{T}(A_2) = \mathcal{T}(A_1 \cup A_2).$$

(b) Let $(A_n)_{n\in\mathbb{N}}$ be a family of domains, with $A_1 \subset A_2 \subset A_3 \subset \ldots$ then

$$\cup_{n\in\mathbb{N}} \mathcal{T}(A_n) = \mathcal{T}(\cup_{n\in\mathbb{N}} A_n).$$

Proof. For (a), Let $z \in \mathcal{T}(A_1)$, if $z \in A_1$ then $z \in A_2 \subset \mathcal{T}(A_2)$. If z is in the bounded complementary component of A_1, it is sufficient to show that z does not belong to any unbounded complementary component of A_2. For suppose it does. Then there exits an arc in A_2^c joining z to ∞. As $A_1 \subset A_2$, so this arc lies in A_1^c. Consequently z lies in unbounded complementary component of A_1. This contradiction proves (a). The other results are simple set theoretic consequences of (a). □

Note: The conditions imposed on A_1, A_2 in Lemma 2.5 are necessary, for instance, let

$$\begin{aligned} A_1 =& \{(x,y) : -2 \leq x \leq 1,\ -2 \leq y \leq -1\} \cup \{(x,y) : -2 \leq x \leq 1,\ 1 \leq y \leq 2\} \cup \\ & \{(x,y) : -2 \leq x \leq -1,\ -1 \leq y \leq 1\} \end{aligned}$$

and

$$\begin{aligned} A_2 =& \{(x,y) : 0 \leq x \leq 2,\ -2 \leq y \leq -1\} \cup \{(x,y) : 0 \leq x \leq 2,\ 1 \leq y \leq 2\} \cup \\ & \{(x,y) : 1 \leq x \leq 2,\ -1 \leq y \leq 1\}. \end{aligned}$$

Then

$$\mathcal{T}(A_1) = A_1 \text{ and } \mathcal{T}(A_2) = A_2,$$

whereas

$$\mathcal{T}(A_1 \cup A_2) = \{(x,y) : -2 \leq x \leq 2,\ -2 \leq y \leq 2\},$$

so that $\mathcal{T}(A_1) \cup \mathcal{T}(A_2) \neq \mathcal{T}(A_1 \cup A_2)$.

Proof of Theorem 1.4. Let $(S, S_n)_{n\in\mathbb{N}}$ be a spider's web. Let $H_n = \mathcal{T}(f(S_n))$ and denote $H = f(S)$. Clearly H_n are bounded simply connected domains being open continuous image of bounded simply connected domains. Now $\partial H_n = \partial(\mathcal{T}(f(S_n))) \subset f(\partial S_n) \subset f(S) = H$. Further $H_n = \mathcal{T}(f(S_n)) \subset \mathcal{T}(f(S_{n+1})) = H_{n+1}$ for $n \in \mathbb{N}$, by Lemma 2.5. Hence in order to show that $f(S)$ is a spider's web it only remains that $\cup_{n=1}^{\infty} H_n = \mathbb{C}$.
For this consider any $z \in \mathbb{C}$, then there exists some $\zeta \in \mathbb{C}$ such that $z = f(\zeta)$, f being surjective. Also as $\cup_{n\in\mathbb{N}} S_n = \mathbb{C}$, it follows that $\zeta \in S_n$ for some $n \in \mathbb{N}$ and consequently $z = f(\zeta) \in f(S_n) \subset \mathcal{T}(f(S_n)) = H_n$. Thus $\cup_{n\in\mathbb{N}} H_n = \mathbb{C}$.
So $(f(S), H_n)_{n\in\mathbb{N}}$ is a spider's web. □

Proof of Corollary 1.1. Being analytic, the function f is open and continuous. Also f would map bounded domain to a bounded domain, for suppose D is a bounded domain with $f(D)$ unbounded, then there exists a curve Γ tending to ∞ in $f(D)$ and consequently a curve γ in D tending to some α in $\overline{D}$ such that $f(\gamma) = \Gamma$, so that α is an asymptotic value for f, contradicting f by hypothesis, has no asymptotic value.

Next if z is not exceptional value Picard for f, then there exist (infinitely many) $\xi \in \mathbb{C}$ such that $f(\xi) = z$, and the proof now follows as in the previous theorem. □

3. Bounded Fatou Components and Spider's Web

Regularity conditions and growth of transcendental entire function also play an important role in transcendental dynamics. In this section we discuss a result related to growth and regularity of transcendental entire function. We start with the following well known definitions of order ρ_f and lower order λ_f of entire functions respectively given by:

$$\rho_f = \overline{\lim}_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f = \underline{\lim}_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Clearly we have $0 \le \lambda_f \le \rho_f \le \infty$ and given any $\rho, (0 \le \rho \le \infty)$, there exists an entire function of order ρ. We shall also need the following regularity condition:

Definition 3.1. [**11**] *Let $c > 0$. A transcendental entire function f is said to be* log*-regular with constant c, if the function $\phi(t) = M(e^t, f)$ satisfies*

$$\frac{\phi'(t)}{\phi(t)} \ge \frac{1+c}{t}, \qquad \text{for large } t.$$

Further (see [**13**]) if a function f is log-regular then it satisfies Lemma 3.2 (b) (mentioned below) for all $m > 1$.

We shall also need few lemmas from [**6**], [**11**] and [**13**].

Lemma 3.1. [**11**] *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \ge R$ and let $A_R(f)$ be a spider's web, then f has no unbounded Fatou components*

Lemma 3.2. [**13**] *Let f be a transcendental entire function. Let $R > 0$ be such that $M(r, f) > r$ for $r \ge R$. Then $A_R(f)$ is a spider's web if for some $m > 1$,*

(a) *there exists* $R_0 > 0$ *such that for all* $r \geq R_0$ *there is a simply connected domain G=G(r) with*

$$B(0,r) \subset G \subset B(0,r^m) \text{ and } |f(z)| \geq M(r,f) \text{ for z} \in \partial\text{G}. \tag{3.1}$$

(b) f *has a regular growth in the sense that there exists a sequence* $(r_n)_{n\geq 0}$ *with*

$$r_n > M^n(R,f) \text{ and } M(r_n,f) \geq r_{n+1}^m \text{ for n} \geq 0. \tag{3.2}$$

Lemma 3.3. [**13**] *Let* $f_1, f_2, \ldots, f_k$ *be non constant transcendental entire functions. Suppose that for all* $j \in \{1,2,\ldots,k\}$, f_j *satisfies Lemma 3.2 (a) with* $m = m_j > 1$. *Let* $g = f_1 \circ f_2 \circ \cdots \circ f_k$, *then* g *satisfies Lemma 3.2 (a) with* $m = m_1 m_2 \ldots m_k$.

Lemma 3.4. [**6**] *If* f *is a transcendental entire function with finite order and positive lower order then* f *is* log *regular.*

Lemma 3.5. [**13**] *If* f *is a transcendental entire function with order less than 1/2, then* f *satisfies Lemma 3.2 (a) for some* $m > 1$.

Lemma 3.6. [**13**] *Let* $f_1, f_2, \ldots, f_k$ *be non constant transcendental entire functions. Suppose that for some* $j \in \{1,2,\ldots,k\}$, f_j *is log-regular. Then* $g = f_1 \circ f_2 \circ \cdots \circ f_k$ *is also log-regular.*

Theorem 3.1. *Let* $h = f_1 \circ f_2 \circ \cdots \circ f_n$ *where* $f_i (i = 1,2,\ldots,n)$ *are transcendental entire functions, each having order less than 1/2. If there is a number* $j \in \{1,2,3,\ldots,n\}$, *such that* f_j *has positive lower order, then* $A_R(h)$ *is a spider's web.*

Remark 3.1. (i) If further $0 < \rho_{f_k} < \frac{1}{2}$ for $k \neq j$ above, then by a well known theorem of Polya, h is of infinite order, and thus this gives an example of a transcendental entire function h of infinite order for which $A_R(h)$ is a spider's web.

(ii) Theorem 3.1 with Lemma 3.1 immediately yields Theorem B of [**12**].

Proof. In order to prove the Theorem, it is sufficient to show that h satisfies the two conditions of Lemma 3.2. Now by Lemma 3.5, each f_i will satisfy Lemma 3.2(a) with $m = m_i > 1$. Hence h satisfies Lemma 3.2(a) by Lemma 3.3 with $m = m_1 m_2 \ldots m_n$.

Also for some $j \in \{1,2,\ldots,n\}$ the lower order λ_j of f_j is greater than zero and so f_j is log regular, by Lemma 3.4. Hence h is log regular by Lemma 3.6, and so $A_R(h)$ is a spider's web by Lemma 3.2. □

References

[1] I.N. Baker, *Repulsive fixed points of entire functions*, Math. Z., **104** (1968), 252–256.

[2] W. Bergweiler, *Iteration of meromorphic functions*, Bull. Amer. Math. Soc., **29** (1993), 151–188.

[3] W. Bergweiler and A. Hinkkanen, *On semiconjugation of entire functions*, Math. Proc. Camb. Phil. Soc., **126** (1999), 565–574.

[4] A.E. Eremenko, *On the iteration of entire functions*, from *Dynamical Systems and Ergodic Theory*, Banach Center Publications 23, Polish Scientific Publishers, Warsaw, 1989, 339–345.

[5] H. Mihaljević-Brandt and J. Peter, *Poincaré functions with spiders' webs*, Proc. Amer. Math. Soc. **140** (2012), 3193–3205.

[6] A. Hinkkanen, *Entire functions with bounded Fatou components,* from *Transcendental Dynamics and Complex Analysis*, London Math. Soc. 348, Lecture Note Ser., Cambridge University Press, 2008, 187–216.

[7] S. Morosawa, Y. Nishimura, M. Taniguchi, and T. Ueda, *Holomorphic Dynamics,* Cambridge Studies in Advanced Mathematics, 66, 2000.

[8] J.W. Osborne, *The structure of spider's web fast escaping sets,* Bull. London Math. Soc., **3** (44) (2012), 503–519.

[9] L. Rempe, *On a question of Eremenko concerning escaping components of entire functions*, Bull. London Math. Soc., **39** (2007), 661–666.

[10] P.J. Rippon and G.M. Stallard, *On questions of Fatou and Eremenko*, Proc. Amer. Math. Soc., **133** (2005), 1119–1126.

[11] P.J. Rippon and G.M. Stallard, *Fast escaping points of entire functions,* Proc. London Math. Soc., **105** (2012), 787–820.

[12] A.P. Singh, *Composite entire functions with no unbounded Fatou components*, J. Math. Anal. Appl. **335** (2) (2007), 907–914.

[13] D.J. Sixsmith, *Entire functions for which the escaping set is a spider's web,* Math. Proc. Camb. Phil. Soc., **151** (2011), 551–571.

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