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YAMABE SOLITONS IN CONTACT GEOMETRY

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Abstract. It is shown that the scalar curvature of a Yamabe soliton as a Sasakian manifold is constant and the soliton vector field is Killing. The same conclusion is shown to hold for a Yamabe soliton as a K-contact manifold M^{2n+1} if any one of the following conditions hold: (i) its scalar curvature is constant along the soliton vector field V, (ii) V is an eigenvector of the Ricci operator with eigenvalue $2n$, (iii) V is gradient.

1. Introduction

The evolution of a Riemannian metric g on a smooth manifold M to a metric $g(t)$ in time t through the equation

$$\frac{\partial}{\partial t} g(t) = -r(t)g(t), \quad g(0) = g,$$

where $r(t)$ denotes the scalar curvature of $g(t)$, is called the Yamabe flow, and was introduced by Hamilton [**8**]. The Yamabe flow is a natural geometric deformation to metrics of constant scalar curvature, and corresponds to the fast diffusion case of the porous medium equation (the plasma equation) in mathematical physics (Burchard et al. [**3**]). Just as a Ricci soliton is a special solution of the Ricci flow, a Yamabe soliton is a special solution of the Yamabe flow that moves by a one parameter family of diffeomorphisms ϕ_t generated by a time-dependent vector field W_t on M, and homotheties, i.e., $g(t) = \sigma(t)\phi_t^* g$, where σ is a positive real valued function of the parameter t. Substituting the foregoing equation in the Yamabe flow equation and setting $\sigma(0) = 1, -\dot{\sigma}(0) = c$ gives the equation

$$\pounds_V g = 2(c - r)g, \tag{1.1}$$

where g is the initial metric $g(0)$ of the Yamabe flow, V is a vector field on M such that $W_t = \frac{1}{2\sigma(t)} V$, $\pounds$ the Lie-derivative operator, r the scalar curvature of g, and c is a real constant defined earlier. The Riemannian manifold (M, g) with a vector field V and a constant c, satisfying the equation (1.1) is called a Yamabe soliton. The Yamabe soliton is said to be shrinking, steady, or expanding when $c > 0, c = 0$, or $c < 0$ respectively. In particular, if $V = Df$ (up to the addition of a Killing vector field) for a smooth function f, where D denotes the gradient operator of

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g, then a Yamabe soliton is called a gradient Yamabe soliton in which case (1.1) assumes the form

$$\nabla\nabla f = (c - r)g. \tag{1.2}$$

The gradient Yamabe soliton is trivial when f is constant and r is constant. In [**6**], Daskalopoulos and Sesum showed that a compact gradient Yamabe soliton has constant scalar curvature (i.e., its metric is a Yamabe metric). Chen and Deshmukh [**5**] obtained some sufficient conditions on Yamabe solitons to be of constant scalar curvature. Ghosh [**7**] has shown that a Yamabe soliton on a Kenmotsu manifold has constant scalar curvature. In [**15**], Sharma studied Yamabe solitons in contact Riemannian geometry, and proved that a 3-dimensional Yamabe soliton whose metric is Sasakian has constant scalar curvature and the soliton vector field V is Killing. In this paper, we generalize this result in any dimension in the form of the following result.

Theorem 1.1. Let (M, g) be a Yamabe soliton with soliton vector field V. If g is a Sasakian metric, then it has constant scalar curvature and V is Killing.

Furthermore, we show that the same conclusion (as in the above theorem) holds on a $(2n + 1)$-dimensional K-contact manifold M (a generalization of Sasakian manifold) under certain conditions specified in the following result. More precisely, we establish

Theorem 1.2. Let (M, g) be a $(2n + 1)$-dimensional Yamabe soliton with soliton vector field V, and g be a K-contact metric. Then its scalar curvature is constant and V is Killing, if any one of the following conditions hold:
(i) The scalar curvature is constant along V.
(ii) V is an eigenvector of the Ricci operator with eigenvalue $2n$.
(iii) V is gradient.
In case (iii), the Yamabe soliton becomes trivial (i.e., f is constant).

Remarks.
1. An odd dimensional analogue of Kaehler geometry is the Sasakian geometry. The Kaehler cone over a Sasakian Einstein manifold is a Calabi-Yau manifold which has application in superstring theory based on a 10-dimensional manifold that is the product of the 4-dimensional space-time and a 6-dimensional Ricci-flat Kaehler (Calabi-Yau) manifold (see Candelas et al. [**4**]). Sasakian geometry has been extensively studied since its recently perceived relevance in string theory. Sasakian Einstein metrics have received a lot of attention in physics, for example, p-brane solutions in superstring theory, Maldacena conjecture (AdS/CFS duality) [**11**].

2. The condition (ii) in Theorem 1.2 is motivated by the fact that, if we take V as the Reeb vector field ξ, then the Yamabe soliton equation (1.1) implies $r = c$ because ξ is Killing for a K-contact metric.

2. A Brief Review Of Contact Geometry

A $(2n + 1)$-dimensional smooth manifold M is said to be contact if it has a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ on M. For a contact 1-form η there exists a unique vector field ξ (Reeb vector field) such that $d\eta(\xi, .) = 0$ and $\eta(\xi) = 1$. Henceforth X, Y, Z will denote arbitrary vector field on M. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, we obtain a Riemannian metric g and a (1,1)-tensor field φ such that

$$d\eta(X,Y) = g(X,\varphi Y),\ \eta(X) = g(X,\xi),\ \varphi^2 = -I + \eta \otimes \xi \tag{2.1}$$

g is called an associated metric of η and (φ, η, ξ, g) a contact metric structure. The name contact seems to be due to Sophus Lie [**10**] and is natural in view of the simple example of Huygens' principle (cited in Blair [**2**]). Tangent wave fronts are mapped to tangent wave fronts through a contact transformation. According to Gibbs, the geometrical structure of thermodynamics is described by a contact manifold equipped with a contact form whose zeros define the laws of thermodynamics (Arnold [**1**]).

A contact metric structure is said to be K-contact if ξ is Killing with respect to g. The following formulas are valid on a K-contact manifold.

$$\nabla_X \xi = -\varphi X, \tag{2.2}$$

$$Ric(X,\xi) = 2ng(X,\xi), i.e., \quad Q\xi = 2n\xi, \tag{2.3}$$

$$R(X,\xi)\xi = X - \eta(X)\xi, \tag{2.4}$$

where ∇, R, Ric and Q denote respectively, the Riemannian connection, curvature tensor, Ricci tensor and Ricci operator of g. A contact metric manifold (M, g) is said to be Sasakian if the almost Kaehler structure on the cone manifold $(M \times R^+, r^2 g + dr^2)$ over M is Kaehler. Sasakian manifolds are K-contact and the converse is true only in dimension 3. For a Sasakian manifold, the following formula holds:

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

For details, we refer to the standard monograph of Blair [**2**].

It is evident from the defining equation (1.1) of a Yamabe soliton, that the associated vector field V is a conformal vector field with conformal scale function $2(c-r)$. A vector field V on an m-dimensional Riemannian manifold (M, g) is said to be a conformal vector field if

$$\pounds_V g = 2\rho g \tag{2.5}$$

for a smooth function ρ on M. Denoting the gradient vector field of ρ by $D\rho$, the Laplacian $-div.D\rho$ by $\Delta\rho$, and $R(X,Y,Z) = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$, we have the following integrability conditions for the conformal vector field V (Yano [**17**]):

$$(\pounds_V R)(X,Y,Z) = -(\nabla\nabla\rho)(Y,Z)X + (\nabla\nabla\rho)(X,Z)Y - g(Y,Z)\nabla_X D\rho + g(X,Z)\nabla_Y D\rho \tag{2.6}$$

$$(\pounds_V Ric)(X,Y) = -(m-2)(\nabla\nabla\rho)(X,Y) + (\Delta\rho)g(X,Y) \tag{2.7}$$

$$\pounds_V r = 2(m-1)\Delta\rho - 2r\rho. \tag{2.8}$$

3. Proofs of The Results

Proof of Theorem 1.1 First, we note that this theorem was already proved in dimension 3 by Sharma [**15**]. So, we now prove it in higher dimensions. The crucial idea of the proof is based on the following results of Okumura [**12**] (Theorems 3.1 and 3.3): *A conformal vector field V (defined by (2.5) on a Sasakian manifold of dimension $2n+1 > 3$ differs from $-D\rho$ by a Killing vector field, i.e., $V = W - D\rho$*

where W is a Killing vector field on M, and the conformal scale function ρ satisfies the concircular equation:

$$\nabla\nabla\rho = -\rho g.$$

For a Yamabe soliton (1.1) we note that $\rho = c - r$, and therefore the above equation can be expressed as

$$\nabla_X Dr = (c - r)X. \tag{3.1}$$

Computing $R(X,Y)Dr = \nabla_X\nabla_Y Dr - \nabla_Y\nabla_X Dr - \nabla_{[X,Y]}Dr$ using the above equation we get

$$R(X,Y)Dr = (Yr)X - (Xr)Y.$$

Contracting it over X yields

$$QDr = 2nDr.$$

Differentiating it along an arbitrary vector field X and using equation (3.1) gives

$$(\nabla_X Q)Dr + (c - r)QX = 2n(c - r)X.$$

Contracting this equation over X, using (3.1) along with twice contracted Bianchi's second identity: $divQ = \frac{1}{2}dr$, we get

$$|Dr|^2 = 2(c - r)[2n(2n + 1) - r]. \tag{3.2}$$

On the other hand, we see from (3.1) that $\nabla_X|Dr|^2 = 2(c - r)Xr$. This, in conjunction with (3.2), yields

$$[2c - 3r + 2n(2n + 1)]Dr = 0. \tag{3.3}$$

So, either (i) $Dr = 0$ on M, i.e., r is constant on M, or (ii) $Dr \neq 0$ on an open dense subset $\mathcal{U}$ of M. In case (ii), equation (3.3) implies $2c - 3r + 2n(2n + 1) = 0$. Using it in equation (3.2) yields $|Dr|^2 + 4(r - c)^2 = 0$. Hence $Dr = 0$ on $\mathcal{U}$, a contradiction. Thus, r is constant on M. Hence, from (3.1), $r = c$. Thus, from (1.1), V is Killing. This completes the proof.

Proof of Theorem 1.2 Taking the Lie-derivative of the K-contact formula (2.3) along the Yamabe vector field V and using the integrability equation (2.7) with $m = 2n + 1$, we have

$$(\Delta\rho - 4n\rho)\xi - (2n - 1)\nabla_\xi D\rho + Q\pounds_V\xi - 2n\pounds_V\xi = 0. \tag{3.4}$$

At this point, we compute the term $\nabla_\xi D\rho$ as follows. As $\rho = c - r$, this term is $-\nabla_\xi Dr$. To compute its value, we take the Lie-derivative of $dr(X) = g(Dr, X)$ along ξ and noting that the Lie-derivative operator commutes with the exterior derivative operator, we find that $(d\pounds_\xi r)(X) = g(\nabla_\xi Dr - \nabla_{Dr}\xi, X)$. We note that $\xi r = 0$ because, by definition, ξ is Killing for a K-contact manifold. As a consequence, the use of the formula (2.2) provides $\nabla_\xi Dr = -\varphi Dr$, i.e., $\nabla_\xi D\rho = \varphi Dr$. Hence equation (3.4) takes the form

$$(\Delta\rho - 4n\rho)\xi - (2n - 1)\varphi Dr + Q\pounds_V\xi - 2n\pounds_V\xi = 0. \tag{3.5}$$

Its inner product with ξ and the use of (2.3) provides

$$\Delta\rho = 4n\rho. \tag{3.6}$$

We see from equation (3.6) and $\rho = c - r$ that

$$-\Delta r = 4n(c - r). \tag{3.7}$$

Let us prove part (i). Using the hypothesis: $\pounds_V r = 0$ in (2.8) with $\rho = c - r$, we get

$$-4n\Delta r = 2r(c-r). \tag{3.8}$$

The equations (3.7) and (3.8) imply that $(c-r)(r-8n^2) = 0$. So, either (i) $r = c$ on M, or (ii) $r \neq c$ and $r - 8n^2 = 0$ on an open dense subset $\mathcal{U}$ of M. Substituting the value of r from (ii) in (3.7) immediately gives $r = c$ on $\mathcal{U}$, a contradiction. Hence $r = c$ on M and V is Killing, proving part (i).

To prove part (ii), we assume the hypothesis: $QV = 2nV$ and operate it by $\pounds_\xi$, and noting that ξ is Killing and hence $\pounds_\xi Q = 0$, we find that $Q\pounds_\xi V = 2n\pounds_\xi V$. On the other hand, equations (3.5) and (3.6) provide

$$(2n-1)\varphi Dr = Q\pounds_V \xi - 2n\pounds_V \xi.$$

Combined use of these two results, and that $\pounds_V \xi = -\pounds_\xi V$ shows that $\varphi Dr = 0$. Operating it by φ and using the last equation in (2.1) in conjunction with $\xi r = 0$ (because ξ is Killing) we conclude that $Dr = 0$, i.e., r is constant. Hence V is homothetic. But a homothetic vector field on a K-contact manifold is Killing [**14**], and hence V is Killing. This proves part (ii).

In order to prove part (iii), we let $V = Df$ up to the sum of a Killing vector field. So, we can present equation (1.2) in the form

$$\nabla_Y Df = (c-r)Y. \tag{3.9}$$

This implies that

$$R(X,Y)Df = (Yr)X - (Xr)Y. \tag{3.10}$$

Substituting ξ for X, taking the inner product of (3.10) with ξ, and subsequently using the formula (2.4) we get

$$Yf - (\xi f)\eta(Y) = Yr$$

which can be expressed as

$$d(f-r) = (\xi f)\eta. \tag{3.11}$$

Applying the exterior derivation, using Poincare formula: $d^2 = 0$, and taking its wedge product with η provides $(\xi f)\eta \wedge d\eta = 0$. Since $\eta \wedge d\eta$ is nowhere zero on M, by definition of the contact structure, we conclude that $\xi f = 0$. Hence $Xf = Xr$, i.e., f differs from r by a constant. Thus, we note, in passing, that equation (3.9) is same as (3.1). Virtually following the argument used in the proof of Theorem 1.1 from equation (3.1) onwards, we conclude that r is constant. As f differs from r by a constant, and r is constant, it follows that f is constant and therefore the Yamabe soliton is trivial, completing the proof.

Concluding Remarks.

1. In the proof of Theorem 1.2, we came across equation (3.6) which shows for a Yamabe soliton as a K-contact manifold, that the conformal scale function $\rho = c - r$ is an eigenfunction of the Laplacian with eigenvalue $4n$. This implies that, if the eigenvalues of the Laplacian are different from $4n$, then $r = c$, i.e., the metric of the K-contact Yamabe soliton in dimension > 3 is a Yamabe metric. This assumption

holds for the unit sphere S^{2n+1} (which is Sasakian and hence K-contact) because the spectral values of the Laplacian acting on functions on S^{2n+1} are $k(k+2n)$ for $k = 0, 1, 2, ..., \infty$ (Tanno, p. 91 in [**16**]) and hence do not include $4n$.

2. Let us recall the formula $\pounds_{[X,Y]} = \pounds_X \pounds_Y - \pounds_Y \pounds_X$ (Kobayashi-Nomizu, p. 32 in [**9**]). Substituting $X = V, Y = \xi$ in it, and operating on the K-contact metric g we find $\pounds_{[V,\xi]} g = -\pounds_\xi \pounds_V g$, because ξ is Killing. But $\pounds_V g = 2(c-r)g$, and as ξ is Killing, $\xi r = 0$. Thus $\pounds_\xi \pounds_V g = 0$, and hence $\pounds_{[V,\xi]} g = 0$, i.e., $[V, \xi]$ is Killing.

3. If the Yamabe soliton vector field V on any contact metric manifold is pointwise either collinear with, or orthogonal to the Reeb vector field ξ, then by Proposition 1 of [**13**], V is Killing and hence r is constant, equal to c.

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References

[1] V. I. Arnold, *Contact geometry: The geometrical method of Gibb's thermodynamics*, Proc. Gibbs Symposium, Yale University, 1989, 163-179.

[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, second ed., Progress in Mathematics 203, Birkhäuser, Boston, MA, 2010. Doi: 10.1007/978-0-8176-4959-3

[3] A. Burchard, R. J. McCann and A. Smith, *Explicit Yamabe flow of an asymmetric cigar*, Methods Appl. Anal. **15** (2008), 65–80. Doi: 10.4310/MAA.2008.v15.n1.a7

[4] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys., B **258** (1985), 46–74. Doi: 10.1016/0550-3213(85)90602-9

[5] S. Deshmukh and B. Y. Chen, *A note on Yamabe solitons*, Balkan J. Geom. Appl. **23** (2018), 37–43.

[6] P. Daskalopoulos and N. Sesum, *The classification of locally conformally flat Yamabe solitons*, Adv. Math. **240** (2013), 346–369. Doi: 10.1016/j.aim.2013.03.011

[7] A. Ghosh, *Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold*, Math. Slovaca **70** (2020), 151–160. Doi: 10.1515/ms-2017-0340

[8] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity, Contemp. Math. **71** (1988), 237–262.

[9] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Volume I*, Interscience Tracts in Pure and Applied Mathematics 15, Interscience Publishers, New York, NY, 1963.

[10] S. Lie, *Theorie der Transformationsgruppen*, Vol. 2, B. G. Teubner, Leipzig, 1890.

[11] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, Int. J. Theor. Phys. **38** (1999), 1113–1133. Doi: 10.1023/A:1026654312961

[12] M. Okumura, *On infinitesimal conformal and projective transformations of normal contact spaces*, Tôhoku Math. J. (2) **14** (1962), 398–412. Doi: 10.2748/tmj/1178244076

[13] R. Sharma, *Conformal and curvature symmetries of contact metric manifolds*, C. R. Math. Acad. Sci., Soc. R. Can. **12** (1990), 235–240.

[14] R. Sharma, *Addendum to our paper "Conformal motion of contact manifolds with characteristic vector field in the k-nullity distribution"*, Ill. J. Math. **42** (1998), 673–677.

[15] R. Sharma, *A 3-dimensional Sasakian metric as a Yamabe soliton*, Int. J. Geom. Methods Mod. Phys. **9** (2012), 1220003, 5 p.

[16] S. Tanno, *Promenades on spheres*, Lecture Notes, Tokyo Institute of Technology, Tokyo, 1996.

[17] K. Yano, *Integral Formulas in Riemannian Geometry*, Pure and Applied Mathematics 1, Marcel Dekker, Inc., New York, NY, 1970.

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