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BLOCKS OF THE GROTHENDIECK RING OF EQUIVARIANT BUNDLES ON A FINITE GROUP

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Abstract. If G is a finite group, the Grothendieck group $\mathbf{K}_G(G)$ of the category of G-equivariant $\mathbb{C}$-vector bundles on G (for the action of G on itself by conjugation) is endowed with a structure of (commutative) ring. If K is a sufficiently large extension of $\mathbb{Q}_p$ and $\mathcal{O}$ denotes the integral closure of $\mathbb{Z}_p$ in K, the K-algebra $K\mathbf{K}_G(G) = K \otimes_{\mathbb{Z}} \mathbf{K}_G(G)$ is split semisimple. The aim of this paper is to describe the $\mathcal{O}$-blocks of the $\mathcal{O}$-algebra $\mathcal{O}\mathbf{K}_G(G)$.

1. Notation, Introduction

1.1. Groups. We fix in this paper a finite group G, a prime number p and a finite extension K of the p-adic field $\mathbb{Q}_p$ such that KH is split for all subgroups H of G. We denote by $\mathcal{O}$ the integral closure of $\mathbb{Z}_p$ in K, by $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$, by k the residue field of $\mathcal{O}$ (i.e. $k = \mathcal{O}/\mathfrak{p}$) We denote by $\mathrm{Irr}(KG)$ the set of irreducible characters of G (over K).

A *p-element* (respectively *p'-element*) of G is an element whose order is a power of p (respectively prime to p). If $g \in G$, we denote by g_p and $g_{p'}$ the unique elements of G such that $g = g_p g_{p'} = g_{p'} g_p$, g_p is a p-element and $g_{p'}$ is a p'-element. The set of p-elements (respectively p'-elements) of G is denoted by G_p (respectively $G_{p'}$).

If X is a G-set (i.e. a set endowed with a left G-action), we denote by $[G\backslash X]$ a set of representatives of G-orbits in X. The reader can check that we will use formulas like

$$\sum_{x \in [G\backslash X]} f(x)$$

(or families like $(f(x))_{x \in [G\backslash X]}$) only whenever $f(x)$ does not depend on the choice of the representative x in its G-orbit. If X is a set-G (i.e. a set endowed with a right G-action), we define similarly $[X/G]$ and will use it according to the same principles.

1.2. Blocks. A *block idempotent* of kG (respectively $\mathcal{O}G$) is a primitive idempotent of the center $\mathrm{Z}(kG)$ (respectively $\mathrm{Z}(\mathcal{O}G)$) of $\mathcal{O}G$. We denote by $\mathrm{Blocks}(kG)$ (respectively $\mathrm{Blocks}(\mathcal{O}G)$) the set of block idempotents of kG (respectively $\mathcal{O}G$). Reduction modulo $\mathfrak{p}$ induces a bijection $\mathrm{Blocks}(\mathcal{O}G) \xrightarrow{\sim} \mathrm{Blocks}(kG)$, $e \mapsto \bar{e}$ (whose inverse is denoted by $e \mapsto \tilde{e}$).

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A *p-block* of G is a subset $\mathcal{B}$ of $\mathrm{Irr}(G)$ such that $\mathcal{B} = \mathrm{Irr}(KGe)$, for some block idempotent e of $\mathcal{O}G$.

1.3. Fourier coefficients. Let

$$\mathrm{IrrPairs}(G) = \{(g,\gamma) \mid g \in G \text{ and } \gamma \in \mathrm{Irr}(KC_G(g))\}$$

and
$$\mathrm{BlPairs}_p(G) = \{(s,e) \mid s \in G_{p'} \text{ and } e \in \mathrm{Blocks}(\mathcal{O}C_G(s))\}.$$

The group G acts (on the left) on these two sets by conjugation. We set

$$\mathcal{M}(G) = [G\backslash \mathrm{IrrPairs}(G)] \qquad \text{and} \qquad \mathcal{M}^p(G) = [G\backslash \mathrm{BlPairs}(G)].$$

If (g,γ), $(h,\eta) \in \mathrm{IrrPairs}(G)$, we define, following Lusztig [**3**, 2.5(a)],

$$\big\{\, (g,\gamma),(h,\eta)\,\big\} = \frac{1}{|C_G(g)|\cdot|C_G(h)|} \sum_{\substack{x\in G \\ xhx^{-1}\in C_G(g)}} \gamma(xhx^{-1})\eta(x^{-1}g^{-1}x).$$

Note that $\big\{\, (g,\gamma),(h,\eta)\,\big\}$ depends only on the G-orbit of (g,γ) and on the G-orbit of (h,η).

1.4. Vector bundles. Except from Proposition 2.3 below, all the definitions, all the results in this subsection can be found in [**3**, §2]. We denote by $\mathcal{B}un_G(G)$ the category of G-equivariant finite dimensional K-vector bundles on G (for the action of G by conjugation). Its Grothendieck group $\mathbf{K}_G(G)$ is endowed with a ring structure. For each $(g,\gamma) \in \mathcal{M}(G)$, let $V_{g,\gamma}$ be the isomorphism class (in $\mathbf{K}_G(G)$) of the simple object in $\mathcal{B}un_G(G)$ associated with (g,γ), as in [**3**, §2.5] (it is denoted $U_{g,\gamma}$ there). Then

$$\mathbf{K}_G(G) = \bigoplus_{(g,\gamma)\in\mathcal{M}(G)} \mathbb{Z}V_{g,\gamma}.$$

The K-algebra $K\mathbf{K}_G(G) = K \otimes_{\mathbb{Z}} \mathbf{K}_G(G)$ is split semisimple and commutative. Its simple modules (which have dimension one) are also parametrized by $\mathcal{M}(G)$: if $(g,\gamma) \in \mathcal{M}(G)$, the K-linear map

$$\Psi_{g,\gamma} : K\mathbf{K}_G(G) \longrightarrow K$$

defined by

$$\Psi_{g,\gamma}(V_{h,\eta}) = \frac{|C_G(g)|}{\gamma(1)} \cdot \big\{\, (h^{-1},\eta),(g,\gamma)\,\big\}$$

is a morphism of K-algebras and all morphisms of K-algebras $K\mathbf{K}_G(G) \longrightarrow K$ are obtained in this way.

We define similarly block idempotents of $k\mathbf{K}_G(G)$ and $\mathcal{O}\mathbf{K}_G(G)$, as well as *p-blocks of* $\mathcal{M}(G) \xleftrightarrow{\sim} \mathrm{Irr}(K\mathbf{K}_G(G))$.

1.5. Brauer maps. Let Λ denote one of the two rings $\mathcal{O}$ or k. If $g \in G$ (and if we set $s = g_{p'}$), we denote by Br_g^Λ the Λ-linear map

$$\mathrm{Br}_g^\Lambda : \Lambda C_G(s) \longrightarrow \Lambda C_G(g)$$

such that

$$\mathrm{Br}_g^\Lambda(h) = \begin{cases} h & \text{if } h \in C_G(g), \\ 0 & \text{if } h \notin C_G(g), \end{cases}$$

for all $h \in C_G(s)$. Recall [**2**, Lemma 15.32] that

(1.1) Br_g^k *induces a* ***morphism of algebras*** $\mathrm{Z}(kC_G(s)) \to \mathrm{Z}(kC_G(g))$.

Therefore, if $e \in \mathrm{Blocks}(\mathcal{O}C_G(s))$, then $\mathrm{Br}_g^k(e)$ is an idempotent of $\mathrm{Z}(kC_G(g))$ (possibly equal to zero) and we can write it a sum $\mathrm{Br}_g^k(e) = e_1 + \cdots + e_n$, where $e_1, \ldots, e_n$ are pairwise distinct block idempotents of $kC_G(g)$. We then set

$$\beta_g^{\mathcal{O}}(e) = \sum_{i=1}^{n} \tilde{e}_i.$$

It is an idempotent (possibly equal to zero, possibly non-primitive) of $\mathrm{Z}(\mathcal{O}C_G(g))$.

1.6. The main result. In order to state more easily our main result, it will be more convenient (though it is not strictly necessary) to fix a particular set of representatives of conjugacy classes of G.

Hypothesis and notation. *From now on, and until the end of this paper, we denote by:*

- $[G_{p'}/\sim]$ *a set of representatives of conjugacy classes of p'-elements in G.*
- $[G/\sim]$ *a set of representatives of conjugacy classes of elements of G such that, for all $g \in [G/\sim]$, $g_{p'} \in [G_{p'}/\sim]$.*

We also assume that, if $(g,\gamma) \in \mathcal{M}(G)$ or $(s,e) \in \mathcal{M}^p(G)$, then $g \in [G/\sim]$ and $s \in [G_{p'}/\sim]$.

If $(s,e) \in \mathcal{M}^p(G)$, we define $\mathcal{B}_G(s,e)$ to be the set of pairs $(g,\gamma) \in \mathcal{M}(G)$ such that:

(1) $g_{p'} = s$.
(2) $\gamma \in \mathrm{Irr}(KC_G(g)\beta_g^{\mathcal{O}}(e))$.

Theorem 1.2. *The map $(s,e) \mapsto \mathcal{B}_G(s,e)$ induces a bijection between $\mathcal{M}^p(G)$ to the set of p-blocks of $\mathcal{M}(G)$.*

2. Proof of Theorem 1.2

2.1. Central characters and congruences. If $(g,\gamma) \in \mathrm{IrrPairs}(G)$, we denote by $\omega_{g,\gamma} : \mathrm{Z}(KC_G(g)) \to K$ the *central character* associated with γ (if $z \in \mathrm{Z}(KC_G(g))$, then $\omega_{g,\gamma}(z)$ is the scalar through which z acts on an irreducible $KC_G(g)$-module affording the character γ). It is a morphism of algebras: when restricted to $\mathrm{Z}(\mathcal{O}C_G(g))$, it has values in $\mathcal{O}$.

If $h \in C_G(g)$, we denote by $\Sigma_g(h)$ conjugacy class of h in $C_G(g)$ and we set

$$\hat{\Sigma}_g(h) = \sum_{v \in \Sigma_g(h)} v \in \mathrm{Z}(\mathcal{O}C_G(g)).$$

We have

$$\omega_{g,\gamma}\big(\hat{\Sigma}_g(h)\big) = \frac{|\Sigma_g(h)| \cdot \gamma(h)}{\gamma(1)}. \tag{2.1}$$

We also recall the following classical results:

Proposition 2.2. *If $g \in G$ and γ, γ' are two irreducible characters of $C_G(g)$, then γ and γ' lie in the same p-block of $C_G(g)$ if and only if*

$$\omega_{g,\gamma}(\hat{\Sigma}_g(h)) \equiv \omega_{g,\gamma'}(\hat{\Sigma}_g(h)) \mod \mathfrak{p}$$

for all $h \in C_G(g)$.

Proposition 2.3. *Let (g,γ) and (g',γ') be two elements of $\mathcal{M}(G)$. Then (g,γ) and (g',γ') belong to the same p-block of $\mathcal{M}(G)$ if and only if*

$$\Psi_{g,\gamma}(V_{h,\eta}) \equiv \Psi_{g',\gamma'}(V_{h,\eta}) \mod \mathfrak{p}$$

for all $(h,\eta) \in \mathcal{M}(G)$.

2.2. Around the Brauer map. As Brauer maps are morphisms of algebras, we have

$$\sum_{e \in \mathrm{Blocks}(kC_G(g_{p'}))} \mathrm{Br}_g^p(e) = 1,$$

and so

(2.4) *The family $\big(\mathcal{B}_G(g,e)\big)_{(g,e)\in\mathcal{M}^p(G)}$ is a partition of $\mathcal{M}(G)$.*

Now, let $(g,\gamma) \in \mathcal{M}(G)$ and let $s = g_{p'}$. If $e \in \mathrm{Blocks}(\mathcal{O}C_G(s))$ is such that $\gamma \in \mathrm{Irr}(KC_G(g)\beta_g^{\mathcal{O}}(e))$, and if $\sigma \in \mathrm{Irr}(KC_G(s)e)$, then [**2**, Lemma 15.44]

$$\omega_{s,\sigma}(z) \equiv \omega_{g,\gamma}\big(\mathrm{Br}_g^{\mathcal{O}}(z)\big) \mod \mathfrak{p} \tag{2.5}$$

for all $z \in \mathrm{Z}(\mathcal{O}C_G(s))$.

2.3. Rearranging the formula for $\Psi_{g,\gamma}$. If (g,γ), $(h,\eta) \in \mathrm{IrrPairs}(g)$ then

(2.6)
$$\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{\substack{x\in[C_G(g)\backslash G/C_G(h)]\\ xhx^{-1}\in C_G(g)}} \eta(x^{-1}gx)\omega_{g,\gamma}\big(\hat{\Sigma}_g(xhx^{-1})\big).$$

Proof. By definition,

$$\begin{aligned}\Psi_{g,\gamma}(V_{h,\eta}) &= \frac{1}{\gamma(1)\cdot|C_G(h)|}\sum_{\substack{x\in G\\ xhx^{-1}\in C_G(g)}} \eta(x^{-1}gx)\gamma(xhx^{-1})\\ &= \frac{1}{\gamma(1)}\sum_{\substack{x\in[G/C_G(h)]\\ xhx^{-1}\in C_G(g)}} \eta(x^{-1}gx)\gamma(xhx^{-1}).\end{aligned}$$

Now, if $x \in G$ is such that $xhx^{-1} \in C_G(g)$ and if $u \in C_G(g)$, then

$$\eta\big((ux)^{-1}g(ux)\big)\gamma\big((ux)h(ux)x^{-1}\big) = \eta(x^{-1}gx)\gamma(xhx^{-1}).$$

So we can gather the terms in the last sum according to their $C_G(g)$-orbit. We get

$$\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{\substack{x\in[C_G(g)\backslash G/C_G(h)]\\ xhx^{-1}\in C_G(g)}} \eta(x^{-1}gx)\frac{|C_G(g)|}{|C_G(g)\cap xC_G(h)x^{-1}|}\cdot\frac{\gamma(xhx^{-1})}{\gamma(1)}.$$

But, for x in G such that $xhx^{-1} \in C_G(g)$,

$$\frac{|C_G(g)|}{|C_G(g)\cap xC_G(h)x^{-1}|} = |\Sigma_g(xhx^{-1})|,$$

so the result follows from 2.1. □

Corollary 2.7. *Let $g \in [G/\sim]$ and let γ, $\gamma' \in \mathrm{Irr}(KC_G(g))$ lying in the same p-block of $C_G(g)$. Then (g,γ) and (g,γ') lie in the same p-block of $\mathcal{M}(G)$.*

Proof. This follows from 2.6 and Proposition 2.3. □

2.4. p'-part. Fix $(g,\gamma) \in \mathcal{M}(G)$. Then it follows from 2.6 that, for all $\chi \in \mathrm{Irr}(KG)$,

(2.8)
$$\Psi_{g,\gamma}(V_{1,\chi}) = \chi(g).$$

Proposition 2.9. *Let (g,γ) and (h,η) be two elements in $\mathcal{M}(G)$ which lie in the same p-block. Then $g_{p'} = h_{p'}$.*

Proof. By Proposition 2.3 and Equality 2.8, it follows from the hypothesis that

$$\chi(g) \equiv \chi(h) \mod \mathfrak{p}$$

for all $\chi \in \mathrm{Irr}(KG)$. Hence $g_{p'}$ and $h_{p'}$ are conjugate in G (see [**1**, Proposition 2.14]), so they are equal according to our conventions explained in §1.6. □

Proposition 2.10. *Let $s \in G_{p'}$ and let σ, $\sigma' \in \mathrm{Irr}(KC_G(s))$. Then (s,σ) and (s,σ') lie in the same p-block if and only if σ and σ' lie in the same p-block of $C_G(s)$.*

Proof. The if part has been proved in Corollary 2.7. Conversely, assume that (s,σ) and (s,σ') lie in the same p-block. Fix $h \in C_G(s)$. Then $s \in C_G(h)$. Let $\eta_{s,h} : C_G(h) \to K$ be the class function on $C_G(h)$ defined by:

$$\eta_{s,h}(g) = \begin{cases} 1 & \text{if } g_{p'} \text{ and } s \text{ are conjugate in } C_G(h), \\ 0 & \text{otherwise.} \end{cases}$$

It follows from [**1**, Proposition 2.20] that $\eta_{s,h} \in \mathcal{O}\,\mathrm{Irr}(KC_G(h))$. Therefore, by 2.6 and Proposition 2.3,

(#)

$$\sum_{\substack{x \in [C_G(s)\backslash G/C_G(h)] \\ xhx^{-1} \in C_G(s)}} \eta_{s,h}(x^{-1}sx)\Big(\omega_{s,\sigma}\big(\hat{\Sigma}_g(xhx^{-1})\big) - \omega_{s,\sigma'}\big(\hat{\Sigma}_g(xhx^{-1})\big)\Big) \equiv 0 \mod \mathfrak{p}.$$

Now, let $x \in G$ be such that $xhx^{-1} \in C_G(s)$. Since $x^{-1}sx$ is also a p'-element, $\eta_{s,h}(x^{-1}sx) = 1$ if and only if s and $x^{-1}sx$ are conjugate in $C_G(h)$ that is, if and only if $x \in C_G(s)C_G(h)$. So it follows from (#) that

$$\omega_{s,\sigma}(\hat{\Sigma}_g(h)) \equiv \omega_{s,\sigma'}(\hat{\Sigma}_g(h)) \mod \mathfrak{p}$$

for all $h \in C_G(s)$. This shows that σ and σ' lie in the same p-block of $C_G(s)$. □

2.5. Last step.

We shall prove here the last intermediate result:

Proposition 2.11. *Let $(s,e) \in \mathcal{M}^p(G)$ and let (g,γ), $(g',\gamma') \in \mathcal{B}_G(s,e)$. Then (g,γ) and (g',γ') are in the same p-block of $\mathcal{M}(G)$.*

Proof. We fix $\sigma \in \mathrm{Irr}(KC_G(s)e)$. It is sufficient to show that (g,γ) and (s,σ) are in the same p-block of $\mathcal{M}(G)$. For this, let $(h,\eta) \in \mathcal{M}(G)$. By Proposition 2.9, we have $g_{p'} = s$, so $C_G(g) \subset C_G(s)$. So 2.6 can be rewritten:

$$\Psi_{g,\gamma}(V_{h,\eta}) = \sum_{x \in [C_G(s)\backslash G/C_G(h)]} \sum_{\substack{y \in [C_G(g)\backslash C_G(s)xC_G(h)/C_G(h)] \\ yhy^{-1} \in C_G(g)}} \eta(y^{-1}gy)\omega_{g,\gamma}(\hat{\Sigma}_g(yhy^{-1})).$$

Now, let $x \in [C_G(s)\backslash G/C_G(h)]$ and $y \in [C_G(g)\backslash C_G(s)xC_G(h)/C_G(h)]$ be such that $yhy^{-1} \in C_G(g)$. Then $yhy^{-1} \in C_G(s)$ and so $xhx^{-1} \in C_G(s)$. Moreover $y^{-1}sy$ is conjugate to $x^{-1}sx$ in $C_G(h)$. Finally, it is well-known (and easy to prove) that $\eta(y^{-1}hy) \equiv \eta(y^{-1}sy) \mod \mathfrak{p}$ (see for instance [**1**, Proposition 2.14]). Therefore:

($\diamondsuit$) $\quad \Psi_{g,\gamma}(V_{h,\eta}) \equiv$

$$\sum_{\substack{x \in [C_G(s)\backslash G/C_G(h)] \\ xhx^{-1} \in C_G(s)}} \eta(x^{-1}sx)\ \omega_{g,\gamma}\Big(\sum_{\substack{y \in [C_G(g)\backslash C_G(s)xC_G(h)/C_G(h)] \\ yhy^{-1} \in C_G(g)}} \hat{\Sigma}_g(yhy^{-1})\Big) \mod \mathfrak{p}.$$

Now, let $x \in [C_G(s)\backslash G/C_G(h)]$ be such that $xhx^{-1} \in C_G(s)$. Then, by definition of the Brauer map,

$$(\heartsuit) \qquad \mathrm{Br}_g^{\mathcal{O}}(\hat{\Sigma}_s(xhx^{-1})) = \sum_{\substack{z\in[C_G(g)\backslash C_G(s)/(C_G(s)\cap C_G(xhx^{-1}))] \\ z(xhx^{-1})z^{-1}\in C_G(g)}} \hat{\Sigma}_g((zx)h(zx)^{-1}).$$

But $(zx)_{z\in[C_G(g)\backslash C_G(s)/(C_G(s)\cap C_G(xhx^{-1}))]}$ is a set of representatives of double classes in $C_G(g)\backslash C_G(s)xC_G(h)/C_G(h)$. So it follows from ($\diamondsuit$) and ($\heartsuit$) that

$$\Psi_{g,h}(V_{h,\eta}) \equiv \sum_{\substack{x\in[C_G(s)\backslash G/C_G(h)] \\ xhx^{-1}\in C_G(s)}} \eta(x^{-1}sx)\ \omega_{g,\gamma}\big(\mathrm{Br}_g^{\mathcal{O}}(\hat{\Sigma}_s(xhx^{-1}))\big).$$

Using now 2.5 and 2.6, we obtain that

$$\Psi_{g,h}(V_{h,\eta}) \equiv \Psi_{s,\sigma}(V_{h,\eta}) \mod \mathfrak{p},$$

as desired. □

Proof of Theorem 1.2. Let (s,e) and (s',e') be two elements of $\mathcal{M}^p(G)$ such that $\mathcal{B}_G(s,e)$ and $\mathcal{B}_G(s',e')$ are contained in the same p-block of $\mathcal{M}(G)$ (see Proposition 2.11). Let $\sigma \in \mathrm{Irr}\big(KC_G(s)e\big)$ and $\sigma' \in \mathrm{Irr}\big(KC_G(s')e'\big)$.

Then (s,σ) and (s',σ') are in the same p-block, so it follows from Proposition 2.9 that $s = s'$ and it follows from Proposition 2.10 that γ and γ' are in the same p-block of $C_G(s)$, that is $e = e'$. This completes the proof of Theorem 1.2. □

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