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A CHARACTERIZATION OF MINIMAL ASCREEN NULL HYPERSURFACES OF (LCS)-SPACE FORMS

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Abstract. In the present paper, we study nontotally geodesic minimal ascreen null hypersurface, M, of a Lorentzian concircular structure (LCS)-space form of constant curvature 0 or 1. We prove that; if the Ricci tensor of M is parallel with respect to a given leaf of its screen distribution, then M is isometric to a product of a null curve and spheres.

1. Introduction and Results

By a null hypersurface, in a spacetime, we mean a smooth codimension one submanifold such that the ambient metric *degenerates* when restricted to it. Null hypersurfaces play an important role in general relativity, as they represent horizons of various sorts (event horizon of a black hole, killing horizon, etc.). Among the simplest examples of null hypersurfaces are the null cones. The main challenge in understanding their geometric behaviors is the degeneracy of the induced metric. This degeneracy hinders the existence of a metric connection as well as a volume form, among other objects induced by the metric, on a null hypersurface. Some attempts to overcome this difficulty have had remarkable success. In [**5, 6**], the approach consists in fixing a geometric data formed by a null section and a screen distribution on the null hypersurface. This allows to induce some geometric objects such as a connection (not necessarily metric connection), a null second fundamental form and Gauss-Codazzi type equations. In [**20**] the author uses the quotient vector bundle $TM/TM^{\perp}$ to "get rid" of the degeneracy of the induced metric. His approach is essentially intrinsic, while that of [**5**] (or [**6**]) is extrinsic.

In both approaches, the authors succeeds in describing the topology of a null manifold under certain geometric conditions. For example, [**20**] shows that the event horizons of Schwarzschild, Reissner and Kerr spacetimes are *stationary* semi-Riemannian manifolds with the following structure: Let $M = \mathbb{R}\times H$, where $H = \mathbb{S}^2$ with a nondegenerate metric tensor $\tilde{g}$ of type $(0, 2)$. Now let $\psi : \mathbb{R} \times \mathbb{S}^2 \longrightarrow \mathbb{S}^2$ be the projection. Then the event horizons of the above solutions are the semi-Riemannian manifolds (M, g) with $g = \psi^{*}\tilde{g}$. By stationary the author means that the complementary degenerate bundle to the factor bundle $TM/TM^{\perp}$ is a killing distribution (see Definition 3.1.3 of [**20**, p. 41]). Note that when a null manifold M is imbedded into a semi-Riemannian manifold, the stationary condition is equivalent to M being totally geodesic, see for instance [**6**]. The topological structures above are also important in understanding the expanding nature of black

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hole horizons. For instance, Duggal-Sahin [**6**, p. 116] defines a *Non-Expanding Horizon* (in short; NEH) as a null hypersurface of a spacetime, (a) with a topology of $\mathbb{R}\times\mathbb{S}^2$, (b) expansion free (i.e., minimal) and (c) its stress energy tensor obeys the null dominant energy condition.

Back to the extrinsic approach of [**5**], they showed that an $(n{+}1)$-dimensional null cone M^{n+1} of a semi-Euclidean space $\mathbb{R}_1^{n+2}$ is totally umbilic and screen conformal (see Proposition 5.4). Moreover, such a cone is locally isometric to $\mathcal{C}_\xi\times\mathbb{S}^n$, where $\mathcal{C}_\xi$ is a null curve tangent to the normal bundle of M and $\mathbb{S}^n$ is an n-sphere (see Proposition 5.7 of [**5**] and Remark 5 of [**7**] for details). Classifications of null hypersurfaces as general product manifolds has been done briefly in [**6**, Theorems 2.5.15 and 2.5.17] under the assumptions that M is Einstein.

In the present paper, we give an explicit structure of minimal ascreen null hypersurfaces in Lorentzian concircular structure (LCS)-space forms of constant curvature $c=0,1$. *From now on, we denote by* $\mathbb{S}^m(r)$ *the* m*-sphere of radius* r *in an Euclidean space* $\mathbb{R}^{m+1}$. To that end, we prove the following main result.

Theorem 1.1. *Let* (M^{n+1},g) *be a nontotally geodesic minimal ascreen null hypersurface of a* (LCS)*-space form* $\overline{M}^{n+2}(c)$*. If the Ricci tensor,* Ric*, of* M *is parallel with respect to each* n*-dimensional leaf* M' *of the screen distribution* $S(TM)$*, then the following hold;*

(1) *if* $c=0$ *then, for all* $0<k<\frac{n}{4}$*,* M *is locally isometric to*
$$\mathcal{C}_\xi\times\mathbb{S}^k\left(\frac{k}{\alpha\sqrt{(n-2k)(n-4k)}}\right)\times\mathbb{S}^{n-k}\left(\frac{n-k}{\alpha\sqrt{(n-2k)(3n-4k)}}\right),$$

(2) *if* $\alpha=1$ *then* $c=1$*, and* M *is locally isometric to*
$$\mathcal{C}_\xi\times\mathbb{S}^k\left(\frac{k}{n}\right)\times\mathbb{R}^{n-k},\qquad 0<k<n,$$

where $\mathcal{C}_\xi$ *is a null curve tangent to the normal bundle of* M *and* α *is a nonzero function of* $\overline{M}$*.*

Following the approach of [**5**] (or [**6**]) many researchers have investigated the geometry of hull submanifolds which include, but not limited to, [**1**], [**2**], [**4**], [**9**], [**9**], [**11**], [**12**], [**13**], [**14**] and [**19**]. In a recent paper [**13**], the author initiated the study of ascreen null hypersurfaces of Lorentzian concircular structure (LCS)-manifolds, in which it was discovered that such hypersurfaces admits a symmetric Ricci tensor. The rest of the paper is arranged as follows; Section 2 focusses on the basic notions of null hypersurfaces and Lorentzian concircular structure (LCS)-manifolds needed in the rest of the paper. In Section 3, we prove some results needed to prove our main result. Finally, in Section 4, we prove our main result (Theorem 1.1).

2. Preliminaries

An $(n+2)$-dimensional Lorentzian manifold $\overline{M}$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $\overline{g}$, that is, $\overline{M}$ admits a smooth tensor field $\overline{g}$ of type $(0,2)$ such that, for each point $p\in\overline{M}$, the tensor $\overline{g}_p:T_p\overline{M}\times T_p\overline{M}\longrightarrow\mathbb{R}$ is a *non-degenerate* inner product of signature $(-,+,\dots,+)$, where $T_p\overline{M}$ denotes the tangent vector space of $\overline{M}$ at p and $\mathbb{R}$ is the real number space. A non-zero vector field $v \in T_p\overline{M}$ is said to be timelike (resp., non-spacelike, null and spacelike) if it satisfies $\overline{g}_p(v,v) < 0$ (resp., ≤ 0, $= 0$ and > 0) [**18**]. A vector field V defined by $\overline{g}(X,V) = A(X)$, for any X tangent to $\overline{M}$, is said to be a *concircular* [**16, 17**] vector field if, for any X, Y tangent to $\overline{M}$, we have $(\overline{\nabla}_X A)Y = \alpha[\overline{g}(X,Y) - \omega(X)A(Y)]$, where α is a non-vanishing smooth function and ω is a closed 1-form. Here, $\overline{\nabla}$ denotes the Levi-Civita connection of $\overline{M}$ with respect to $\overline{g}$.

Let $\overline{M}$ be an $(n+2)$-dimensional Lorentzian manofold admitting a unit timelike concircular vector field ζ, called the *characteristic* vector field of the manifold. Then we have

$$\overline{g}(\zeta,\zeta) = -1. \tag{2.1}$$

As ζ is a unit concircular vector field, it follows that there exists a non-zero 1-form θ such that for

$$\overline{g}(X,\zeta) = \theta(X), \tag{2.2}$$

the following equations hold

$$(\overline{\nabla}_X\theta)(Y) = \alpha[\overline{g}(X,Y) + \theta(X)\theta(Y)] \ (\alpha \neq 0), \tag{2.3}$$

for all vector fields X, Y tangent to $\overline{M}$, where $\overline{\nabla}$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $\overline{g}$ and α is a non-zero scalar function satisfying

$$\overline{\nabla}_X\alpha = X\alpha = d\alpha(X) = \rho\theta(X), \tag{2.4}$$

ρ being a certain scalar function given by $\rho = -\zeta\alpha$. Throughout this paper, $\Gamma(E)$ will denote the $\mathcal{F}(\overline{M})$-module of differentiable sections of a vector bundle E. Let

$$\overline{\phi}X = \frac{1}{\alpha}\overline{\nabla}_X\zeta, \quad \forall\, X \in \Gamma(T\overline{M}). \tag{2.5}$$

Then, by (2.3) and (2.5), we have

$$\overline{\phi}X = X + \theta(X)\zeta, \tag{2.6}$$

which follows that $\overline{\phi}$ is a symmetric $(1,1)$ tensor field called the *structure* tensor field of the manifold. Thus, the Lorentzian manifold $\overline{M}$ together with the unit timelike concircular vector field ζ, its associated 1-form θ and a $(1,1)$ tensor field $\overline{\phi}$ is said to be a *Lorentzian concircular structure* manifold (briefly, an (LCS)-manifold) [**16, 17**]. In particular, if $\alpha = 1$, then we obtain the LP-Sasakian structure of Matsumoto [**15**]. In an (LCS)-manifold, the following relations [**16**] hold for all vector fields $X, Y, Z \in \Gamma(T\overline{M})$:

$$\overline{\phi}^2X = X+\theta(X)\zeta, \quad \overline{\phi}\zeta = 0, \ \ \theta\circ\overline{\phi} = 0, \quad \theta(\zeta) = -1, \tag{2.7}$$

$$\overline{g}(\overline{\phi}X,\overline{\phi}Y) = \overline{g}(X,Y) + \theta(X)\theta(Y), \tag{2.8}$$

$$(\overline{\nabla}_X\overline{\phi})(Y) = \alpha(\overline{g}(X,Y)\zeta + 2\theta(X)\theta(Y)\zeta + \theta(Y)X), \ \ \overline{\nabla}_X\zeta = \alpha\overline{\phi}X, \tag{2.9}$$

$$\theta(\overline{R}(X,Y)Z) = (\alpha^2 - \rho)(\overline{g}(Y,Z)\theta(X) - \overline{g}(X,Z)\theta(Y)), \tag{2.10}$$

where $\overline{R}$ denotes the curvature tensor of $\overline{M}$.

Next, we give an example of (LCS)-manifold.

Example 2.1 ([**16**])**.** Consider the 4-dimensional manifold

$$\overline{M} = \{(v, x, y, z) \in \mathbb{R}^4 : Z \neq 0\},$$

where (v, x, y, z) are the standard coordinates in $\mathbb{R}^4$. Let $\{Z_1, Z_2, Z_3, Z_4\}$ be linearly independent global frame on $\overline{M}$ given by

$$Z_1 = z\left(\frac{\partial}{\partial v} + x\frac{\partial}{\partial x}\right), \quad Z_2 = z\frac{\partial}{\partial x}, \;\; Z_3 = z\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \;\; Z_4 = z^3\frac{\partial}{\partial z}.$$

Let us define $\overline{\phi}$, ζ, θ and $\overline{g}$ by $\overline{\phi}Z_1 = Z_1$, $\overline{\phi}Z_2 = Z_2$, $\overline{\phi}Z_3 = Z_3$, $\overline{\phi}Z_4 = 0$; $\zeta = Z_4$, $\theta(X) = \overline{g}(X, Z_4)$, for all $X \in \Gamma(T\overline{M})$, $\overline{g}(Z_1, Z_1) = \overline{g}(Z_2, Z_2) = \overline{g}(Z_3, Z_3) = 1$, $\overline{g}(Z_4, Z_4) = -1$, $\overline{g}(Z_i, Z_j) = 0$, $i \neq j$ for all $1 \leq i, j \leq 4$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric $\overline{g}$. Then we have

$$[Z_1, Z_2] = -zZ_2, \;\; [Z_1, Z_4] = -z^2 Z_1, \;\; [Z_2, Z_4] = -z^2 Z_2 \;\; [Z_3, Z_4] = -z^2 Z_3.$$

Taking $Z = \zeta$, and using Koszul's formula (see Theorem 11 of [**18**, p. 61]), we have

$$\begin{aligned}
&\overline{\nabla}_{Z_1} Z_4 = -z^2 Z_1, \;\; \overline{\nabla}_{Z_2} Z_1 = zZ_2, \;\; \overline{\nabla}_{Z_1} Z_1 = -z^2 Z_4 \;\; \overline{\nabla}_{Z_2} Z_4 = -z^2 Z_2 \\
&\overline{\nabla}_{Z_3} Z_4 = -z^2 Z_3, \;\; \overline{\nabla}_{Z_3} Z_3 = -z^2 Z_4 \;\; \overline{\nabla}_{Z_2} Z_2 = -z^2 Z_4 - zZ_1.
\end{aligned}$$

From the above it can be easily seen that $(\overline{\phi}, \zeta, \theta, \overline{g})$ is a (LCS)-structure on $\overline{M}$. Consequently $(\overline{M}, \overline{\phi}, \zeta, \theta, \overline{g})$ is a (LCS)-manifold with $\alpha = -z^2 \neq 0$ such that $X\alpha = \rho\theta(X)$, where $\rho = 2z^4$.

Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional semi-Riemannian manifold and let M be a hypersurface of $\overline{M}$. Let g be the induced tensor field by $\overline{g}$ on M. Then, M is called a *null hypersurface* of $\overline{M}$ if g is of constant rank n [**5**]. Consider the vector bundle $TM^{\perp}$ whose fibers are defined by $T_xM^{\perp} = \{Y_x \in T_x\overline{M} : \overline{g}_x(X_x, Y_x) = 0$, for all $X_x \in T_xM\}$, for any $x \in M$. Hence, a hypersurface M of $\overline{M}$ is null if and only if $TM^{\perp}$ is a distribution of rank 1 on M. Let M be a null hypersurface, we consider the complementary distribution $S(TM)$ to $TM^{\perp}$ in TM, which is called a *screen distribution.* It is well-known that $S(TM)$ is non-degenerate (see [**5**]). Thus,

$$TM = S(TM) \perp TM^{\perp}. \tag{2.11}$$

As $S(TM)$ is non-degenerate with respect to $\overline{g}$, we have $T\overline{M} = S(TM) \perp S(TM)^{\perp}$, where $S(TM)^{\perp}$ is the complementary vector bundle to $S(TM)$ in $T\overline{M}|_M$. Let (M, g) be a null hypersurface of $(\overline{M}, \overline{g})$. Then, there exists a unique vector bundle $tr(TM)$, called the *null transversal bundle* [**5**] of M with respect to $S(TM)$, of rank 1 over M such that for any non-zero section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying $\overline{g}(\xi, N) = 1$, $\overline{g}(N, N) = \overline{g}(N, Z) = 0$, for any section Z of $S(TM)$. Consequently, we have the following decomposition of $T\overline{M}$.

$$T\overline{M}|_M = S(TM) \perp \{TM^{\perp} \oplus tr(TM)\} = TM \oplus tr(TM). \tag{2.12}$$

Let ∇ and ∇^* denote the induced connections on M and $S(TM)$, respectively, and P be the projection of TM onto $S(TM)$, then the local Gauss-Weingarten equations of M and $S(TM)$ are the following [**5**];

$$\overline{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.13}$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N, \tag{2.14}$$

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.15}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \quad A^*_\xi \xi = 0, \tag{2.16}$$

for all $X, Y \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$, where $\overline{\nabla}$ is the Levi-Civita connection on $\overline{M}$. In the above setting, B is the local second fundamental form of M and C is the local second fundamental form on $S(TM)$. A_N and A^*_ξ are the shape operators on TM and $S(TM)$ respectively, while τ is a 1-form on TM. The above shape operators are related to their local fundamental forms by

$$g(A^*_\xi X, Y) = B(X,Y), \quad g(A_N X, PY) = C(X, PY), \tag{2.17}$$

for any $X, Y \in \Gamma(TM)$. Moreover, $\overline{g}(A^*_\xi X, N) = 0$, and $\overline{g}(A_N X, N) = 0$, for all $X \in \Gamma(TM)$. From these relations, we notice that A^*_ξ and A_N are both screen-valued operators. Let $\vartheta = \overline{g}(N, \cdot)$ be a 1-form metrically equivalent to N defined on $\overline{M}$. Take $\eta = i^*\vartheta$ to be its restriction on M, where $i : M \to \overline{M}$ is the inclusion map. Then it is easy to show that

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.18}$$

for all $X, Y, Z \in \Gamma(TM)$. Consequently, ∇ is generally *not* a metric connection with respect to g. However, the induced connection ∇^* on $S(TM)$ is a metric connection.

Denote by $\overline{R}$, R and R^* the curvature tensors of the connection $\overline{\nabla}$ on $\overline{M}$, and the induced linear connections ∇ and ∇^* on M and $S(TM)$, respectively. Using the Gauss-Weingarten formulae, we obtain the following Gauss-Codazzi equations for M and $S(TM)$ (see details in [**5, 6**]).

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z, PW) =& g(R(X,Y)Z, PW) + B(X,Z)C(Y,PW)\\ &- B(Y,Z)C(X,PW),\end{aligned} \tag{2.19}$$

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z, \xi) =& (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z)\\ &- \tau(Y)B(X,Z),\end{aligned} \tag{2.20}$$

$$\begin{aligned}g(R(X,Y)PZ, PW) =& g(R^*(X,Y)PZ, PW) + C(X,PZ)B(Y,PW)\\ &- C(Y,PZ)B(X,PW),\end{aligned} \tag{2.21}$$

$$\overline{g}(\overline{R}(X,Y)N, \xi) = B(Y, A_N X) - B(X, A_N Y) - 2d\tau(X,Y), \tag{2.22}$$

where $2d\tau(X,Y) = X(\tau(Y)) - Y(\tau(X)) - \tau([X,Y])$, for all $X, Y, Z, W \in \Gamma(TM)$, $\xi \in \Gamma(TM^\perp)$ and $N \in \Gamma(tr(TM))$.

Suppose π is a non-degenerate plane of $T_p\overline{M}$, for $p \in \overline{M}$. Then, the associated matrix G_p of $\overline{g}_p$, with respect to an arbitrary basis $\{u, v\}$, is of rank 2 given by (1.2.15) of [**6**, p. 16]. Define a real number $K(\pi) = K_p(u,v) = \overline{R}(u,v,v,u)$, where $\overline{R}(u,v,v,u)$ is the 4-linear mapping on $T_p\overline{M}$ by the curvature tensor. The smooth function K, which assigns to each non-degenerate tangent plane π the real number $K(\pi)$ is called the *sectional curvature* of $\overline{M}$, which is independent of the basis

$\{u, v\}$. If K is a constant c at every point of $p \in \overline{M}$ then $\overline{M}$ is of constant sectional curvature c, denote by $\overline{M}(c)$, whose curvature tensor field $\overline{R}$ is given by

$$\overline{R}(X, Y)Z = c\{\overline{g}(Y, Z)X - \overline{g}(X, Z)Y\}, \tag{2.23}$$

for any $X, Y, Z \in \Gamma(T\overline{M})$ (see [**18**] for details). In particular, if $K = 0$, then $\overline{M}$ is called a *flat manifold* for which $\overline{R} = 0$.

In what follows, we give an example of (LCS)-space form.

Example 2.2. Consider the 3-dimensional manifold $\overline{M} = \{(x, y, z) \in \mathbb{R}^3 : z > 0\}$, where (x, y, z) are standard coordinates of $\mathbb{R}^3$. Let Z_1, Z_2 and Z_3 be the vector fields on $\overline{M}$ given by

$$Z_1 = z\frac{\partial}{\partial x}, \qquad Z_2 = z\frac{\partial}{\partial y}, \quad \text{and} \quad Z_3 = z\frac{\partial}{\partial z}, \tag{2.24}$$

which are linearly independent at each point $p \in \overline{M}$, and hence form a basis of $T_p\overline{M}$. Let us define a Lorentzian metric $\overline{g}$ on $\overline{M}$ as

$$\overline{g}(Z_1, Z_1) = \overline{g}(Z_2, Z_2) = 1, \;\; \overline{g}(Z_3, Z_3) = -1, \;\; \text{and} \;\; \overline{g}(Z_i, Z_j) = 0, \tag{2.25}$$

for all $i \neq j$, for all $1 \leq i, j \leq 3$. Let θ be a differential 1-form on $\overline{M}$ defined by $\theta(X) = \overline{g}(X, Z_3) = \overline{g}(X, \zeta)$, for all $X \in \Gamma(T\overline{M})$, and let $\overline{\phi}$ be the $(1, 1)$-tensor field on $\overline{M}$ defined as

$$\overline{\phi}Z_1 = Z_1, \quad \overline{\phi}Z_2 = Z_2, \quad \text{and} \quad \overline{\phi}Z_3 = 0. \tag{2.26}$$

Applying the linearity of $\overline{\phi}$ and $\overline{g}$, we have

$$\begin{aligned} \theta(\zeta) = \overline{g}(\zeta, \zeta) = -1, \quad \overline{\phi}^2 X = X + \theta(X)\zeta, \quad \theta(\overline{\phi}X) = 0, \\ \overline{g}(X, \zeta) = \theta(X), \quad \text{and} \quad \overline{g}(\overline{\phi}X, \overline{\phi}Y) = \overline{g}(X, Y) + \theta(X)\theta(Y), \end{aligned} \tag{2.27}$$

for all $X, Y \in \Gamma(T\overline{M})$. Let $\overline{\nabla}$ be the Levi-Civita connection with respect to the Lorentzian metric $\overline{g}$. Then we have

$$[Z_1, Z_2] = 0, \quad [Z_1, Z_3] = -Z_1, \quad \text{and} \quad [Z_2, Z_3] = -Z_2. \tag{2.28}$$

Next, using Koszul's formula (see Theorem 11 of [**18**, p. 61]), we have

$$\begin{aligned} &\overline{\nabla}_{Z_1}Z_1 = -Z_3 = -\zeta, \quad \overline{\nabla}_{Z_1}Z_2 = 0, \quad \overline{\nabla}_{Z_1}Z_3 = -Z_1, \\ &\overline{\nabla}_{Z_2}Z_1 = 0, \quad \overline{\nabla}_{Z_2}Z_2 = -Z_3, \quad \overline{\nabla}_{Z_2}Z_3 = -Z_2, \\ \text{and} \quad &\overline{\nabla}_{Z_3}Z_1 = \overline{\nabla}_{Z_3}Z_2 = \overline{\nabla}_{Z_3}Z_3 = 0. \end{aligned} \tag{2.29}$$

It follows from (2.27) and (2.29) that $(\overline{M}, \overline{\phi}, \zeta = Z_3, \theta, \overline{g})$ is a 3-dimensional LCS-manifold, with $\alpha = -1$ and $\rho = 0$. Moreover, using (2.29) and the definition of the curvature tensor $\overline{R}$ of $\overline{\nabla}$, i.e., $\overline{R}(X, Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z$, for all $X, Y, Z \in \Gamma(T\overline{M})$, we obtain

$$\begin{aligned} &\overline{R}(Z_1, Z_2)Z_1 = -Z_2, \quad \overline{R}(Z_1, Z_3)Z_1 = -Z_3, \quad \overline{R}(Z_2, Z_3)Z_1 = 0, \\ &\overline{R}(Z_1, Z_2)Z_2 = Z_1 \quad \overline{R}(Z_1, Z_3)Z_2 = 0, \quad \overline{R}(Z_2, Z_3)Z_2 = -Z_3, \\ &\overline{R}(Z_1, Z_2)Z_3 = 0, \quad \overline{R}(Z_1, Z_3)Z_3 = -Z_1, \quad \text{and} \quad \overline{R}(Z_2, Z_3)Z_3 = -Z_2. \end{aligned} \tag{2.30}$$

From (2.30), we deduce that

$$\overline{R}(X, Y)Z = \overline{g}(Y, Z)X - \overline{g}(X, Z)Y, \tag{2.31}$$

for all $X, Y, Z \in \Gamma(T\overline{M})$. It then follows from (2.31) that $\overline{M}$ is a space of constant curvature $c = 1$. We note that $\overline{M}$ is locally isometric to the pseudo sphere $\mathbb{S}^3_1(1)$.

3. Intermediate Results

Let (M, g) be a null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$. Then, the timelike characteristic vector field ζ of $\overline{M}$ can be decomposed as follows

$$\zeta = W + a\xi + bN, \tag{3.1}$$

where a and b are smooth functions, given by $a = \theta(N)$ and $b = \theta(\xi)$, and W is a smooth section of $S(TM)$. It was proved in [**13**] that ζ is *never* tangent or transversal to M. Furthermore, it was shown, in the same paper, that $\overline{\phi}TM^{\perp}$ and $\overline{\phi}tr(TM)$ *can not* be considered as subbundles of $S(TM)$ as it is often done in the Sasakian case (see [**6**] for detailes). In this section, we consider *ascreen* null hypersurfaces of $\overline{M}$. More precisely, a null hypersurface (M, g) will be called ascreen if the characteristic vector field ζ, of the (LCS)-manifold $\overline{M}$, belongs to $TM^{\perp} \oplus tr(TM)$. These hypersurfaces were first considered by Jin [**8**] in Sasakian ambient spaces. For an ascreen null hypersurface (M, g) of a (LCS)-manifold $\overline{M}$, ζ in (3.1) reduces to

$$\zeta = a\xi + bN, \tag{3.2}$$

where $a = \theta(N)$ and $b = \theta(\xi)$ both non-vanishing smooth functions. Moreover, we have the following result.

Theorem 3.1. *Let (M, g) be a null hypersurface of a (LCS)-manifold $\overline{M}$. Then M is an ascreen null hypersurface of $\overline{M}$ if and only if $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$.*

Proof. Suppose that M is ascreen null hypersurface. Applying $\overline{\phi}$ to (3.2) and using the fact that $\overline{\phi}\zeta = 0$, we get $a\overline{\phi}\xi + b\overline{\phi}N = 0$. Thus, one gets $\overline{\phi}\xi = \omega\overline{\phi}N$, where $\omega = -\frac{b}{a} \neq 0$, a non vanishing smooth function. This implies that $\overline{\phi}TM^{\perp} \cap \overline{\phi}tr(TM) \neq \{0\}$. Since rank $\overline{\phi}TM^{\perp} =$ rank $\overline{\phi}tr(TM) = 1$, it follows that $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$. Conversely, suppose that $\overline{\phi}TM^{\perp} = \overline{\phi}tr(TM)$. Then, there exists a non-vanishing smooth function ω such that $\overline{\phi}\xi = \omega\overline{\phi}N$. Taking the inner product of this relation with respect to $\overline{\phi}\xi$ and $\overline{\phi}N$ in turn, we get $b^2 = \omega(ab + 1)$ and $\omega a^2 = ab + 1$, respectively. Since $\omega \neq 0$, we have $a \neq 0$, $b \neq 0$ and $b^2 = (\omega a)^2$. The latter gives $b = \pm\omega a$. The case $b = \omega a$ implies that $ab = \omega a^2 = ab + 1$, which is a contradiction. Thus $b = -\omega a$, from which $2ab = -1$. Since $\omega = -\frac{b}{a}$, $a \neq 0$ and $\overline{\phi}\xi = \omega\overline{\phi}N$, it is easy to see that $a\overline{\phi}\xi + b\overline{\phi}N = 0$. Applying $\overline{\phi}$ to this equation, and using $b^2 = \omega(ab + 1)$ together with $2ab = -1$, we get $\zeta = a\xi + bN$. Therefore, M is ascreen null hypersurface of $\overline{M}$, which completes the proof. □

Let (M, g) is an ascreen null hypersurface of $\overline{M}$. Differentiating (3.2) and using (2.14) and (2.16), we get

$$\alpha\overline{\phi}X = -aA^*_{\xi}X - bA_NX + [Xa - a\tau(X)]\xi + [Xb + b\tau(X)]N, \tag{3.3}$$

for any $X \in \Gamma(TM)$. Taking the inner product of (3.3) with N and ξ, in turn, we get

$$Xa - a\tau(X) = \alpha\eta(X) + \alpha a\theta(X), \quad Xb + b\tau(X) = \alpha b\theta(X), \tag{3.4}$$

in which we have used the fact $X = PX + \eta(X)\xi$, for any $X \in \Gamma(TM)$. On the other hand, taking the inner product of (3.3) with PY, where $Y \in \Gamma(TM)$, we have

$$aB(X, PY) + bC(X, PY) = -\alpha g(PX, PY). \tag{3.5}$$

Setting $X = \xi$ in (3.5) and using (2.16) together with the fact $b \neq 0$, we get

$$C(\xi, PY) = 0, \quad \forall\, Y \in \Gamma(TM). \tag{3.6}$$

As B is symmetric, it is easy to see from (3.5) that C is also symmetric.

Next, we give an example of an ascreen null hypersurface of a (LCS)-manifold.

Example 3.2. Let us consider a hypersurface (M, g) of a (LCS)-space form $(\overline{M}(1), \overline{\phi}, \zeta, \theta, \overline{g})$ in Example 2.2 given by $M = \{(x, y, z) \in \mathbb{R}^3 : x + z = 0\}$. Thus, the tangent space TM is spanned by $W_1 = Z_1 + \zeta$ and $W_2 = Z_2$. Observe that $\overline{g}(W_1, W_1) = 0$ and $\overline{g}(W_1, W_2) = 0$, which implies that W_1 is a null vector field tangent to M. It then follows that the 1-dimensional normal bundle $TM^\perp \subset TM$ is spanned by

$$\xi := W_1 = Z_1 + \zeta, \tag{3.7}$$

hence M is 2-dimensional null hypersurface $\overline{M}$. Also note that $S(TM)$ is spanned by the spacelike vector field W_2. The corresponding transversal bundle $tr(TM)$ is spanned by N, where

$$N = \frac{1}{2}(Z_1 - \zeta). \tag{3.8}$$

From (3.7) and (3.8) and the definition of $\overline{\phi}$, we have

$$\overline{\phi}\xi = Z_1 \quad \text{and} \quad \overline{\phi}N = \frac{1}{2}Z_1. \tag{3.9}$$

In view of (3.9), we see that $\overline{\phi}TM^\perp = \overline{\phi}tr(TM)$. Hence, by Theorem 3.1, (M, g) is an ascreen null hypersurface of an a (LCS)-space form $(\overline{M}(1), \overline{\phi}, \zeta, \theta, \overline{g})$, with

$$\zeta = \frac{1}{2}\xi - N. \tag{3.10}$$

In fact, from (3.2), (3.9) the fact $\overline{\phi}\zeta = 0$, we have $2a + b = 0$. Since ζ is a unit timelike vector field, (3.2) gives $2ab + 1 = 0$. From these equations, we get $a = \frac{1}{2}$ and $b = -1$, proving (3.10).

Using the functions a and b, it was shown in [**13**] that the Ricci tensor, with respect to the induced connection ∇, of an ascreen null hypersurface of a (LCS)-manifold is actually symmetric. In the following result, we supply a different proof to that in [**13**].

Theorem 3.3. *Let (M, g) be an ascreen null hypersurface of a (LCS)-manifold $(\overline{M}, \overline{g})$. Then the Ricci type tensor of M is symmetric, the screen distribution $S(TM)$ is integrable and M is locally a product manifold $\mathcal{C}_\xi \times M'$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M' is a leaf of $S(TM)$.*

Proof. Setting $Z = \xi$ in (2.10), we get $\theta(\overline{R}(X, Y)\xi) = 0$, for all $X, Y \in \Gamma(TM)$. As M is ascreen, $\zeta = a\xi + bN$ and hence, the previous relation simplifies as $0 = \theta(\overline{R}(X, Y)\xi) = a\overline{g}(\overline{R}(X, Y)\xi, \xi) + b\overline{g}(\overline{R}(X, Y)\xi, N)$. Using the properties of $\overline{R}$ and the fact that $b \neq 0$, we get $\overline{g}(\overline{R}(X, Y)\xi, N) = 0$, for all $X, Y \in \Gamma(TM)$. Considering the last relation in (2.22), we get $B(Y, A_N X) - B(X, A_N Y) = 2d\tau(X, Y)$. From (3.5) and (3.6), we have $A_N X = -\frac{a}{b} A^*_\xi X - \frac{\alpha}{b} PX$, for all $X \in \Gamma(TM)$. Thus, using these two relations, together with the fact that A^*_ξ is symmetric with respect to B, we get $2d\tau(X, Y) = 0$ or simply $d\tau = 0$. As τ is closed, by a well-known argument in [**5**] or [**6**], the Ricci tensor of M is symmetric. Next, the symmetry of C follows from (3.5) by the fact that B is symmetric. Consequently, M is screen integrable and by Remark 5 of [**7**, 215], M is locally a product manifold $\mathcal{C}_\xi \times M'$, where $\mathcal{C}_\xi$ is a null curve tangent to $TM^\perp$ and M' is a leaf of $S(TM)$. Hence the proof is completed. □

Remark 3.4. If the Ricci tensor of (M, g) is symmetric, then there exists a null pair $\{\xi, N\}$ such that the corresponding 1-form τ satisfies $\tau = 0$ [**5**], which is called a *canonical* null pair of M. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $S(TM)^\natural = TM/TM^\perp$ of Kupeli [**20**]. This implies that all screen distribution are mutually isomorphic. For this reason, we consider only ascreen null hypersurfaces M endowed with the canonical null pair $\{\xi, N\}$.

Let $x \in M$ be an arbitrary point and let $\{Z_i\}_{i=1}^n$ be an othonormal frame of $S(TM)_x$ such that $A^*_\xi Z_i = \lambda_i Z_i$. For each λ, let $P_\lambda = \{Z : A^*_\xi Z = \lambda Z\}$. Then P_λ is called the *principle distribution*, which is completely integrable [**14**]. Moreover, as $\tau = 0$, each *principle curvature* λ is constant along each principal distribution P_λ, whose leaves are totally geodesic in $S(TM)$ and totally umbilic in $\overline{M}$ (see [**14**] for details). The following result is fundamental to our main result.

Proposition 3.5. *Let (M, g) be an ascreen null hypersurface of a (LCS)-space form $\overline{M}(c)$. Let $\{Z_i\}_{i=1}^n$ be an othonormal frame of $S(TM)_x$ such that $A^*_\xi Z_i = \lambda_i Z_i$. If A^*_ξ is parallel along the leaves M' of $S(TM)$, i.e., $\nabla^* A^*_\xi = 0$, then*

$$(\lambda_i \lambda_j - \alpha b(\lambda_i + \lambda_j) + cb^2)(\lambda_i - \lambda_j) = 0, \tag{3.11}$$

for all $1 \leq i, j \leq n$.

Proof. Fix a point $x \in M$ and let $\lambda_1, \ldots, \lambda_m$ be the distinct screen principal curvatures of A^*_ξ at x. Denote by E_k the space of principal vectors corresponding to λ_k. Let $\mathcal{H}_x$ be the holonomy group of $S(TM)_x$ at $x \in M$ considered as a group of non-singular, linear transformations of $S(TM)_x$. Then, we have $A^*_\xi H = HA^*_\xi$, for all $H \in \mathcal{H}_x$ [**10**]. Moreover, it is easy to prove that $HE_k \subseteq E_k$, for all $H \in \mathcal{H}_x$ and $1 \leq k \leq m$. As in [**10**], the Lie algebra, $\mathcal{B}_x$ of $\mathcal{H}_x$is generated by elements of the form $\varepsilon^{-1} \circ R^*(u, v) \circ \varepsilon$, where $u, v \in S(TM)_x$, R^* denotes the curvature tensor of leaves of $S(TM)$ and ε corresponds to parallel displacement along a piece-wise smooth closed curve at x. Choosing $\varepsilon =$ identity we must have that $R^*(u, v)E_k \subseteq E_k$ for each k. Hence, the sectional curvatures

$$g(R^*(Z_i, Z_j)Z_j, Z_i) = 0, \quad \forall\, Z_i \in E_i \quad \text{and} \quad Z_j \in E_j, \tag{3.12}$$

where $i \neq j$. Next, considering (2.19), (2.21) and (2.23), we have

$$\begin{aligned} g(R^*(X, Y)PZ, PW) = {} & c(g(Y, PZ)g(X, PW) - g(X, PZ)g(Y, PW)) \\ & + B(Y, PZ)C(X, PW) + C(Y, PZ)B(X, PW) \\ & - C(X, PZ)B(Y, PW) - B(X, PZ)C(Y, PW), \end{aligned} \tag{3.13}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Setting $X = W = Z_i$ and $Y = Z = Z_j$ in (3.13) and using (3.5), we get

$$\begin{aligned} g(R^*(Z_i, Z_j)Z_j, Z_i) = c(1-\delta_{ij}^2) - \frac{2a}{b}\lambda_i\lambda_j - \frac{\alpha}{b}(\lambda_i + \lambda_j) \\ + \frac{2a}{b}\delta_{ij}^2\lambda_i^2 + \frac{2\alpha}{b}\delta_{ij}^2\lambda_i. \end{aligned} \tag{3.14}$$

As M is ascreen, (3.1) and $\overline{g}(\zeta, \zeta) = -1$ implies that $2ab+1 = 0$. Using this relation in (3.14) and considering $i \neq j$, we get

$$g(R^*(Z_i, Z_j)Z_j, Z_i) = c + \frac{1}{b^2}\lambda_i\lambda_j - \frac{\alpha}{b}(\lambda_i + \lambda_j). \tag{3.15}$$

Then, our assertions follows from (3.12) and (3.15), hence the proof. □

Let (M, g) be a null hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. We say that M is totally umbilic [**5**] if $B = \delta \otimes g$, for some smooth function δ on a neighborhood $\mathcal{U} \subset M$. When $\delta = 0$, M is said to be totally geodesic. The trace of A_ξ^* (or B) is called the null mean curvature of M, explicitly given by

$$S_x = \sum_{i=1}^{n} g(A_\xi^* Z_i, Z_i) = \sum_{i=1}^{n} B(Z_i, Z_i), \tag{3.16}$$

where $\{Z_i\}_{i=1}^n$ is an orthonormal basis of $S(TM)$ at $x \in M$. We say that M is a *minimal* if S_x vanishes. Generally, we say that M is a *constant mean curvature (CMC)* null hypersurface if S_x is a constant function at each $x \in M$.

Example 3.6. Let us consider the ascreen null hypersurface of Example 3.2. By a direct calculation, using (3.7), (3.8) and (2.29), we have

$$\begin{aligned} &\overline{\nabla}_\xi \xi = -\xi, \quad \overline{\nabla}_\xi W_2 = 0, \quad \overline{\nabla}_{W_2}\xi = -W_2, \quad \nabla^*_{W_2} W_2 = 0 \\ \text{and} \quad &\overline{\nabla}_\xi N = N, \quad \overline{\nabla}_{W_2} N = \frac{1}{2}W_2, \quad \overline{\nabla}_{W_2} W_2 = -\zeta, \quad \nabla^*_\xi W_2 = 0. \end{aligned} \tag{3.17}$$

Then applying the Gauss-Weingarten equations (2.13)–(2.16) to (3.17), leads to

$$A_\xi^* W_2 = W_2, \quad A_N \xi = 0, \quad A_N W_2 = -\frac{1}{2}W_2. \tag{3.18}$$

Moreover, $\tau(\xi) = 1$ and $\tau(W_2) = 0$. From (3.18) we see that (M, g) is a nontotally geodesic ascreen null hypersurface of a (LCS)-space form $(\overline{M}(1), \overline{\phi}, \zeta, \theta, \overline{g})$. In fact, M has constant mean curvature $S = 1$.

In view of Proposition 3.5 and (3.16), we have the following.

Corollary 3.7. *A_ξ^* has at most two distinct screen principal curvatures at each point. Moreover, if $A_\xi^* \neq 0$ and M is minimal null hypersurface of a (LCS)-space form of constant curvature c, then the screen principal curvatures λ_1 and λ_2 are given by;*

(1) *for $c = 0$;*

$$\lambda_1 = \frac{n - 2n_1}{n_1}\alpha b \quad \text{and} \quad \lambda_2 = -\frac{n - 2n_1}{n_2}\alpha b, \tag{3.19}$$

(2) *for $\alpha = 1$, we have $c = 1$ and;*

$$\lambda_1 = -\frac{n_2}{n_1}b, \quad \lambda_2 = b \quad \text{or} \quad \lambda_1 = b, \quad \lambda_2 = -\frac{n_1}{n_2}b, \tag{3.20}$$

where $n_1 + n_2 = n$ and λ_k has multiplicity $n_k \geq 1$. Moreover, each leaf M' of $S(TM)$ is locally isometric to the product manifold $M_{\lambda_1} \times M_{\lambda_2}$, where M_{λ_1} and M_{λ_2} are the leaves of the principal distributions P_{λ_1} and P_{λ_2}, respectively.

Proof. As M is minimal, we have $S_x = 0$ by (3.16). But $S_x = n_1\lambda_1 + n_2\lambda_2$. Thus $n_1\lambda_1 + n_2\lambda_2 = 0$ and (3.19), (3.20) follows from this relation and (3.11) of Proposition 3.5 by simple calculations.

Notice that A^*_ξ has the same principal curvatures with the same multiplicities at all points of $S(TM)_x$, where $x \in M$. Let $\{Z_i\}_{i=1}^n$ be a basis of $S(TM)_x$ consisting of principal vectors of $(A^*_\xi)_x$ with eigenvalues $\mu_1, \dots, \mu_n$, respectively. Consider $x' \in M$. Join x to x' by a piecewise differentiable curve γ, and define vector fields $\mathcal{Z}_1, \dots, \mathcal{Z}_n$ by parallel extension of $Z_1, \dots, Z_n$ along γ. It is obvious that $A^*_\xi\mathcal{Z}_1, \dots, A^*_\xi\mathcal{Z}_n$ are also parallel along γ and the equation $A^*_\xi\mathcal{Z}_k = \mu_k\mathcal{Z}_k$ must hold at x'. Hence, if $(A^*_\xi)_x = 0$, then $A^*_\xi = 0$ on M since $A^*_\xi\xi = 0$. Therefore M is totally geodesic. We therefore assume that $(A^*_\xi)_x$ has two distinct eigenvalues, λ_1 and λ_2, of multiplicities n_1 and n_2 respectively, and we define

$$P_{\lambda_1} = \{X \in S(TM)_x : x \in M,\ \ A^*_\xi X = \lambda_1 X\},$$
$$P_{\lambda_2} = \{X \in S(TM)_x : x \in M,\ \ A^*_\xi X = \lambda_2 X\}.$$

The two principal distributions P_{λ_1}, P_{λ_2} are smooth. Moreover, as $\tau = 0$ and λ_1 (resp. λ_2) is constant along P_{λ_1} (resp. P_{λ_2}) (see [**14**]), we derive $A^*_\xi\nabla^*_X Y = \nabla^*_X A^*_\xi Y = \lambda_1\nabla^*_X Y$, for all $Y \in \Gamma(P_{\lambda_1})$ and $X \in \Gamma(S(TM))$, in which the parallel assumption of A^*_ξ along $S(TM)$ has been used. Thus P_{λ_1} is parallel. It can be shown in the same way that P_{λ_2} is parallel. Clearly, these distributions are integrable. Let M_{λ_1} and M_{λ_2} be their leaves of P_{λ_1} and P_{λ_2}, respectively. These leaves are totally geodesic in $S(TM)$, and totally umbilic in $\overline{M}$ (see [**14**] for details). By straightforward argument used originally by de Rham [**3**] now shows that each leaf M' of $S(TM)$ is isometric to the product $M_{\lambda_1} \times M_{\lambda_2}$. □

Moreover, we have the following.

Corollary 3.8. *For an ascreen null hypersurface of a (LCS)-space form $\overline{M}(c)$, every integral leaf $M_{\lambda_1}(c_1)$ (resp. $M_{\lambda_2}(c_2)$) is a manifold of constant sectional curvature*

$$c_1 = \frac{1}{b^2}\lambda_1^2 - \frac{2\alpha}{b}\lambda_1 + c \quad (\textit{resp. } c_2 = \frac{1}{b^2}\lambda_2^2 - \frac{2\alpha}{b}\lambda_2 + c).$$

Moreover, the following hold;

(1) *if $c = 0$, then*

$$c_1 = \frac{\alpha^2}{n_1^2}(n - 2n_1)(n - 4n_1) \quad \textit{and} \quad c_2 = \frac{\alpha^2}{(n - n_1)^2}(n - 2n_1)(3n - 4n_1),$$

(2) *if $\alpha = 1$ then $c = 1$; $c_1 = \frac{n^2}{n_1^2}$, $\ \ c_2 = 0$ or $c_1 = 0$, $\ \ c_2 = \frac{n^2}{(n-n_1)^2}$.*

(3) *under (1) and (2) above, there exist a unique mapping*

$$\phi : M' \longrightarrow \mathbb{S}^{n_1}\left(\frac{\sqrt{c_1}}{c_1}\right) \times \mathbb{S}^{n-n_1}\left(\frac{\sqrt{c_2}}{c_2}\right), \ \ 0 < n_1 < \frac{n}{4},$$

$$\textit{and } \ \phi : M' \longrightarrow \mathbb{S}^{n_1}\left(\frac{\sqrt{c_1}}{c_1}\right) \times \mathbb{R}^{n-n_1}, \ \ 0 < n_1 < n,$$

respectively.

Proof. Let $\{Z_k\}_{k=1}^{n_1}$ be an orthonormal basis of M_{λ_1}. Then, by (3.14), we have

$$g(R^*(Z_i, Z_j)Z_j, Z_i) = c + \frac{1}{b^2}\lambda_1^2 - \frac{2\alpha}{b}\lambda_1, \quad \forall\, i, j \in \{1, \dots, n_1\}, \tag{3.21}$$

in which we have used the fact $2ab + 1 = 0$. Then c_1 follows from (3.21) easily by definition of sectional curvature. In the same way c_2 follows. Notice that λ_1 is constant along P_{λ_1} (see [**14**] for details). We are now left to verify the constancy of b and α. Since M is ascrren and $\tau = 0$, the second relation of (3.4) gives $Xb = \alpha b\theta(X)$, for all $X \in \Gamma(TM)$. This implies that $PXb = 0$ since M is ascreen. On the other hand, using (2.4), we have $PX\alpha = \rho\theta(PX) = 0$. Therefore, both functions are constant along leaves of $S(TM)$. Consequently, c_1 is constant along $M_{\lambda 1}$ since c is a constant. Using similar arguments, the constancy of c_2 can be established. Next, (1) and (2) follows easily from Corollary 3.7 and the expressions of c_1 and c_2. Finally, (3) follows easily as in Theorem 4 of [**10**] by utilizing the fact that A_ξ^* is parallel, which completes the proof. □

4. Proof of Main Result: Theorem 1.1

Using the tools developed in the previous section, we now proceed to the proof of Theorem 1.1. Consider the quasi-orthonormal frame $\{\xi, Z_i\}$ on M, where $TM^\perp = \mathrm{Span}\{\xi\}$ and $S(TM) = \mathrm{Span}\{Z_i\}_{i=1}^n$ and let $\{\xi, N, Z_i\}$ be the corresponding frames field on $\overline{M}$. As M is ascreen, by Theorem 3.3, its Ricci tensor 'Ric' is symmetric. Moreover, by the method of [**6**], we have

$$\mathrm{Ric}(X, Y) = \sum_{i=1}^{n} g(R(X, Z_i)Y, Z_i) + \overline{g}(R(X, \xi)Y, N), \tag{4.1}$$

for all $X, Y \in \Gamma(TM)$. Using (2.19) and (3.5), we have

$$\begin{aligned} g(R(X, Z_i)Y, Z_i) &= \overline{g}(\overline{R}(X, Z_i)Y, Z_i) + B(X, Y)C(Z_i, Z_i) \\ &\quad - B(Y, Z_i)C(X, Z_i) \\ &= \overline{g}(\overline{R}(X, Z_i)Y, Z_i) - \frac{a}{b}B(X, Y)B(Z_i, Z_i) \\ &\quad - \frac{\alpha}{b}B(X, Y)g(Z_i, Z_i) + \frac{a}{b}g(A_\xi^* X, Z_i)g(Z_i, A_\xi^* Y) \\ &\quad + \frac{\alpha}{b}B(Y, Z_i)g(Z_i, X). \end{aligned} \tag{4.2}$$

Replacing (4.2) in (4.1), we get

$$\begin{aligned} \mathrm{Ric}(X, Y) &= \sum_{i=1}^{n} \overline{g}(\overline{R}(X, Z_i)Y, Z_i) - \frac{a}{b}B(X, Y)\mathrm{tr}A_\xi^* \\ &\quad - \frac{\alpha n}{b}B(X, Y) + \frac{a}{b}g(A_\xi^* X, A_\xi^* Y) + \frac{\alpha}{b}B(Y, X), \end{aligned} \tag{4.3}$$

for all $X, Y \in \Gamma(TM)$, where $\mathrm{tr}(\cdot)$ is the trace operator. As $\overline{M}$ is a space constant of constant curvature c and M is minimal, (4.3) gives

$$\mathrm{Ric}(X, Y) = c(1 - n)g(X, Y) + \frac{\alpha}{b}(1 - n)B(X, Y) + \frac{a}{b}g(A_\xi^* X, A_\xi^* Y). \tag{4.4}$$

Suppose that Ric is parallel along $S(TM)$, then (4.4) implies that the operator $A := \frac{\alpha}{b}(1-n)A^*_\xi + \frac{a}{b}(A^*_\xi)^2$ satisfies $\nabla^* A = 0$. More precisely, by (4.4) we have

$$\operatorname{Ric}(X,Y) = c(1-n)g(X,Y) + g(AX,Y), \tag{4.5}$$

and the assumption $(\nabla^*_X \operatorname{Ric})(PY,PZ) = 0$ leads to

$$\nabla^*_X \operatorname{Ric}(PY,PZ) - \operatorname{Ric}(\nabla^*_X PY, PZ) - \operatorname{Ric}(PY, \nabla^*_X PZ) = 0. \tag{4.6}$$

As ∇^* is a metric connection, (4.5) and (4.6) gives

$$Xg(AY,PZ) - g(A\nabla^*_X PY, PZ) - g(AY, \nabla^*_X PZ) = 0. \tag{4.7}$$

Furthermore, the first term in (4.7) simplifies as

$$Xg(AY,PZ) = g(\nabla^*_X AY, PZ) + g(AY, \nabla^*_X PZ). \tag{4.8}$$

Putting (4.7) and (4.8) together, we get $g((\nabla^*_X A)PY, PZ) = 0$, and the fact that $S(TM)$ is Riemannian implies that $\nabla^* A = 0$.

It then follows that the eigenvalues of A are constant in value and multiplicity over $S(TM)$. It is easy to see that the eigenvalues of A^*_ξ are also constant in value and multiplicity over $S(TM)$. In fact, let λ be an eigenvalue of A^*_ξ for a given eigenvector $X \in \Gamma(S(TM))$. Then, $\tilde{\lambda} := \frac{\alpha}{b}(1-n)\lambda + \frac{a}{b}\lambda^2$ is an eigenvalue of A with respect to X. As $\tilde{\lambda}$ is constant along $S(TM)$, we have $\frac{\alpha}{b}(1-n)Y\lambda + \frac{2a}{b}\lambda(Y\lambda) = 0$, and thus $[\lambda - \frac{\alpha}{2a}(n-1)](Y\lambda) = 0$, for all $Y \in \Gamma(S(TM))$. Thus, $\lambda = \frac{\alpha}{2a}(n-1)$ or $Y\lambda = 0$.

Case 1: $\lambda \neq \frac{\alpha}{2a}(n-1)$**.** Let $\lambda_1, \ldots, \lambda_k$ be the distinct principal curvatures of A^*_ξ, and set $P_{\lambda_j}(x) = \{X \in S(TM)_x : A^*_\xi X = \lambda_j X, \quad x \in M, \quad j = 1, \ldots, k\}$. As $A := \frac{\alpha}{b}(1-n)A^*_\xi + \frac{a}{b}(A^*_\xi)^2$ satisfy $\nabla^* A = 0$, we have

$$\frac{\alpha}{a}(1-n)(\nabla^*_X A^*_\xi) + (\nabla^*_X A^*_\xi) \circ A^*_\xi + A^*_\xi \circ (\nabla^*_X A^*_\xi) = 0, \tag{4.9}$$

on $S(TM)_x$. Next, we claim that $(\nabla^*_X A^*_\xi)^2 Y = 0$, for each $X \in P_{\lambda_i}(x)$ and $Y \in P_{\lambda_j}(x)$, for $1 \leq i, j \leq n$. We prove our claim as follows. Using (4.9) we have

$$\begin{aligned} A^*_\xi((\nabla^*_X A^*_\xi)Y) &= \frac{\alpha}{a}(n-1)(\nabla^*_X A^*_\xi)Y - (\nabla^*_X A^*_\xi)A^*_\xi Y \\ &= \frac{\alpha}{a}(n-1)(\nabla^*_X A^*_\xi)Y - (\nabla^*_X A^*_\xi)(\lambda_j Y) \\ &= -[2\alpha b(n-1) + \lambda_j](\nabla^*_X A^*_\xi)Y, \end{aligned} \tag{4.10}$$

in which we have used the fact $2ab + 1 = 0$. Thus, from (4.10) we see that

$$(\nabla^*_X A^*_\xi)Y \in P_{-\tilde{\mu}_j}(x), \quad \text{where } \tilde{\mu}_j := 2\alpha b(n-1) + \lambda_j. \tag{4.11}$$

On the other hand, as $\tau = 0$ and $\overline{M}$ is a space of constant curvature c, (2.20) gives

$$(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z), \tag{4.12}$$

for all $X, Y, Z \in \Gamma(TM)$. Then, applying (2.15), (2.16) and (2.17) to (4.12) we derive

$$(\nabla^*_X A^*_\xi)Y = (\nabla^*_Y A^*_\xi)X, \quad \forall X, Y \in \Gamma(S(TM)). \tag{4.13}$$

If $\lambda_i \neq \lambda_j$, (4.11) and (4.13) implies that

$$(\nabla^*_X A^*_\xi)Y \in P_{-\tilde{\mu}_j}(x) \cap P_{-\tilde{\mu}_i}(x), \tag{4.14}$$

and hence $(\nabla^*_X A^*_\xi)Y = 0$. Next, let $\lambda_i = \lambda_j \neq 0$ then $(\nabla^*_X A^*_\xi)Y \in P_{\tilde{\mu}_i}(x)$ and thus, $(\nabla^*_X A^*_\xi)^2 Y = (\nabla^*_X A^*_\xi)((\nabla^*_X A^*_\xi)Y) = 0$. Furthermore, suppose $\lambda_i = \lambda_j = 0$. Extend Y locally to a vector field Y' by parallel translation along geodesics originating from $x \in M$. Thus, $A^{*2}_\xi Y' = 0$, and since A^*_ξ is symmetric and $S(TM)$ is Riemannian, we get $A^*_\xi Y' = 0$. Therefore, $(\nabla^*_X A^*_\xi)Y' = \nabla^*_X A^*_\xi Y' - A^*_\xi \nabla^*_X Y' = 0$. Let $\{Z_i\}_{i=1}^n$ be an orthonormal basis of $S(TM)_x$ such that each Z_i belongs to some $P_{\lambda_i}(x)$. Then, by the symmetry of $(\nabla_{Z_i} A^*_\xi)$, we have

$$\begin{aligned} g(\nabla^* A^*_\xi, \nabla^* A^*_\xi) &= \sum_{i=1}^n \sum_{j=1}^n g((\nabla^*_{Z_i} A^*_\xi)Z_j, (\nabla^*_{Z_i} A^*_\xi)Z_j) \\ &= \sum_{i=1}^n \sum_{j=1}^n g(Z_j, (\nabla^*_{Z_i} A^*_\xi)^2 Z_j) = 0. \end{aligned} \tag{4.15}$$

Thus, as $S(TM)$ is Riemannian, (4.15) implies that $\nabla^* A^*_\xi = 0$.

Case 2: $\lambda = \frac{\alpha}{2a}(n-1)$**.** We construct the geodesic γ through x with initial tangent vector X and we extend Y by parallel translation along γ. Now, since a, b and α are constant along $S(TM)$, we have

$$\nabla^*_X \left(\frac{a}{b} A^{*2}_\xi Y - \frac{\alpha}{b}(n-1) A^*_\xi Y \right) = \left(\frac{a}{b} A^{*2}_\xi - \frac{\alpha}{b}(n-1) A^*_\xi \right) \nabla^*_X Y. \tag{4.16}$$

But $\nabla^*_X Y = 0$ along γ. We therefore conclude from (4.16) that the vector field $A^{*2}_\xi Y - \frac{\alpha}{a}(n-1) A^*_\xi Y$ is parallel along γ. Moreover, the value of this vector at x is

$$\frac{\alpha^2}{4a^2}(n-1)^2 Y - \frac{\alpha}{a}(n-1) \cdot \frac{\alpha}{2a}(n-1) Y = -\frac{\alpha^2}{4a^2}(n-1)^2 Y. \tag{4.17}$$

But the vector $-\frac{\alpha^2}{4a^2}(n-1)^2 Y$ in (4.17) is also parallel along γ. Therefore,

$$A^{*2}_\xi Y - \frac{\alpha}{a}(n-1) A^*_\xi Y = -\frac{\alpha^2}{4a^2}(n-1)^2 Y. \tag{4.18}$$

From (4.18), we have $\left(A^*_\xi - \frac{\alpha}{2a}(n-1) I \right)^2 Y = 0$ along γ. Since $A^*_\xi - \frac{\alpha}{2a}(n-1) I$ is symmetric, we have that $A^*_\xi Y = \frac{\alpha}{2a}(n-1) Y$ along γ. Hence, along γ,

$$(\nabla^*_X A^*_\xi)Y = \nabla^*_X A^*_\xi Y - A^*_\xi \nabla^*_X Y = \nabla^*_X \left(\frac{\alpha}{2a}(n-1) \right) Y = 0. \tag{4.19}$$

We have shown that $(\nabla^*_X A^*_\xi)Y = 0$ for any pair of screen principal vectors X and Y at any point $x \in M$. Since the principal vectors span $S(TM)_x$, we have shown that $\nabla^* A^*_\xi = 0$. Our result then follows from Theorem 3.3, Proposition 3.5, and its corollaries, which ends the proof.

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