Abstract. If $G$ is a finite group, the Grothendieck group $K_G(G)$ of the category of $G$-equivariant $\mathbb{C}$-vector bundles on $G$ (for the action of $G$ on itself by conjugation) is endowed with a structure of (commutative) ring. If $K$ is a sufficiently large extension of $\mathbb{Q}_p$ and $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in $K$, the $K$-algebra $K K_G(G) = K \otimes_{\mathbb{Z}_p} K_G(G)$ is split semisimple. The aim of this paper is to describe the $\mathcal{O}$-blocks of the $\mathcal{O}$-algebra $\mathcal{O} K_G(G)$.

1. Notation, Introduction

1.1. Groups. We fix in this paper a finite group $G$, a prime number $p$ and a finite extension $K$ of the $p$-adic field $\mathbb{Q}_p$ such that $K H$ is split for all subgroups $H$ of $G$. We denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}_p$ in $K$, by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, by $k$ the residue field of $\mathcal{O}$ (i.e. $k = \mathcal{O}/\mathfrak{p}$) We denote by $\text{Irr}(KG)$ the set of irreducible characters of $G$ (over $K$).

A $p$-element (respectively $p'$-element) of $G$ is an element whose order is a power of $p$ (respectively prime to $p$). If $g \in G$, we denote by $g_p$ and $g_{p'}$ the unique elements of $G$ such that $g = g_p g_{p'} = g_{p'} g_p$, $g_p$ is a $p$-element and $g_{p'}$ is a $p'$-element. The set of $p$-elements (respectively $p'$-elements) of $G$ is denoted by $G_p$ (respectively $G_{p'}$).

If $X$ is a $G$-set (i.e. a set endowed with a left $G$-action), we denote by $[G\backslash X]$ a set of representatives of $G$-orbits in $X$. The reader can check that we will use formulas like

$$\sum_{x \in [G\backslash X]} f(x)$$

(or families like $(f(x))_{x \in [G\backslash X]}$) only whenever $f(x)$ does not depend on the choice of the representative $x$ in its $G$-orbit. If $X$ is a set-$G$ (i.e. a set endowed with a right $G$-action), we define similarly $[X/G]$ and will use it according to the same principles.

1.2. Blocks. A block idempotent of $kG$ (respectively $\mathcal{O}G$) is a primitive idempotent of the center $Z(kG)$ (respectively $Z(\mathcal{O}G)$) of $\mathcal{O}G$. We denote by $\text{Blocks}(kG)$ (respectively $\text{Blocks}(\mathcal{O}G)$) the set of block idempotents of $kG$ (respectively $\mathcal{O}G$). Reduction modulo $\mathfrak{p}$ induces a bijection $\text{Blocks}(\mathcal{O}G) \xrightarrow{\sim} \text{Blocks}(kG)$, $e \mapsto \bar{e}$ (whose inverse is denoted by $e \mapsto \tilde{e}$).

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A $p$-block of $G$ is a subset $B$ of $\text{Irr}(G)$ such that $B = \text{Irr}(KG_e)$, for some block idempotent $e$ of $OG$.

1.3. Fourier coefficients. Let

$$\text{IrrPairs}(G) = \{(g, \gamma) \mid g \in G \text{ and } \gamma \in \text{Irr}(KC_G(g))\}$$

and

$$\text{BlPairs}_p(G) = \{(s, e) \mid s \in G_p' \text{ and } e \in \text{Blocks}(OC_G(s))\}.$$ 

The group $G$ acts (on the left) on these two sets by conjugation. We set

$$M(G) = \left[G\backslash \text{IrrPairs}(G)\right] \quad \text{and} \quad M^p(G) = \left[G\backslash \text{BlPairs}(G)\right].$$

If $(g, \gamma), (h, \eta) \in \text{IrrPairs}(G)$, we define, following Lusztig [3, 2.5(a)],

$$\{ (g, \gamma), (h, \eta) \} = \frac{1}{|C_G(g)| \cdot |C_G(h)|} \sum_{xh^{-1} \in C_G(g)} \gamma(xh^{-1})\eta(x^{-1}g^{-1}x).$$

Note that $\{ (g, \gamma), (h, \eta) \}$ depends only on the $G$-orbit of $(g, \gamma)$ and on the $G$-orbit of $(h, \eta)$.

1.4. Vector bundles. Except from Proposition 2.3 below, all the definitions, all the results in this subsection can be found in [3, §2]. We denote by $\mathcal{B}un_G(G)$ the category of $G$-equivariant finite dimensional $K$-vector bundles on $G$ (for the action of $G$ by conjugation). Its Grothendieck group $K_G(G)$ is endowed with a ring structure. For each $(g, \gamma) \in \mathcal{M}(G)$, let $V_{g, \gamma}$ be the isomorphism class (in $K_G(G)$) of the simple object in $\mathcal{B}un_G(G)$ associated with $(g, \gamma)$, as in [3, §2.5] (it is denoted $U_{g, \gamma}$ there). Then

$$K_G(G) = \bigoplus_{(g, \gamma) \in \mathcal{M}(G)} \mathbb{Z}V_{g, \gamma}.$$ 

The $K$-algebra $KK_G(G) = K \otimes_{\mathbb{Z}} K_G(G)$ is split semisimple and commutative. Its simple modules (which have dimension one) are also parametrized by $\mathcal{M}(G)$: if $(g, \gamma) \in \mathcal{M}(G)$, the $K$-linear map

$$\Psi_{g, \gamma} : KK_G(G) \to K$$

defined by

$$\Psi_{g, \gamma}(V_{h, \eta}) = \frac{|C_G(g)|}{\gamma(1)} \cdot \{ (h^{-1}, \eta), (g, \gamma) \}$$

is a morphism of $K$-algebras and all morphisms of $K$-algebras $KK_G(G) \to K$ are obtained in this way.

We define similarly block idempotents of $kK_G(G)$ and $OK_G(G)$, as well as $p$-blocks of $\mathcal{M}(G) \longleftrightarrow \text{Irr}(KK_G(G))$. 
1.5. Brauer maps. Let $\Lambda$ denote one of the two rings $\mathcal{O}$ or $k$. If $g \in G$ (and if we set $s = g p'$), we denote by $\text{Br}_g^\Lambda$ the $\Lambda$-linear map

$$\text{Br}_g^\Lambda : \Lambda C_G(s) \to \Lambda C_G(g)$$

such that

$$\text{Br}_g^\Lambda(h) = \begin{cases} h & \text{if } h \in C_G(g), \\ 0 & \text{if } h \not\in C_G(g), \end{cases}$$

for all $h \in C_G(s)$. Recall [2, Lemma 15.32] that

$$\text{Br}_g^k \text{ induces a morphism of algebras } \mathcal{Z}(kC_G(s)) \to \mathcal{Z}(kC_G(g)).$$

Therefore, if $e \in \text{Blocks}(\mathcal{O}C_G(s))$, then $\text{Br}_g^k(e)$ is an idempotent of $\mathcal{Z}(kC_G(g))$ (possibly equal to zero) and we can write it a sum $\text{Br}_g^k(e) = e_1 + \cdots + e_n$, where $e_1, \ldots, e_n$ are pairwise distinct block idempotents of $kC_G(g)$. We then set

$$\beta_g^\mathcal{O}(e) = \sum_{i=1}^n \tilde{e}_i.$$ 

It is an idempotent (possibly equal to zero, possibly non-primitive) of $\mathcal{Z}(\mathcal{O}C_G(g))$.

1.6. The main result. In order to state more easily our main result, it will be more convenient (though it is not strictly necessary) to fix a particular set of representatives of conjugacy classes of $G$.

**Hypothesis and notation.** From now on, and until the end of this paper, we denote by:

- $[G_{p'}/\sim]$ a set of representatives of conjugacy classes of $p'$-elements in $G$.
- $[G/\sim]$ a set of representatives of conjugacy classes of elements of $G$ such that, for all $g \in [G/\sim], g_{p'} \in [G_{p'}/\sim]$.

We also assume that, if $(g, \gamma) \in \mathcal{M}(G)$ or $(s, e) \in \mathcal{M}_p(G)$, then $g \in [G/\sim]$ and $s \in [G_{p'}/\sim]$.

If $(s, e) \in \mathcal{M}_p(G)$, we define $\mathcal{B}_G(s, e)$ to be the set of pairs $(g, \gamma) \in \mathcal{M}(G)$ such that:

1. $g_{p'} = s$.
2. $\gamma \in \text{Irr}(kC_G(g) \beta_g^\mathcal{O}(e))$.

**Theorem 1.2.** The map $(s, e) \mapsto \mathcal{B}_G(s, e)$ induces a bijection between $\mathcal{M}_p(G)$ to the set of $p$-blocks of $\mathcal{M}(G)$.
2. Proof of Theorem 1.2

2.1. Central characters and congruences. If \((g, \gamma) \in \text{IrrPairs}(G)\), we denote by \(\omega_{g, \gamma} : \mathbb{Z}(KC_G(g)) \to K\) the central character associated with \(\gamma\) (if \(z \in \mathbb{Z}(KC_G(g))\), then \(\omega_{g, \gamma}(z)\) is the scalar through which \(z\) acts on an irreducible \(KC_G(g)\)-module affording the character \(\gamma\)). It is a morphism of algebras: when restricted to \(\mathbb{Z}(O_C G(g))\), it has values in \(O\).

If \(h \in C_G(g)\), we denote by \(\Sigma_g(h)\) conjugacy class of \(h\) in \(C_G(g)\) and we set \(\hat{\Sigma}_g(h) = \sum_{v \in \Sigma_g(h)} v \in \mathbb{Z}(O_C G(g))\).

We have

\[
2.1 \omega_{g, \gamma}(\hat{\Sigma}_g(h)) = |\Sigma_g(h)| \cdot \gamma(h) \gamma(1) \mod p
\]

We also recall the following classical results:

**Proposition 2.2.** If \(g \in G\) and \(\gamma, \gamma'\) are two irreducible characters of \(C_G(g)\), then \(\gamma\) and \(\gamma'\) lie in the same \(p\)-block of \(C_G(g)\) if and only if 
\[
\omega_{g, \gamma}(\hat{\Sigma}_g(h)) \equiv \omega_{g, \gamma'}(\hat{\Sigma}_g(h)) \mod p
\]
for all \(h \in C_G(g)\).

**Proposition 2.3.** Let \((g, \gamma)\) and \((g', \gamma')\) be two elements of \(M(G)\). Then \((g, \gamma)\) and \((g', \gamma')\) belong to the same \(p\)-block of \(M(G)\) if and only if 
\[
\Psi_{g, \gamma}(V_{h, \eta}) \equiv \Psi_{g', \gamma'}(V_{h, \eta}) \mod p
\]
for all \((h, \eta) \in M(G)\).

2.2. Around the Brauer map. As Brauer maps are morphisms of algebras, we have
\[
\sum_{e \in \text{Blocks}(kC_G(g'))} \text{Br}_g^p(e) = 1,
\]
and so
\[
2.4 \text{The family } (\text{Br}_G(g, e))_{(g, e) \in M^p(G)} \text{ is a partition of } M(G).
\]

Now, let \((g, \gamma) \in M(G)\) and let \(s = g_{g'}\). If \(e \in \text{Blocks}(O_C G(s))\) is such that \(\gamma \in \text{Irr}(KC_G(g)o_G^G(e))\), and if \(\sigma \in \text{Irr}(KC_G(s)e)\), then [2, Lemma 15.44]
\[
2.5 \omega_{s, \sigma}(z) \equiv \omega_{g, \gamma}(\text{Br}_G^G(z)) \mod p
\]
for all \(z \in \mathbb{Z}(O_C G(s))\).
2.3. Rearranging the formula for $\Psi_{g,\gamma}$. If $(g, \gamma), (h, \eta) \in \text{IrrPairs}(g)$ then

\begin{equation}
\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{x \in [C_G(g)\setminus G/C_G(h)] \atop xh^{-1} \in C_G(g)} \eta(x^{-1}gx)\omega_{g,\gamma}(\Sigma_g(xhx^{-1})).
\end{equation}

Proof. By definition,

$$
\Psi_{g,\gamma}(V_{h,\eta}) = \frac{1}{\gamma(1) \cdot |C_G(h)|} \sum_{x \in G \atop xh^{-1} \in C_G(g)} \eta(x^{-1}gx)\gamma(xhx^{-1})
$$

Now, if $x \in G$ is such that $xhx^{-1} \in C_G(g)$ and if $u \in C_G(g)$, then

$$
\eta((ux)^{-1}g(ux))\gamma((ux)h(ux)x^{-1}) = \eta(x^{-1}gx)\gamma(xhx^{-1}).
$$

So we can gather the terms in the last sum according to their $C_G(g)$-block. Then

$$
\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{x \in [G/C_G(h)] \atop xh^{-1} \in C_G(g)} \eta(x^{-1}gx)\gamma(xhx^{-1}).
$$

But, for $x$ in $G$ such that $xhx^{-1} \in C_G(g)$,

$$
\frac{|C_G(g)|}{|C_G(g) \cap xC_G(h)x^{-1}|} = |\Sigma_g(xhx^{-1})|,
$$

so the result follows from 2.1.

\begin{corollary}
Let $g \in [G/\sim]$ and let $\gamma, \gamma' \in \text{Irr}(KC_G(g))$ lying in the same $p$-block of $C_G(g)$. Then $(g, \gamma)$ and $(g, \gamma')$ lie in the same $p$-block of $\mathcal{M}(G)$.
\end{corollary}

Proof. This follows from 2.6 and Proposition 2.3.

\begin{proposition}
Let $(g, \gamma)$ and $(h, \eta)$ be two elements in $\mathcal{M}(G)$ which lie in the same $p$-block. Then $g^p = h^p$.
\end{proposition}

Proof. By Proposition 2.3 and Equality 2.8, it follows from the hypothesis that

$$
\chi(g) \equiv \chi(h) \mod p
$$

for all $\chi \in \text{Irr}(KG)$. Hence $g^p$ and $h^p$ are conjugate in $G$ (see [1, Proposition 2.14]), so they are equal according to our conventions explained in §1.6.
Proposition 2.10. Let \( s \in G \) and let \( \sigma, \sigma' \in \text{Irr}(KC_G(s)) \). Then \((s, \sigma)\) and \((s, \sigma')\) lie in the same \( p \)-block if and only if \( \sigma \) and \( \sigma' \) lie in the same \( p \)-block of \( C_G(s) \).

Proof. The if part has been proved in Corollary 2.7. Conversely, assume that \((s, \sigma)\) and \((s, \sigma')\) lie in the same \( p \)-block. Fix \( h \in C_G(s) \). Then \( s \in C_G(h) \). Let \( \eta_{s,h} : C_G(h) \to K \) be the class function on \( C_G(h) \) defined by:

\[
\eta_{s,h}(g) = \begin{cases} 1 & \text{if } g \equiv \sigma \text{ and } s \text{ are conjugate in } C_G(h), \\ 0 & \text{otherwise}. \end{cases}
\]

It follows from [1, Proposition 2.20] that \( \eta_{s,h} \in \mathcal{O}\text{Irr}(KC_G(h)) \). Therefore, by 2.6 and Proposition 2.3,

\[
\sum_{x \in [C_G(s) \backslash G/C_G(h)] \atop xhx^{-1} \in C_G(s)} \eta_{s,h}(x^{-1}sx)\left(\omega_{s,\sigma} (\hat{\Sigma}_g(xhx^{-1})) - \omega_{s,\sigma'} (\hat{\Sigma}_g(xhx^{-1}))\right) \equiv 0 \mod p.
\]

Now, let \( x \in G \) be such that \( xhx^{-1} \in C_G(s) \). Since \( x^{-1}sx \) is also a \( p' \)-element, \( \eta_{s,h}(x^{-1}sx) = 1 \) if and only if \( s \) and \( x^{-1}sx \) are conjugate in \( C_G(h) \) that is, if and only if \( x \in C_G(s)C_G(h) \). So it follows from \((\#)\) that

\[
\omega_{s,\sigma} (\hat{\Sigma}_g(h)) \equiv \omega_{s,\sigma'} (\hat{\Sigma}_g(h)) \mod p
\]

for all \( h \in C_G(s) \). This shows that \( \sigma \) and \( \sigma' \) lie in the same \( p \)-block of \( C_G(s) \). \( \square \)

2.5. Last step. We shall prove here the last intermediate result:

Proposition 2.11. Let \((s,e) \in \mathcal{M}^p(G)\) and let \((g,\gamma)\), \((g',\gamma') \in \mathcal{B}_G(s,e)\). Then \((g,\gamma)\) and \((g',\gamma')\) are in the same \( p \)-block of \( \mathcal{M}(G) \).

Proof. We fix \( \sigma \in \text{Irr}(KC_G(s)e) \). It is sufficient to show that \((g,\gamma)\) and \((s,\sigma)\) are in the same \( p \)-block of \( \mathcal{M}(G) \). For this, let \((h,\eta) \in \mathcal{M}(G) \). By Proposition 2.9, we have \( g\sigma = s \), so \( C_G(g) \subset C_G(s) \). So 2.6 can be rewritten:

\[
\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{x \in [C_G(s) \backslash G/C_G(h)] \atop y \in [C_G(g) \backslash C_G(s)xC_G(h)/C_G(h)]} \eta(y^{-1}gy)\omega_{g,\gamma} (\hat{\Sigma}_g(yhy^{-1})).
\]

Now, let \( x \in [C_G(s) \backslash G/C_G(h)]\) and \( y \in [C_G(g) \backslash C_G(s)xC_G(h)/C_G(h)] \) be such that \( yhy^{-1} \in C_G(g) \). Then \( yhy^{-1} \in C_G(s) \) and so \( xhx^{-1} \in C_G(s) \). Moreover \( y^{-1}sy \) is conjugate to \( x^{-1}sx \) in \( C_G(h) \). Finally, it is well-known (and easy to prove) that \( \eta(y^{-1}hy) \equiv \eta(y^{-1}sy) \mod p \) (see for instance [1, Proposition 2.14]). Therefore:

\[
\psi_{g,\gamma}(V_{h,\eta}) \equiv \sum_{x \in [C_G(s) \backslash G/C_G(h)] \atop xhx^{-1} \in C_G(s)} \eta(x^{-1}sx)\omega_{g,\gamma} \left( \sum_{y \in [C_G(g) \backslash C_G(s)xC_G(h)/C_G(h)] \atop yhy^{-1} \in C_G(g)} \hat{\Sigma}_g(yhy^{-1}) \right) \mod p.
\]
Now, let $x \in [C_G(s) \setminus G/C_G(h)]$ be such that $xhx^{-1} \in C_G(s)$. Then, by definition of the Brauer map,

\[ \langle \land \rangle \quad \text{Br}_g^G(\hat{\Sigma}_x(xhx^{-1})) = \sum_{z \in \{C_G(g) \setminus (C_G(s) \cap C_G(xhx^{-1}))\}} \hat{\Sigma}_g((zx)h(zx)^{-1}). \]

But $(zx)z \in \{C_G(g) \setminus C_G(s) \setminus (C_G(s) \cap C_G(xhx^{-1}))\}$ is a set of representatives of double classes in $\{C_G(g) \setminus C_G(s) \setminus C_G(h)\}$. So it follows from ($\langle \land \rangle$) and ($\langle \land \rangle$) that

\[ \Psi_{g,h}(V_{h,\eta}) \equiv \sum_{z \in [C_G(s) \setminus G/C_G(h)]} \eta(x^{-1}sz) \omega_{g,\gamma}(\text{Br}_g^G(\hat{\Sigma}_x(xhx^{-1}))). \]

Using now 2.5 and 2.6, we obtain that

\[ \Psi_{g,h}(V_{h,\eta}) \equiv \Psi_{s,\sigma}(V_{h,\eta}) \mod p, \]

as desired. \qed

Proof of Theorem 1.2. Let $(s,e)$ and $(s',e')$ be two elements of $\mathcal{M}(G)$ such that $B_G(s,e)$ and $B_G(s',e')$ are contained in the same $p$-block of $\mathcal{M}(G)$ (see Proposition 2.11). Let $\sigma \in \text{Irr}(KC_G(s)e)$ and $\sigma' \in \text{Irr}(KC_G(s')e')$.

Then $(s,\sigma)$ and $(s',\sigma')$ are in the same $p$-block, so it follows from Proposition 2.9 that $s = s'$ and it follows from Proposition 2.10 that $\gamma$ and $\gamma'$ are in the same $p$-block of $C_G(s)$, that is $e = e'$. This completes the proof of Theorem 1.2. \qed