

WEIGHTED FIRST MOMENTS OF SOME SPECIAL QUADRATIC DIRICHLET L -FUNCTIONS

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(Received August 7, 2019)

Abstract. In this paper, we obtain asymptotic formulas for weighted first moments of central values of families of primitive quadratic Dirichlet L -functions whose conductors comprise only primes that split in a given quadratic number field. We then deduce a non-vanishing result of these L -functions at the point $s = 1/2$.

1. Introduction

It is a conjecture due to S. Chowla [2] that a Dirichlet L -function is never zero at the central point $s = 1/2$. One way to address this problem is by studying the moments of central values of L -functions. For the family of quadratic Dirichlet L -functions, M. Jutila [10] obtained the first and second moments of $L(1/2, \chi_d)$ with χ_d being the Kronecker symbol. The error term in the asymptotic formula for the first moment in [10] was later improved in [7, 18, 17]. For the second and third moment of this quadratic family, K. Soundararajan obtained asymptotic formulas with power savings in [16]. The error term for the third moment was improved by A. Diaconu, D. Goldfeld and J. Hoffstein [4] and later further improved by M. P. Young [19]. More recently, an explicit lower order term in the third moment was found in [5] and under the assumption of the generalized Riemann hypothesis for Dirichlet L -function, an asymptotic formula for the fourth moment was proved in [15]. For families of Dirichlet L -functions associated with characters of higher orders, we note that S. Baier and M. P. Young studied the first and second moments of $L(1/2, \chi)$ for cubic Dirichlet L -functions in [1]. With the knowledge of these moments, one can deduce, in manners not unlike the proof of Corollary 1.2, results on the non-vanishing of the L -functions under consideration.

In this paper, we study the first moments of central values of certain subfamilies of quadratic Dirichlet L -functions. Our result is motivated by the class field theory, which implies that when a number field is Galois over \mathbb{Q} , then the set of prime numbers in \mathbb{Q} that split completely in it determines the number field uniquely. For this reason, it is interesting to study the families of primitive quadratic Dirichlet L -functions whose conductors comprise only primes that split in a given number field.

2010 *Mathematics Subject Classification* 11M06, 11M41.

Key words and phrases: quadratic Dirichlet characters, quadratic Hecke characters, quadratic Dirichlet L -functions, Hecke L -functions.

We now let K be a quadratic number field and let $S(K)$ be the set of odd rational integers that comprises only prime factors that split completely in K , i.e.

$$S(K) = \{q \in \mathbb{Z} : (q, 2) = 1, p|q \Rightarrow p \text{ splits completely in } K\}.$$

For technical reasons, other than a smooth weight, we consider the average of the central values of L -functions with an extra weight which essentially measures the number of distinct rational prime factors of the conductor of a given character. For $q \in \mathbb{Z}$, let $\omega(q)$ be the number of distinct rational prime factors of q . Then, our result is

Theorem 1.1. *Let K be a quadratic number field and let $w : (0, \infty) \rightarrow \mathbb{R}$ be a smooth, compactly supported function. Then for any $\varepsilon > 0$,*

$$\sum_{q \in S(K)} \sum_{\substack{\chi \bmod q \\ \chi^2 = \chi_0}}^* 2^{\omega(q)} L(1/2, \chi) w\left(\frac{q}{Q}\right) = Q P_K(\log Q) + O\left(Q^{1-\delta_0+\varepsilon} + Q^{3/4+\varepsilon}\right), \quad (1.1)$$

where $P_K(x)$ is a linear function whose coefficients depend only on K and w (see (3.4) below for the expression for $P_K(x)$), δ_0 is the currently best known constant in the subconvexity bound for a degree two L -function over \mathbb{Q} (see (3.8) below). Here the $*$ on the sum over χ restricts the sum to primitive characters and the implicit constant in the error term depends on K , w and ε .

As alluded to earlier, one can readily deduce the following non-vanishing result from the above theorem.

Corollary 1.2. *Let $Q \in \mathbb{N}$ and sufficiently large. We have*

$$\#\{\chi : \chi^2 = \chi_0, \chi \text{ with conductor } q, q \in S(K) \cap [1, Q], L(1/2, \chi) \neq 0\} \gg \frac{Q}{\log^{17} Q}.$$

Proof. It is well-known that a primitive quadratic Dirichlet character with odd conductor q coincides with the Jacobi symbol modulo q and q must be square-free (see [3, p.40]). Note also that $2^{\omega(q)} = \tau(q)$ if q is square-free. Here $\tau(q)$ is the divisor function. Therefore, using Theorem 1.1 and Hölder's inequality and choose a smooth function w with support in $[0, 1]$, we get

$$Q \log Q \ll \left(\sum_{\substack{q \in S(K) \\ q \leq Q}} \sum_{\substack{\chi \bmod q \\ \chi^2 = \chi_0 \\ L(1/2, \chi) \neq 0}}^* 1 \right)^{1/4} \left(\sum_{\substack{q \leq Q \\ \chi^2 = \chi_0}} \sum_{\chi \bmod q}^* \tau^4(q) \right)^{1/4} \left(\sum_{\substack{q \leq Q \\ \chi^2 = \chi_0}} \sum_{\chi \bmod q}^* L^2(1/2, \chi) \right)^{1/2}.$$

The second factor above is $O(Q^{1/4} \log^{15/4} Q)$ (see [9, (1.80)]). Using Theorem 2 of [10], the third factor is $O(Q^{1/2} \log^{3/2} Q)$. The corollary follows from these estimates. \square

Before presenting the proof of Theorem 1.1, we will give a brief summary of it. We start by applying the approximate functional equation to the left-hand side of (1.1). Those Dirichlet characters under consideration can be identified with Hecke characters in K (see Lemma 2.2). Using Mellin inversion, we are led to study sums involving Hecke L -functions. Moving the contour to the left, we will, in certain

cases, encounter some poles whose residues will give rise to the main term in (1.1). The remaining terms can all be estimated to give admissible error terms. Among other things, a subconvexity bound for Hecke L -functions is needed in the analysis.

1.3. Notations. The following notations and conventions are used throughout the paper.

K denotes a quadratic number field.

\mathcal{O}_K denotes the ring of integers in K .

D_K denotes the discriminant of K .

$f = O(g)$, $f \ll g$ or $g \gg f$ means $|f| \leq cg$ for some unspecified positive constant c .

μ_K denotes the Möbius function on K .

$\zeta_K(s)$ is the Dedekind zeta function for K .

2. Preliminaries

In this section, we enumerate the tools used throughout the paper.

2.1. Quadratic symbol and primitive quartic Dirichlet characters. For any prime ideal $\mathfrak{p} \subset \mathcal{O}_K$ which is co-prime to (2), we define for $a \in \mathcal{O}_K$, $(a, \mathfrak{p}) = 1$ by $\left(\frac{a}{\mathfrak{p}}\right)_K \equiv a^{(N(\mathfrak{p})-1)/2} \pmod{\mathfrak{p}}$, with $\left(\frac{a}{\mathfrak{p}}\right)_K \in \{\pm 1\}$. When $\mathfrak{p}|(a)$, we set $\left(\frac{a}{\mathfrak{p}}\right)_K = 0$. Then this symbol, $\left(\frac{\cdot}{\mathcal{A}}\right)_K$, can be extended multiplicatively to any ideal $\mathcal{A} \subset \mathcal{O}_K$ with $(\mathcal{A}, 2) = 1$ and is called the quadratic residue symbol in K .

For any $m \in \mathbb{Z}$ and any ideal $\mathcal{A} \subset \mathcal{O}_K$ with $(\mathcal{A}, 2D_K) = 1$, it follows from [11, Proposition 4.2] that

$$\left(\frac{m}{\mathcal{A}}\right)_K = \left(\frac{m}{N(\mathcal{A})}\right)_{\mathbb{Q}}, \quad (2.1)$$

where $(\cdot)_{\mathbb{Q}}$ denotes the Jacobi symbol in \mathbb{Q} .

In particular, if p is an odd prime in \mathbb{Q} that splits completely in K and \mathfrak{p} is a prime ideal in \mathcal{O}_K lying above (p) , then for any $m \in \mathbb{Z}$,

$$\left(\frac{m}{\mathfrak{p}}\right)_K = \left(\frac{m}{p}\right)_{\mathbb{Q}}. \quad (2.2)$$

When K is a quadratic number field, it is well-known from algebraic number theory (see [6, pp. 111, 117]) that a prime ideal (p) in \mathbb{Z} can either ramify, split (completely) or stay inert in \mathcal{O}_K . Moreover, a prime ideal p in \mathbb{Z} ramifies in K if and only if p divides D_K (see [6, Theorem 22]). It follows from this and (2.2) that we have the following classification of all the primitive quadratic Dirichlet characters of conductor $q \in S(K)$:

Lemma 2.2. *Primitive quadratic Dirichlet characters of conductor $q \in S(K)$ are of the form $\chi_{\mathcal{A}} : m \rightarrow \left(\frac{m}{\mathcal{A}}\right)_K$ for some ideal $\mathcal{A} \subset \mathcal{O}_K$, \mathcal{A} square-free, co-prime to $2D_K$ and not divisible by any rational primes, with norm $N(\mathcal{A}) = q$. Moreover, there are precisely $2^{\omega(q)}$ different ideals in \mathcal{O}_K satisfying the above conditions that give rise to the same Dirichlet character.*

2.3. The approximate functional equation. We have the following approximate functional equation from [9, Theorem 5.3]):

Proposition 2.4. *Let χ be a primitive Dirichlet character χ of conductor q . For any $\alpha \in \mathbb{C}, j \in \{\pm 1\}$, let*

$$a_j = \frac{1-j}{2}, \quad \epsilon(\chi) = i^{-a_{\chi(-1)}} q^{-1/2} \tau(\chi),$$

where $\tau(\chi)$ is the Gauss sum associated with χ . We define

$$V_j(x) = \frac{1}{2\pi i} \int_{(2)} \gamma_j(s) x^{-s} \frac{ds}{s}, \quad \text{where } \gamma_j(s) = \pi^{-s/2} \frac{\Gamma\left(\frac{1/2+a_j+s}{2}\right)}{\Gamma\left(\frac{1/2+a_j}{2}\right)}. \quad (2.3)$$

Furthermore, let A and B be positive real numbers such that $AB = q$. Then we have

$$L(1/2, \chi) = \sum_{m=1}^{\infty} \frac{\chi(m)}{m^{1/2}} V_{\chi(-1)}\left(\frac{m}{A}\right) + \epsilon(\chi) \sum_{m=1}^{\infty} \frac{\bar{\chi}(m)}{m^{1/2}} V_{\chi(-1)}\left(\frac{m}{B}\right). \quad (2.4)$$

Note that (see [16, Lemma 2.1]) $V_{\pm 1}(\xi)$ are real-valued and smooth on $[0, \infty)$ and for the l -th derivative of $V_{\pm 1}(\xi)$, we have

$$V_{\pm 1}(\xi) = 1 + O(\xi^{1/2-\epsilon}) \text{ for } 0 < \xi < 1 \quad \text{and} \quad V_{\pm 1}^{(l)}(\xi) = O(e^{-\xi}) \text{ for } \xi > 0, l \geq 0. \quad (2.5)$$

We remark here that the estimates in (2.5) are only proved for V_{+1} in [16] and the proof for V_{-1} is similar and one get the same bounds with different implied constants.

3. Proof of Theorem 1.1

We let

$$\mathcal{M} := \sum_{q \in S(K)} \sum_{\substack{\chi \bmod q \\ \chi^2 = \chi_0}}^* 2^{\omega(q)} L(1/2, \chi) w\left(\frac{q}{Q}\right).$$

Applying the approximate functional equation (2.4) with $A = B = \sqrt{q}$ gives, noting that it follows from [3, Chap. 2] that $\epsilon(\chi) = 1$ when χ is quadratic,

$$\mathcal{M} = 2 \sum_{q \in S(K)} \sum_{\substack{\chi \bmod q \\ \chi^2 = \chi_0}}^* 2^{\omega(q)} \sum_{m=1}^{\infty} \frac{\chi(m)}{\sqrt{m}} V_{\chi(-1)}\left(\frac{m}{\sqrt{q}}\right) w\left(\frac{q}{Q}\right).$$

Applying the above with Lemma 2.2 again, we have $\mathcal{M} = \mathcal{M}^+ + \mathcal{M}^-$, with

$$\mathcal{M}^{\pm} = \sum_{\mathcal{A}}' (1 \pm \chi_{\mathcal{A}}(-1)) \sum_{m=1}^{\infty} \frac{\chi_{\mathcal{A}}(m)}{\sqrt{m}} V_{\chi_{\mathcal{A}}(-1)}\left(\frac{m}{\sqrt{N(\mathcal{A})}}\right) w\left(\frac{N(\mathcal{A})}{Q}\right),$$

where the dash on the sum over \mathcal{A} indicates that the sum runs over square-free ideals of \mathcal{O}_K that are co-prime to $2D_K$ and without rational prime divisor.

It remains to evaluate \mathcal{M}^{\pm} . As the arguments are similar, we will only evaluate \mathcal{M}^+ in the sequel. The results are summarized by

Lemma 3.1. *We have*

$$\mathcal{M}^\pm = QP_K^\pm(\log Q) + O(Q^{1-\delta_0+\varepsilon} + Q^{3/4+\varepsilon}), \quad (3.1)$$

where δ_0 is given as in Theorem 1.1, $P_K^\pm(x)$ are given in (3.4) below.

3.2. Evaluating \mathcal{M}^+ , the main term. We detect the condition that \mathcal{A} has no rational prime divisor using the formula

$$\sum_{\substack{(d)|\mathcal{A} \\ d \in \mathbb{Z}}} \mu_{\mathbb{Z}}(d) = \begin{cases} 1, & \mathcal{A} \text{ has no rational prime divisor,} \\ 0, & \text{otherwise.} \end{cases}$$

Here we define $\mu_{\mathbb{Z}}(d) = \mu(|d|)$, the usual Möbius function. We apply this formula and change variables $\mathcal{A} \rightarrow d\mathcal{A}$ to the sum over \mathcal{A} . Since (d) is square-free as an ideal of \mathcal{O}_K , the condition that $d\mathcal{A}$ is square-free then simply means that \mathcal{A} is square-free and $(d, \mathcal{A}) = 1$. Thus we have $\mathcal{M}^+ = \mathcal{M}_1^+ + \mathcal{M}_2^+$, where

$$\begin{aligned} \mathcal{M}_1^+ &= \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2d\mathcal{D}_K)=1}} \mu_{\mathbb{Z}}(d) \sum_{m=1}^{\infty} \left(\frac{m}{d}\right)_K \frac{1}{\sqrt{m}} \\ &\quad \times \sum_{\substack{\mathcal{A} \\ (A, 2d\mathcal{D}_K)=1}}^* \left(\frac{m}{\mathcal{A}}\right)_K V_1 \left(\frac{m}{Q^{1/2}} \frac{Q^{1/2}}{\sqrt{N(d\mathcal{A})}} \right) w \left(\frac{N(d\mathcal{A})}{Q} \right), \\ \mathcal{M}_2^+ &= \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2d\mathcal{D}_K)=1}} \mu_{\mathbb{Z}}(d) \sum_{m=1}^{\infty} \left(\frac{-m}{d}\right)_K \frac{1}{\sqrt{m}} \\ &\quad \times \sum_{\substack{\mathcal{A} \\ (A, 2d\mathcal{D}_K)=1}}^* \left(\frac{-m}{\mathcal{A}}\right)_K V_1 \left(\frac{m}{Q^{1/2}} \frac{Q^{1/2}}{\sqrt{N(d\mathcal{A})}} \right) w \left(\frac{N(d\mathcal{A})}{Q} \right), \end{aligned}$$

where the asterisks indicate that \mathcal{A} run over square-free ideals of \mathcal{O}_K .

We evaluate \mathcal{M}_1^+ first. Using Möbius inversion to detect the condition that \mathcal{A} is square-free, we get

$$\mathcal{M}_1^+ = \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2d\mathcal{D}_K)=1}} \mu_{\mathbb{Z}}(d) \sum_{\substack{\mathfrak{l} \\ (\mathfrak{l}, 2d\mathcal{D}_K)=1}} \mu_K(\mathfrak{l}) \sum_{m=1}^{\infty} \left(\frac{m}{d\mathfrak{l}^2}\right)_K \frac{1}{\sqrt{m}} \mathcal{M}_1(d, \mathfrak{l}, m),$$

where

$$\mathcal{M}_1(d, \mathfrak{l}, m) = \sum_{\substack{\mathcal{A} \\ (A, 2d\mathcal{D}_K)=1}} \left(\frac{m}{\mathcal{A}}\right)_K V_1 \left(\frac{m}{Q^{1/2}} \frac{Q^{1/2}}{\sqrt{N(d\mathfrak{l}^2\mathcal{A})}} \right) w \left(\frac{N(d\mathfrak{l}^2\mathcal{A})}{Q} \right).$$

Next we use the Mellin transform of the weight function to express the sum over \mathcal{A} as a contour integral involving the Hecke L -function. By Mellin inversion,

$$V_1 \left(\frac{m}{Q^{1/2}} \frac{Q^{1/2}}{\sqrt{N(d\mathfrak{l}^2\mathcal{A})}} \right) w \left(\frac{N(d\mathfrak{l}^2\mathcal{A})}{Q} \right) = \frac{1}{2\pi i} \int_{(2)} \left(\frac{Q}{N(d\mathfrak{l}^2\mathcal{A})} \right)^s \tilde{f}(s) ds,$$

where

$$\tilde{f}(s) = \int_0^\infty V_1 \left(\frac{m}{Q^{1/2}} x^{-1/2} \right) w(x) x^{s-1} dx.$$

Integration by parts and using (2.5) shows $\tilde{f}(s)$ is a function satisfying the bound for all $\operatorname{Re}(s) > 0$, $E \in \mathbb{N}$,

$$\tilde{f}(s) \ll (1 + |s|)^{-E} \left(1 + m/Q^{1/2} \right)^{-E}. \quad (3.2)$$

Here the implied constant depends on E .

We then have

$$\mathcal{M}_1(d, \mathfrak{l}, m) = \frac{1}{2\pi i} \int_{(2)} \left(\frac{Q}{N(d\mathfrak{l}^2)} \right)^s L(s, \psi_{4mD_K^2 d^2}) \tilde{f}(s) ds,$$

where

$$L(s, \psi_{4mD_K^2 d^2}) = \sum_{0 \neq \mathcal{A} \subset \mathcal{O}_K} \psi_{4mD_K^2 d^2}(\mathcal{A}) N(\mathcal{A})^{-s},$$

and

$$\psi_{4mD_K^2 d^2}(\mathcal{A}) := \begin{cases} \left(\frac{4mD_K^2 d^2}{\mathcal{A}} \right)_K & \text{when } (\mathcal{A}, 2D_K) = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (3.3)$$

Note that via (2.1) and the quadratic reciprocity law in \mathbb{Q} , there exists a positive integer e independent of m, d such that $\psi_{4mD_K^2 d^2}(a) = 1$ for any $a \in \mathcal{O}_K$ satisfying $a \equiv 1 \pmod{md^2(2D_K)^e}$. It follows from [14, p. 470] that $\psi_{4mD_K^2 d^2}$ can be regarded as a Hecke character of trivial infinite type $\pmod{md^2(2D_K)^e}$.

We estimate \mathcal{M}_1^+ by moving the contour to the line with $\Re s = 1/2$. When m is a square the Hecke L -function has a pole at $s = 1$. We set \mathcal{M}_0 to be the contribution to \mathcal{M}_1^+ of these residues, and \mathcal{M}'_1 to be the remainder.

We evaluate \mathcal{M}_0 first. Note that

$$\begin{aligned} \mathcal{M}_0 &= \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2D_K) = 1}} \mu_{\mathbb{Z}}(d) \sum_{\substack{\mathfrak{l} \\ (1, 2dD_K) = 1}} \mu_K(\mathfrak{l}) \\ &\quad \times \sum_{m=1}^{\infty} \left(\frac{m}{d\mathfrak{l}^2} \right)_K \frac{1}{\sqrt{m}} \frac{Q}{N(d\mathfrak{l}^2)} \tilde{f}(1) \operatorname{Res}_{s=1} L(s, \psi_{4mD_K^2 d^2}), \end{aligned}$$

where using the Mellin inversion formula yields

$$\tilde{f}(1) = \int_0^\infty V_1 \left(\frac{m}{Q^{1/2}} x^{-1/2} \right) w(x) dx = \frac{1}{2\pi i} \int_{(2)} \left(\frac{Q^{1/2}}{m} \right)^s \tilde{w} \left(1 + \frac{s}{2} \right) \gamma_1(s) \frac{ds}{s},$$

with $\gamma_1(s)$ defined in (2.3) and

$$\tilde{w}(s) = \int_0^\infty w(x) x^{s-1} dx.$$

From our discussions above, it is not difficult to see that $\psi_{4mD_K^2} d^2$ is the principal character only if m is a square, in which case

$$L(s, \psi_{4mD_K^2} d^2) = \zeta_K(s) \prod_{\mathfrak{p}|2dmD_K} (1 - N(\mathfrak{p})^{-s}).$$

Here and in what follows, we use \mathfrak{p} or \mathfrak{p}_i to denote prime ideals in \mathcal{O}_K .

Let $C_{K,1}$ be the residue of $\zeta_K(s)$ at $s = 1$, then

$$\begin{aligned} \mathcal{M}_0 &= C_{K,1} Q \sum_{m=1}^{\infty} \frac{\tilde{f}(1)}{m} \prod_{\mathfrak{p}|2mD_K} (1 - N(\mathfrak{p})^{-1}) \\ &\quad \times \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2mD_K)=1}} \frac{\mu_{\mathbb{Z}}(d)}{d^2} \prod_{\mathfrak{p}|d} (1 - N(\mathfrak{p})^{-1}) \sum_{(l, 2mdD_K)=1} \frac{\mu_K(l)}{N(l^2)}. \end{aligned}$$

Computing the sum over l explicitly, we obtain

$$\begin{aligned} \mathcal{M}_0 &= C_{K,1} \zeta_K^{-1}(2) Q \sum_{m=1}^{\infty} \frac{\tilde{f}(1)}{m} \prod_{\mathfrak{p}|2mD_K} (1 - N(\mathfrak{p})^{-1}) \\ &\quad \times \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2mD_K)=1}} \frac{\mu_{\mathbb{Z}}(d)}{d^2} \prod_{\mathfrak{p}|d} (1 - N(\mathfrak{p})^{-1}) \prod_{\mathfrak{p}|2mdD_K} (1 - N(\mathfrak{p})^{-2})^{-1} \\ &= C_{K,1} \zeta_K^{-1}(2) Q \sum_{m=1}^{\infty} \frac{\tilde{f}(1)}{m} \prod_{\mathfrak{p}|2mD_K} (1 + N(\mathfrak{p})^{-1})^{-1} \\ &\quad \times \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2mD_K)=1}} \frac{\mu_{\mathbb{Z}}(d)}{d^2} \prod_{\mathfrak{p}|d} (1 + N(\mathfrak{p})^{-1})^{-1}. \end{aligned}$$

We define

$$C_{K,2} = \prod_{\mathfrak{p}|2D_K} (1 + N(\mathfrak{p})^{-1})^{-1} \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2D_K)=1}} \frac{\mu_{\mathbb{Z}}(d)}{d^2} \prod_{\mathfrak{p}|d} (1 + N(\mathfrak{p})^{-1})^{-1}.$$

It is clear that $C_{K,2}$ is a constant. Using this and setting $\tilde{m} = m/(m, 2D_K)$, we have

$$\begin{aligned} \mathcal{M}_0 &= C_{K,1} C_{K,2} \zeta_K^{-1}(2) Q \\ &\quad \times \sum_{m=1}^{\infty} \frac{\tilde{f}(1)}{m} \prod_{\mathfrak{p}_1|\tilde{m}} (1 + N(\mathfrak{p}_1)^{-1})^{-1} \prod_{\mathfrak{p}|\tilde{m}} \left(1 - p^{-2} \prod_{\mathfrak{p}_2|p} (1 + N(\mathfrak{p}_2)^{-1})^{-1} \right)^{-1}, \end{aligned}$$

where p runs over rational primes. Let

$$C_K(s) = \zeta^{-1}(s) \sum_{m=1}^{\infty} m^{-s} \prod_{\mathfrak{p}_1|\tilde{m}} (1 + N(\mathfrak{p}_1)^{-1})^{-1} \prod_{\mathfrak{p}|\tilde{m}} \left(1 - p^{-2} \prod_{\mathfrak{p}_2|p} (1 + N(\mathfrak{p}_2)^{-1})^{-1} \right)^{-1},$$

where $\zeta(s)$ is the Riemann zeta-function. Expressing $C_K(s)$ as an Euler product, one checks easily that $C_K(s)$ is holomorphic, converges absolutely for $\operatorname{Re}(s) \geq 1/2 + \delta > 1/2$ and can be analytically continued to $\operatorname{Re}(s) > 1/2$. Then

$$\mathcal{M}_0 = C_{K,1} C_{K,2} \zeta_K^{-1}(2) Q \frac{1}{2\pi i} \int_{(2)} Q^{s/2} C_K(1+s) \zeta(1+s) \tilde{w} \left(1 + \frac{s}{2}\right) \gamma_1(s) \frac{ds}{s}.$$

We move the contour of integration to $-1/2 + \varepsilon$, crossing a pole of order 2 at $s = 0$ only. The new contour contributes $O(Q^{3/4+\varepsilon})$, by noting that we have $\zeta(1+s) \ll |1+s|$ on this line (see (3) on p. 79 of [3]). Using the fact that the Laurent expansion of $\zeta(s)$ at $s = 1$ has the form [13, Corollary 1.16]

$$\zeta(s) = \frac{1}{s-1} + \gamma_0 + \sum_{k=1}^{\infty} a_k (s-1)^k, \quad a_k \in \mathbb{C},$$

where γ_0 is the Euler constant. The pole at $s = 0$ gives $QP_K^+(\log Q)$, where we define

$$P_K(x) = P_K^+(x) + P_K^-(x), \quad \text{with} \quad (3.4)$$

$$\begin{aligned} P_K^+(x) &= C_{K,1} C_{K,2} \zeta_K^{-1}(2) \\ &\quad \times \left(\frac{1}{2} C_K(1) \tilde{w}(1)x + C'_K(1) + \frac{\tilde{w}'(1)}{2} + \gamma'_1(0) + \gamma_0 (C_K(1) + \tilde{w}(1)) \right), \\ P_K^-(x) &= C_{K,1} C_{K,2} \zeta_K^{-1}(2) \\ &\quad \times \left(\frac{1}{2} C_K(1) \tilde{w}(1)x + C'_K(1) + \frac{\tilde{w}'(1)}{2} + \gamma'_{-1}(0) + \gamma_0 (C_K(1) + \tilde{w}(1)) \right). \end{aligned}$$

We then conclude that

$$\mathcal{M}_0 = QP_K^+(\log Q) + O\left(Q^{3/4+\varepsilon}\right). \quad (3.5)$$

3.3. Evaluating \mathcal{M}'_1 and \mathcal{M}'_2 . In this section, we estimate \mathcal{M}'_1 and \mathcal{M}'_2 . Recall that \mathcal{M}'_1 is $\mathcal{M}'_1 - \mathcal{M}_0$. More precisely,

$$\begin{aligned} \mathcal{M}'_1 &= \frac{1}{2\pi i} \sum_{\substack{(d), d \in \mathbb{Z} \\ (d, 2D_K)=1}} \mu_{\mathbb{Z}}(d) \sum_{\substack{(l, 2dD_K)=1}} \mu_K(l) \\ &\quad \times \sum_{m=1}^{\infty} \left(\frac{m}{d^2}\right)_K \frac{1}{\sqrt{m}} \int_{(1/2)} \left(\frac{Q}{N(d^2)}\right)^s L(s, \psi_{4mD_K^2 d^2}) \tilde{f}(s) ds. \end{aligned}$$

By bounding everything with absolute values and using (3.2) to bound \tilde{f} , we see that, for some large $E \in \mathbb{N}$,

$$\begin{aligned} |\mathcal{M}'_1| &\ll \sum_{d \leq c_1 \sqrt{Q}} \sum_{N(l) \leq c_2 \sqrt{Q}} \frac{1}{\sqrt{N(d^2)}} \\ &\quad \times \sum_m \frac{\sqrt{Q}}{\sqrt{m}} \left(1 + m/Q^{1/2}\right)^{-E} \int_0^{\infty} \left|L(1/2 + it, \psi_{4mD_K^2 d^2})\right| (1 + |t|)^{-E} dt. \end{aligned} \quad (3.6)$$

Here c_1 and c_2 are constants, chosen according to the size of the support the weight function w . In view of the factor $(1 + m/Q^{1/2})^{-E}$, we may truncate the sum over m above to $m \leq M \ll Q^{1/2+\varepsilon}$ for $\varepsilon > 0$ with a small error. Indeed, if $m \gg Q^{1/2+\varepsilon}$, then

$$\left(1 + \frac{m}{Q^{1/2}}\right)^{-1} \ll \min\left\{\frac{Q^{1/2}}{m}, \frac{1}{Q^\varepsilon}\right\}.$$

So taking E large so that $\varepsilon E > 10$, using the convexity found for the L -function and summing up all the other variables trivially, we see that the contribution to M'_1 from $m \gg Q^{1/2+\varepsilon}$ is

$$\ll Q^{-5}$$

which is negligible.

To estimate the contribution from the small m 's, we need a better bound for $L\left(1/2 + it, \psi_{4mD_K^2 d^2}\right)$. The character $\psi_{4mD_K^2 d^2}$ is induced by a primitive character ψ' with conductor f satisfying $f|m(2D_K)^2$. Note that from its definition in (3.3), $\psi_{4mD_K^2 d^2}$ is induced by a character, not necessarily primitive, modulo $4mD_K^2$. So the conductor f here is independent of d . It follows that for any $\varepsilon > 0$,

$$L\left(1/2 + it, \psi_{4mD_K^2 d^2}\right) \ll (md)^\varepsilon |L(1/2 + it, \psi')|, \quad (3.7)$$

where the implied constant here depends on ε .

Now, the Hecke L -function $L(s, \psi')$, viewed as a degree two L -function over \mathbb{Q} , has analytic conductor $\ll N(m)(1 + t^2)$ (see [8, Theorem 12.5]). It follows from a result on the subconvexity bound for degree two L -functions over any fixed number field by P. Michel and A. Venkatesh [12] that we have an absolute constant $\delta_0 > 0$, independent of the number field, such that

$$L(1/2 + it, \psi') \ll (N(m)(1 + t^2))^{1/4-\delta_0}. \quad (3.8)$$

Applying (3.7) and (3.8), we deduce the following estimation to bound the sum over m :

$$\sum_{m \leq M} \frac{1}{\sqrt{m}} \left| L\left(1/2 + it, \psi_{4mD_K^2 d^2}\right) \right| \ll d^\varepsilon M^{1-2\delta_0+\varepsilon} (1 + t^2)^{1/4-\delta_0}.$$

We sum trivially over d and l in (3.6) to see that

$$|\mathcal{M}'_1| \ll Q^{1-\delta_0+\varepsilon}. \quad (3.9)$$

Using similar arguments, we obtain the same estimation for \mathcal{M}'_2 as above. Combining (3.9) with (3.5) gives (3.1).

3.4. Conclusion. As one readily deduces (1.1) from Lemma 3.1, this completes the proof of Theorem 1.1.

Acknowledgments. P. G. is supported in part by NSFC grant 11871082 and L. Z. by the FRG grant PS43707 and the Silverstar Fund PS49334 at UNSW. The authors thank the anonymous referee for his/her careful reading of the paper and helpful comments.

References

- [1] S. Baier and M.P. Young, *Mean values with cubic characters*, J. Number Theory, **130** (4) (2010), 879–903.
- [2] S. Chowla, *The Riemann Hypothesis and Hilbert’s Tenth Problem*, Mathematics and its Applications 4, Gordon and Breach Science Publishers, New York-London-Paris, 1965.
- [3] H. Davenport, *Multiplicative Number Theory*, third edition, Graduate Texts in Mathematics, 74, Springer-Verlag, Berlin, etc., 2000.
- [4] A. Diaconu, D. Goldfeld, and J. Hoffstein, *Multiple Dirichlet series and moments of zeta and L-functions*, Compositio Math., **139** (3) (2003), 297–360.
- [5] A. Diaconu and I. Whitehead, *On the third moment of $L(1/2, \chi_d)$ II: the number field case*, preprint. arXiv: 1804.00690.
- [6] A. Fröhlich and M.J. Taylor, *Algebraic Number Theory*, Cambridge Studies in Advanced Mathematics, 27, Cambridge University Press, Cambridge, 1993.
- [7] D. Goldfeld and J. Hoffstein, *Eisenstein series of $\frac{1}{2}$ -integral weight and the mean value of real Dirichlet L-series*, Invent. Math., **80** (2) (1985), 185–208.
- [8] H. Iwaniec, *Topics in classical automorphic forms*, Graduate Studies in Mathematics, 17, American Mathematical Society, Providence, RI, 1997.
- [9] H. Iwaniec and E. Kowalski, *Analytic Number Theory*, American Mathematical Society Colloquium Publications, 53, American Mathematical Society, Providence, RI, 2004.
- [10] M. Jutila, *On the mean value of $L(1/2, \chi)$ for real characters*, Analysis, **1** (2) (1981), 149–161.
- [11] F. Lemmermeyer, *Reciprocity Laws. From Euler to Eisenstein*, Springer-Verlag, Berlin, 2000.
- [12] P. Michel and A. Venkatesh, *The subconvexity problem for GL_2* , Publ. Math. Inst. Hautes Études Sci., **111** (2010), 171–271.
- [13] H.L. Montgomery and R.C. Vaughan, *Multiplicative Number Theory. I. Classical Theory*, Cambridge Studies in Advanced Mathematics, 97, Cambridge University Press, Cambridge, 2007.
- [14] J. Neukirch, *Algebraic Number Theory*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 322, Springer-Verlag, Berlin, 1999.
- [15] Q. Shen, *The fourth moment of quadratic Dirichlet L-functions*, preprint. arXiv: 1907.01107.
- [16] K. Soundararajan, *Nonvanishing of quadratic Dirichlet L-functions at $s = \frac{1}{2}$* , Ann. of Math., **152** (2) (2000), 447–488.
- [17] A.I. Vinogradov and L.A. Takhtadzhyan, *Analogues of the Vinogradov-Gauss formula on the critical line*, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI), **109** (1981), 81–82, 180–181, 182–183.
- [18] M.P. Young, *The first moment of quadratic Dirichlet L-functions*, Acta Arith., **138** (1) (2009), 73–99.
- [19] M.P. Young, *The third moment of quadratic Dirichlet L-functions*, Selecta Math. (N.S.), **19** (2) (2013), 509–543.

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