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A NOTE ON CANCELLATION LAW FOR $p-$CONVEX SETS

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Abstract. In this note we generalize a result of R. Urbański from paper [**2**] which states that for subsets A, B, C of topological vector space X the following implication holds
$$A + B \subset B + C \Longrightarrow A \subset C$$
provided that B is bounded and C closed and convex.

1. Introduction

In this paper the symbols $\mathbb{N}, \mathbb{R}$ mean respectively the sets of natural and reals numbers and by X we denote a real topological vector space. For a subsets A, B of the space X we can define so-called *Minkowski addition of sets* as
$$A + B = \{a + b : a \in A, b \in B\}$$
and *scalar multiplication* by
$$\lambda A = \{\lambda a : a \in A\}$$
where $\lambda \in \mathbb{R}$.

We say that a subset A of X is $p-$*convex* with some $0 < p \leq 1$, if for any $x, y \in A$ and $s, t \in \mathbb{R}$ such that $0 \leq s, t \leq 1$ and $s^p + t^p = 1$ we have $sx + ty \in A$.
It is an obvious fact that if A is $p-$*convex* set and $a_1, a_2, ..., a_j \in A$, $t_1, t_2, ..., t_j > 0$ are such that $t_1^p + t_2^p + ...t_j^p = 1$ then the element
$$t_1a_1 + t_2a_2 + ... + t_ja_j$$
belongs to the set A for any $j \in \mathbb{N}$.
Moreover it is easy to see that convex sets are exactly $1-$convex sets in this notion.

We say that that a subset A of the topological vector space X satisfies $p-$*condition,* $0 < p \leq 1$, if A is $p-$convex and
$$n^{-1+\frac{1}{p}} A \subset A$$
for every $n \in \mathbb{N}$.
Notice that the sets which satisfies $1-$*condition* are exactly the convex subsets of the space X.

In the paper [**3**] R. Urbański show that the *cancellation law* ie.
$$A + B \subset B + C \Longrightarrow A \subset C$$

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in general does not hold true for $p-$convex sets.

In the next section in Theorem 1, we generalize the result of R. Urbański from paper [**2**], to a class of sets that satisfies $p-$condition.

2. The Main Result

In this section we prove the following theorem which is main result of this paper.

Theorem 2.1. *Let X be a real topological vector space. Assume that $A, B, C \subset X$ are subsets such that B is bounded and C is closed set that satisfies $p-$condition. Then the following implication*

$$A + B \subset B + C \Longrightarrow A \subset C$$

holds true.

Proof. Take any $a \in A$ and $b_0 \in B$ then there exists $b_n \in B$ and $c_1 \in C$ such taht

$$a + b_0 = b_1 + c_1.$$

Suppose thet the elements $b_k \in B, c_k \in C$ are defined for $k \leq n$. Then the set

$$D_{n+1} = \{(x, y) \in B \times C : a + b_n = x + y\}$$

is not empty since $A + B \subset B + C$, hence we can take any element $(x_0, y_0) \in D_{n+1}$ and set $b_{n+1} = x_0, c_{n+1} = y_0$.
Thus by induction there are defined two sequences (b_n) and (c_n) such that $b_n \in B$, $c_n \in C$ and

$$a + b_n = b_{n+1} + c_{n+1}$$

for all $n \in \mathbb{N}$.

Now, by the above equality we can write

$$na = \sum_{k=0}^{n-1} a = \sum_{k=0}^{n-1} (b_{k+1} - b_k + c_{k+1}) = b_n - b_0 + \sum_{k=1}^{n} c_k,$$

and thus

$$a = \frac{b_n - b_0}{n} + n^{-1+\frac{1}{p}} \sum_{k=1}^{n} \frac{1}{n^{\frac{1}{p}}} c_k.$$

Denote $q_n = \frac{b_n - b_0}{n}$ and $r_n = n^{-1+\frac{1}{p}} \sum_{k=1}^{n} \frac{1}{n^{\frac{1}{p}}} c_k$. Since the set B is bounded subset of topological vector space X, we have that $\lim_{n\to\infty} q_n = 0$ and since the set C satisfyies $p-$ *condition* we obtain that $r_n \in C$ for all $n \in \mathbb{N}$. Therefore, by the closedness of the set C we have

$$a = \lim_{n\to\infty} (q_n + r_n) = \lim_{n\to\infty} r_n \in C$$

which ends the proof.

□

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