NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 48 (2018), 129-155

A GENERALIZATION OF WEIGHTED STEPANOV-LIKE PSEUDO-ALMOST AUTOMORPHIC SPACE

Mohamed Zitane

(Received 07 September, 2017)

Abstract. In this paper we introduce and study a new class of functions called weighted Stepanov-like pseudo-almost automorphic functions with variable exponents, which generalizes the class of weighted Stepanov-like pseudo-almost automorphic functions. Basic properties of these new spaces are established. The existence of weighted pseudo-almost automorphic solutions to some first-order differential equations with $S^{p,q(x)}$-pseudo-almost automorphic coefficients will also be studied.

1. Introduction

In Diagana [**10**] the concept of Stepanov-like pseudo-almost automorphy was introduced and studied. These spaces, which generalize pseudo-almost automorphic spaces, were then utilized to study the existence of pseudo-almost automorphic solutions to some abstract differential equations.

In Blot *et al.* [**4**], the concept of weighted pseudo-almost automorphy, using theoretical measure theory, is introduced and utilized to study the existence of weighted pseudo-almost automorphic solutions to some abstract differential equations.

In a recent paper by Diagana and Zitane [**13**], the concept of Stepanov-like pseudo-almost automorphy is introduced in the Lebesgue space with variable exponents $L^{p(x)}$. These functions were utilized to study the existence of pseudo-almost automorphic solutions to some differential equations.

In this paper we introduce and study a new class of functions called weighted Stepanov-like pseudo-almost automorphic functions with variable exponents, which generalizes the usual weighted Stepanov-like pseudo-almost automorphic functions. Basic properties of these new spaces are established. Afterwards, we study the existence of pseudo-almost automorphic solutions to the class of abstract nonautonomous differential equations given by

$$\frac{d}{dt}\big[u(t) + f(t, B(t)u(t))\big] = A(t)u(t) + g(t, C(t)u(t)), \quad t \in \mathbb{R}, \tag{1.1}$$

where $A(t)$ for $t \in \mathbb{R}$ is a family of closed linear operators on $D(A(t))$ satisfying the well-known Acquistapace-Terreni conditions, $B(t), C(t) \quad (t \in \mathbb{R})$ are families of (possibly unbounded) linear operators, and $f : \mathbb{R} \times \mathbb{X} \to \mathbb{X}^t_\beta \quad (0 < \alpha < \beta < 1)$ and $g : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$ are jointly continuous satisfying some additional assumptions with $\mathbb{X}^t_\beta$ being a real interpolation space between $\mathbb{X}$ and $D(A(t))$ of order $\alpha \in (0, 1)$.

2010 *Mathematics Subject Classification* 34C27, 35B15, 46E30.
Key words and phrases: weighted pseudo-almost automorphy, Lebesgue space with variable exponents, Stepanov-like pseudo-almost automorphy with variable exponents.

2. μ-Pseudo-Almost Automorphic Functions

Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R}\times\mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R}\times\mathbb{Y} \to \mathbb{X}$). Let $B(\mathbb{X}, \mathbb{Y})$ stand for the Banach space of bounded linear operators from $\mathbb{X}$ into $\mathbb{Y}$ equipped with its natural operator topology; in particular, $B(\mathbb{X}, \mathbb{X})$ is denoted by $B(\mathbb{X})$.

In this section, we recall the concept of μ-pseudo-almost automorphic functions introduced by J. Blot *et al* [**5**].

Definition 2.1 ([**6**])**.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$ there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t + s_n)$$

is well defined for each $t \in \mathbb{R}$ and

$$f(t) = \lim_{n\to\infty} g(t - s_n)$$

for each $t \in \mathbb{R}$.

The collection of all such functions will be denoted by $AA(\mathbb{X})$, which turns out to be a Banach space when it is equipped with the sup-norm.

Proposition 2.2 ([**21**])**.** Assume $f, g : \mathbb{R} \to \mathbb{X}$ are almost automorphic and λ is any scalar. Then the following hold true:
- (a) $f + g, \lambda f, f_\tau(t) := f(t + \tau)$ and $\widehat{f}(t) := f(-t)$ are almost automorphic;
- (b) The range R_f of f is precompact, so f is bounded;
- (c) If $\{f_n\}$ is a sequence of almost automorphic functions and $f_n \to f$ uniformly on $\mathbb{R}$, then f is almost automorphic.

We denote by $\mathcal{B}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{B}$ satisfying $\mu(\mathbb{R}) = \infty$ and $\mu([a, b]) < \infty$, for all $a, b \in \mathbb{R}\,(a \le b)$.

Definition 2.3 ([**4**])**.** Let $\mu \in \mathcal{M}$. A function $f \in BC(\mathbb{R}, \mathbb{X})$ is said to be μ-ergodic if

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|f(t)\|\, d\mu(t) = 0$$

where $Q_r := [-r, r]$. We denote the space of all such functions by $\mathcal{E}(\mathbb{X}, \mu)$.

Proposition 2.4 ([**4**])**.** Let $\mu \in \mathcal{M}$. Then $(\mathcal{E}(\mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.

Theorem 2.5 ([**4**])**.** Let $\mu \in \mathcal{M}$ and I be a bounded interval (eventually $I \neq \emptyset$). Assume that $f \in BC(\mathbb{R}, \mathbb{X})$. Then the following assertions are equivalent:
- (a) $f \in \mathcal{E}(\mathbb{X}, \mu)$;
- (b) $\displaystyle\lim_{r\to\infty} \frac{1}{\mu([-r, r]\setminus I)} \int_{[-r,r]\setminus I} \|f(t)\|\, d\mu(t) = 0$;
- (c) For any $\varepsilon > 0$, $\displaystyle\lim_{r\to\infty} \frac{\mu\big(\{t \in [-r, r]\setminus I : \|f(t)\| > \varepsilon\}\big)}{\mu([-r, r]\setminus I)} = 0$.

Definition 2.6 ([**5**])**.** A function $f \in C(\mathbb{R}, \mathbb{X})$ is called μ-pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{X}, \mu)$. The collection of such functions will be denoted by $PAA(\mathbb{X}, \mu)$.

Let $\mathcal{N}_1$ denotes the set of all positive measure $\mu \in \mathcal{M}$ such that for all a, b and $c \in \mathbb{R}$ such that $0 \le a < b \le c$, there exist $\tau_0 \ge 0$ and $\alpha_0 > 0$ such that

$$|\tau| \ge \tau_0 \Rightarrow \mu((a + \tau, b + \tau)) \ge \alpha_0 \mu([\tau, c + \tau]).$$

And let $\mathcal{N}_2$ denotes the set of all positive measure $\mu \in \mathcal{M}$ such that for all $\tau \in \mathbb{R}$, there exist $\beta > 0$ and a bounded interval I such that

$$\mu(\{a + \tau : a \in A\}) \le \beta \mu(A) \quad \text{for all} \quad A \in \mathcal{B} \quad \text{such that} \quad A \cap I = \emptyset.$$

Theorem 2.7 ([**5**])**.** Let $\mu \in \mathcal{N}_1$. Then the decomposition of a μ-pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{X}, \mu)$ is unique.

Theorem 2.8 ([**5**])**.** Let $\mu \in \mathcal{N}_1$. Then $(PAA(\mathbb{X}, \mu), \|\cdot\|_\infty)$ is a Banach space.

Theorem 2.9 ([**5**])**.** Let $\mu \in \mathcal{N}_2$. Then the space $\mathcal{E}(\mathbb{X}, \mu)$ is translation invariant, therefore $PAA(\mathbb{X}, \mu)$ is also translation invariant, that is, if $f \in PAA(\mathbb{X}, \mu)$ implies $f_\tau = f(\cdot + \tau) \in PAA(\mathbb{X}, \mu)$ for all $\tau \in \mathbb{R}$.

Definition 2.10 ([**19**])**.** A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be almost automorphic if $F(t, u)$ is almost automorphic in $t \in \mathbb{R}$ uniformly for all $u \in K$, where $K \subset \mathbb{Y}$ is an arbitrary bounded subset. The collection of all such functions will be denoted by $AA(\mathbb{Y}, \mathbb{X})$.

Definition 2.11 ([**18**])**.** A function $L \in C(\mathbb{R} \times \mathbb{R}, \mathbb{X})$ is called bi-almost automorphic if for every sequence of real numbers $(s'_n)_n$ we can extract a subsequence $(s_n)_n$ such that

$$H(t, s) := \lim_{n \to \infty} L(t + s_n, s + s_n)$$

is well defined for each $t, s \in \mathbb{R}$, and

$$L(t, s) = \lim_{n \to \infty} H(t - s_n, s - s_n)$$

for each $t, s \in \mathbb{R}$. The collection of all such functions will be denoted by $bAA(\mathbb{R} \times \mathbb{R}, \mathbb{X})$.

Definition 2.12 ([**4**])**.** Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is called μ-ergodic in t uniformly with respect to x in $\mathbb{Y}$ if the following two conditions hold:
(a) for all y in $\mathbb{Y}$, $f(\cdot, y) \in \mathcal{E}(\mathbb{Y}, \mu)$;
(b) f is uniformly continuous on each compact set $K \subset \mathbb{Y}$ with respect to the second variable y.
We denote the space of all such functions by $\mathcal{E}(\mathbb{Y}, \mathbb{X}, \mu)$.

Definition 2.13 ([**5**])**.** Let $\mu \in \mathcal{M}$. A function $f \in C(\mathbb{R} \times \mathbb{X}, \mathbb{Y})$ is called μ-pseudo almost automorphic if it can be expressed as $f = g + \phi$, where $g \in AA(\mathbb{Y}, \mathbb{X})$ and $\phi \in \mathcal{E}(\mathbb{Y}, \mathbb{X}, \mu)$. The collection of such functions will be denoted by $PAA(\mathbb{Y}, \mathbb{X}, \mu)$.

3. Evolution Families

Definition 3.1 ([**1, 2**]). A family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domains $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace–Terreni conditions, if there exist constants $\omega \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, $K, L \geq 0$ and $\mu, \nu \in (0,1]$ with $\mu + \nu > 1$ such that

$$\mathrm{S}_\theta \cup \{0\} \subset \rho(\mathrm{A(t)} - \omega \mathrm{I}), \qquad \|\mathrm{R}(\lambda, \mathrm{A(t)} - \omega\mathrm{I})\| \leq \frac{\mathrm{K}}{1+|\lambda|}, \text{ and} \tag{3.1}$$

$$\|(A(t) - \omega I)R(\lambda, A(t) - \omega I)\left[R(\omega, A(t)) - R(\omega, A(s))\right]\| \leq L\,|t-s|^\mu\,|\lambda|^{-\nu} \tag{3.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg\lambda| \leq \theta\}$.

Among other things, the Acquistapace–Terreni Conditions do ensure the existence of a unique evolution family

$$\mathcal{U} = \{U(t,s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$$

on $\mathbb{X}$ associated with $A(t)$ such that $U(t,s)\mathbb{X} \subseteq D(A(t))$ for all $t, s \in \mathbb{R}$ with $t > s$, and

(a) $U(t,s)U(s,r) = U(t,r)$ for $t, s \in \mathbb{R}$ such that $t \geq r \geq s$;
(b) $U(t,t) = I$ for $t \in \mathbb{R}$ where I is the identity operator of $\mathbb{X}$; and
(c) for $t > s$, the mapping $(t,s) \mapsto U(t,s) \in B(\mathbb{X})$ is continuous and continuously differentiable in t with $\partial_t U(t,s) = A(t)U(t,s)$. Moreover, there exists a constant $C' > 0$ which depends on constants in Eq. (3.1) and Eq. (3.2) such that

$$\|A^k(t)U(t,s)\|_{B(\mathbb{X})} \leq C'(t-s)^{-k} \tag{3.3}$$

for $0 < t - s \leq 1$ and $k = 0, 1$.

Definition 3.2. An evolution family $\mathcal{U} = \{U(t,s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$ is said to have an *exponential dichotomy* if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t,s)P(s) = P(t)U(t,s)$;
(g) the restriction $U_Q(t,s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t,s)$ is invertible (we then set $U_Q(s,t) := U_Q(t,s)^{-1}$) where $Q(t) = I - P(t)$; and
(h) $\|U(t,s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s,t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

If an evolution family $\mathcal{U} = \{U(t,s) : t, s \in \mathbb{R} \ \text{ such that } \ t \geq s\}$ has an exponential dichotomy, we then define

$$\Gamma(t,s) := \begin{cases} U(t,s)P(s), & \text{if } t \geq s, \quad t, s \in \mathbb{R}, \\ \\ -U_Q(t,s)Q(s), & \text{if } s > t, \quad t, s \in \mathbb{R}. \end{cases}$$

This setting requires the introduction of some interpolation spaces for $A(t)$. We refer the reader to the following excellent books [**3**], [**9**], and [**20**] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (Definition 3.1 holds when $A(t)$ is replaced with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{x \in \mathbb{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha\,(A - \omega)\,R\,(r, A - \omega)\,x\| < \infty\right\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \|x\|_0^A := \|x\|,\ \mathbb{X}_1^A := D(A)$$

and $\|x\|_1^A := \|(\omega - A)x\|$. Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we will frequently be using the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X}, \tag{3.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{3.5}$$

for $0 < \alpha < \beta < 1$.

Definition 3.3. Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying Acquistapace-Terreni conditions (Definition 3.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms.

Then the embedding in (3.4) hold with constants independent of $t \in \mathbb{R}$.

These interpolation spaces are of class $\mathcal{J}_\alpha$ ([**20**, Definition 1.1.1]) and it can be shown that

$$\|y\|_\alpha^t \leq K^\alpha L^{1-\alpha} \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)) \tag{3.6}$$

where K, L are the constants appearing in Definition 3.1.

Proposition 3.4 ([**7**])**.** For $x \in \mathbb{X}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:

(i) There is a constant $n(\alpha)$, such that

$$\|U(t,s)P(s)x\|_\alpha^t \leq n(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|, \tag{3.7}$$

(ii) There is a constant $m(\alpha)$, such that

$$\|\widetilde{U}_Q(s,t)Q(t)x\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|, \quad t \leq s. \tag{3.8}$$

4. Weighted Stepanov-Like Pseudo Almost Automorphic Functions with Variable Exponents

In what follows, we recall the notion of Lebesgue spaces with variable exponents $L^{p(x)}(\mathbb{R}, \mathbb{X})$ developed in [**12, 13, 14, 15, 17, 22**].

Let $\Omega \subseteq \mathbb{R}$ be a subset and let $M(\Omega, \mathbb{X})$ denote the collection of all measurable functions $f : \Omega \mapsto \mathbb{X}$. Let us recall that two functions f and g of $M(\Omega, \mathbb{X})$ are equal whether they are equal almost everywhere. Set $m(\Omega) := M(\Omega, \mathbb{R})$ and fix

$p \in m(\Omega)$. Define

$$p^- := \operatorname{ess\,inf}_{x\in\Omega} p(x), \quad p^+ := \operatorname{ess\,sup}_{x\in\Omega} p(x),$$

$$C_+(\Omega) := \Big\{ p \in m(\Omega) : 1 < p^- \leq p(x) \leq p^+ < \infty, \text{ for each } x \in \Omega \Big\},$$

$$D_+(\Omega) := \Bigg\{ p \in m(\Omega) : 1 \leq p^- \leq p(x) \leq p^+ < \infty, \text{ for each } x \in \Omega \Bigg\},$$

$$\rho(u) = \rho_{p(x)}(u) = \int_\Omega \|u(x)\|^{p(x)} dx.$$

We then define the Lebesgue space with variable exponents $L^{p(x)}(\Omega, \mathbb{X})$ with $p \in C_+(\Omega)$, by

$$L^{p(x)}(\Omega, \mathbb{X}) := \Big\{ u \in M(\Omega, \mathbb{X}) : \int_\Omega \|u(x)\|^{p(x)} dx < \infty \Big\}.$$

Define, for each $u \in L^{p(x)}(\Omega, \mathbb{X})$,

$$\|u\|_{p(x)} := \inf \Bigg\{ \lambda > 0 : \int_\Omega \Big\| \frac{u(x)}{\lambda} \Big\|^{p(x)} dx \leq 1 \Bigg\}.$$

It can be shown that $\|\cdot\|_{p(x)}$ is a norm upon $L^{p(x)}(\Omega, \mathbb{X})$, which is referred to as the *Luxemburg norm.*

Remark 4.1. Let $p \in C_+(\Omega)$. If p is constant, then the space $L^{p(\cdot)}(\Omega, \mathbb{X})$, as defined above, coincides with the usual space $L^p(\Omega, \mathbb{X})$.

Proposition 4.2 ([**15, 22**])**.** Let $p \in C_+(\Omega)$. If $u, v \in L^{p(x)}(\Omega, \mathbb{X})$, then the following properties hold,

(a) $\|u\|_{p(x)} \geq 0$, with equality if and only if $u = 0$;
(b) $\rho_p(u) \leq \rho_p(v)$ and $\|u\|_{p(x)} \leq \|v\|_{p(x)}$ if $\|u\| \leq \|v\|$;
(c) $\rho_p(u\|u\|^{-1}_{p(x)}) = 1$ if $u \neq 0$;
(d) $\rho_p(u) \leq 1$ if and only if $\|u\|_{p(x)} \leq 1$;
(e) If $\|u\|_{p(x)} \leq 1$, then

$$\Big[\rho_p(u)\Big]^{1/p^-} \leq \|u\|_{p(x)} \leq \Big[\rho_p(u)\Big]^{1/p^+}.$$

(f) If $\|u\|_{p(x)} \geq 1$, then

$$\Big[\rho_p(u)\Big]^{1/p^+} \leq \|u\|_{p(x)} \leq \Big[\rho_p(u)\Big]^{1/p^-}.$$

Theorem 4.3 ([**15, 17**])**.** Let $p \in C_+(\Omega)$. The space $(L^{p(x)}(\Omega, \mathbb{X}), \|\cdot\|_{p(x)})$ is a Banach space that is separable and uniform convex. Its topological dual is $L^{q(x)}(\Omega, \mathbb{X})$, where $p^{-1}(x) + q^{-1}(x) = 1$. Moreover, for any $u \in L^{p(x)}(\Omega, \mathbb{X})$ and $v \in L^{q(x)}(\Omega, \mathbb{R})$, we have

$$\Big\| \int_\Omega uv dx \Big\| \leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) \|u\|_{p(x)} . \|v\|_{q(x)}.$$

Corollary 4.4 ([**22**])**.** Let $p, r \in D_+(\Omega)$. If the function q defined by the equation

$$\frac{1}{q(x)} = \frac{1}{p(x)} + \frac{1}{r(x)}$$

is in $D_+(\Omega)$, then there exists a constant $C = C(p, r) \in [1, 5]$ such that

$$\|uv\|_{q(x)} \leq C\|u\|_{p(x)}.\, \|v\|_{r(x)},$$

for every $u \in L^{p(x)}(\Omega, \mathbb{X})$ and $v \in L^{r(x)}(\Omega, \mathbb{R})$.

Corollary 4.5 ([**15**])**.** Let $mes\,(\Omega) < \infty$ where $mes\,(\cdot)$ stands for the Lebesgue measure and $p, q \in D_+(\Omega)$. If $q(\cdot) \leq p(\cdot)$ almost everywhere in Ω, then the embedding $L^{p(x)}(\Omega, \mathbb{X}) \hookrightarrow L^{q(x)}(\Omega, \mathbb{X})$ is continuous whose norm does not exceed $2(mes\,(\Omega) + 1)$.

Definition 4.6 ([**10**])**.** The Bochner transform $f^b(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$ of a function $f : \mathbb{R} \to \mathbb{X}$ is defined by $f^b(t, s) := f(t + s)$.

Remark 4.7. (i) A function $\varphi(t, s)$, $t \in \mathbb{R}$, $s \in [0, 1]$, is the Bochner transform of a certain function f, $\varphi(t, s) = f^b(t, s)$, if and only if $\varphi(t + \tau, s - \tau) = \varphi(s, t)$ for all $t \in \mathbb{R}$, $s \in [0, 1]$ and $\tau \in [s - 1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 4.8 ([**10**])**.** The Bochner transform $F^b(t, s, u)$, $t \in \mathbb{R}$, $s \in [0, 1]$, $u \in \mathbb{X}$ of a function $F(t, u)$ on $\mathbb{R} \times \mathbb{X}$, with values in $\mathbb{X}$, is defined by $F^b(t, s, u) := F(t + s, u)$ for each $u \in \mathbb{X}$.

Definition 4.9 ([**10**])**.** Let $p \in [1, \infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions f on $\mathbb{R}$ with values in $\mathbb{X}$ such that $f^b \in L^\infty\big(\mathbb{R}, L^p((0, 1), \mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} = \|f^b\|_{L^\infty(\mathbb{R}, L^p)} = \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p}.$$

Note that for each $p \geq 1$, we have the following continuous inclusion:

$$(BC(\mathbb{X}), \|\cdot\|_\infty) \hookrightarrow (BS^p(\mathbb{X}), \|\cdot\|_{S^p}).$$

Definition 4.10 ([**12**])**.** Let $p \in C_+(\mathbb{R})$. The space $BS^{p(x)}(\mathbb{X})$ consists of all functions $f \in M(\mathbb{R}, \mathbb{X})$ such that $\|f\|_{S^{p(x)}} < \infty$, where

$$\begin{aligned} \|f\|_{S^{p(x)}} &= \sup_{t \in \mathbb{R}} \left[\inf \left\{ \lambda > 0 : \int_0^1 \left\| \frac{f(x+t)}{\lambda} \right\|^{p(x+t)} dx \leq 1 \right\} \right] \\ &= \sup_{t \in \mathbb{R}} \left[\inf \left\{ \lambda > 0 : \int_t^{t+1} \left\| \frac{f(x)}{\lambda} \right\|^{p(x)} dx \leq 1 \right\} \right]. \end{aligned}$$

Note that the space $\Big(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\Big)$ is a Banach space, which, depending on $p(\cdot)$, may or may not be translation-invariant.

Definition 4.11 ([**12**])**.** If $p, q \in C_+(\mathbb{R})$, we then define the space $BS^{p(x),q(x)}(\mathbb{X})$ as follows:

$$\begin{aligned} BS^{p(x),q(x)}(\mathbb{X}) &:= BS^{p(x)}(\mathbb{X}) + BS^{q(x)}(\mathbb{X}) \\ &= \Big\{ f = h + \varphi \in M(\mathbb{R}, \mathbb{X}) : h \in BS^{p(x)}(\mathbb{X}) \ \text{ and } \varphi \in BS^{q(x)}(\mathbb{X}) \Big\}. \end{aligned}$$

We equip $BS^{p(x),q(x)}(\mathbb{X})$ with the norm $\|\cdot\|_{S^{p(x),q(x)}}$ defined by

$$\|f\|_{S^{p(x),q(x)}} := \inf\left\{\|h\|_{S^{p(x)}} + \|\varphi\|_{S^{q(x)}} : \quad f = h + \varphi\right\}.$$

Clearly, $\left(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\right)$ is a Banach space, which, depending on both $p(\cdot)$ and $q(\cdot)$, may or may not be translation-invariant.

Lemma 4.12 ([**12**])**.** Let $p, q \in C_+(\mathbb{R})$. Then the following continuous inclusion holds,

$$\left(BC(\mathbb{R},\mathbb{X}), \|\cdot\|_\infty\right) \hookrightarrow \left(BS^{p(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x)}}\right) \hookrightarrow \left(BS^{p(x),q(x)}(\mathbb{X}), \|\cdot\|_{S^{p(x),q(x)}}\right).$$

Definition 4.13. Let $p \geq 1$ be a constant. A function $f \in BS^p(\mathbb{X})$ is said to be S^p-almost automorphic (or Stepanov-like almost automorphic function) if $f^b \in AA\big(L^p((0,1),\mathbb{X})\big)$. That is, a function $f \in L^p_{\text{loc}}(\mathbb{R},\mathbb{X})$ is said to be Stepanov-like almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_n$, there exists a subsequence $(s_n)_n$ and a function $g \in L^p_{\text{loc}}(\mathbb{R},\mathbb{X})$ such that

$$\Big(\int_0^1 \|f(t+s+s_n)-g(t+s)\|^p ds\Big)^{1/p} \to 0, \quad \Big(\int_0^1 \|g(t+s-s_n)-f(t+s)\|^p ds\Big)^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$. The collection of such functions will be denoted by $S^p_{aa}(\mathbb{X})$.

Remark 4.14. It is clear that if $1 \leq p < q < \infty$ and $f \in L^q_{\text{loc}}(\mathbb{R},\mathbb{X})$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \leq p < \infty$.

Remark 4.15. There are some difficulties in defining $S^{p(x)}_{aa}(\mathbb{X})$ for a function $p \in C_+(\mathbb{R})$ that is not necessarily constant. This is mainly due to the fact that the space $BS^{p(x)}(\mathbb{X})$ is not always translation-invariant.

Taking into account Remark 4.15, we introduce the concept of weighted $S^{p,q(x)}$-pseudo-almost automorphy as follows, which obviously generalizes the notion of weighted S^p-pseudo-almost automorphy.

Definition 4.16. Let $\mu \in \mathcal{M}, p \geq 1$ be a constant and let $q \in C_+(\mathbb{R})$. A function $f \in BS^{p,q(x)}(\mathbb{X})$ is said to be weighted $S^{p,q(x)}$-pseudo-almost automorphic (or weighted Stepanov-like pseudo-almost automorphic with variable exponents $p, q(x)$) if it can be decomposed as $f = h + \varphi$, where $h \in S^p_{aa}(\mathbb{X})$ and $\varphi^b \in \mathcal{E}\left(L^{q^b(x)}((0,1),\mathbb{X}),\mu\right)$, i.e.,

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \Big\|\frac{\varphi(x+t)}{\lambda}\Big\|^{p(x+t)} dx \leq 1\right\} d\mu(t) = 0.$$

The collection of such functions will be denoted by $S^{p,q(x)}_{paa}(\mathbb{X},\mu)$.

Proposition 4.17. Let $r, s \geq 1, p, q \in D_+(\mathbb{R})$, $\mu \in \mathcal{M}$. If $s \leq r, q(\cdot) \leq p(\cdot)$ and $f \in BS^{r,p(x)}(\mathbb{X})$ is weighted $S^{r,p(x)}$-pseudo-almost automorphic, then f is weighted $S^{s,q(x)}$-pseudo-almost automorphic.

Proof. Suppose that $f \in BS^{r,p(x)}(\mathbb{X})$ is $S^{r,p(x)}$-pseudo-almost automorphic. Thus there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$, where $h \in S^r_{aa}(\mathbb{X})$ and $\varphi^b \in \mathcal{E}\left(L^{p^b(x)}((0,1),\mathbb{X}),\mu\right)$. From remark 4.14, h is S^s-almost automorphic.

From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \geq 0$ such that $\mu(Q_r) > 0$ for all $r \geq r_0$. By using Corollary 4.5 and the fact that $q(\cdot) \leq q^+ \leq p^- \leq p(\cdot)$ and $\varphi^b \in \mathcal{E}\left(L^{p^b(x)}((0,1),\mathbb{X}),\mu\right)$, one has

$$\begin{aligned}
&\frac{1}{\mu(Q_r)}\int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{q(x+t)} dx \leq 1\right\} d\mu(t)\\
&\leq \frac{4}{\mu(Q_r)}\int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{p(x+t)} dx \leq 1\right\} d\mu(t).
\end{aligned}$$

that is $\varphi^b \in \mathcal{E}\left(L^{q^b(x)}((0,1),\mathbb{X}),\mu\right)$ and hence f is weighted $S^{s,q(x)}$-pseudo-almost automorphic. □

Proposition 4.18. Let $p \geq 1$ be a constant, $q \in C_+(\mathbb{R})$ and let $\mu \in \mathcal{N}_2$. Then $PAA(\mathbb{X},\mu) \subset S^{p,q(x)}_{paa}(\mathbb{X},\mu)$.

Proof. Let $f \in PAA(\mathbb{X},\mu)$. Thus there exist two functions $h, \varphi : \mathbb{R} \to \mathbb{X}$ such that $f = h + \varphi$, where $h \in AA(\mathbb{X})$ and $\varphi \in \mathcal{E}\left(\mathbb{X},\mu\right)$. Now from remark 4.14, $h \in AA(\mathbb{X}) \subset S^p_{aa}(\mathbb{X})$. The proof of $\varphi^b \in \mathcal{E}\left(L^{q^b(x)}((0,1),\mathbb{X}),\mu\right)$. was given in [**14**]. However for the sake of clarity, we reproduce it here. From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \geq 0$ such that $\mu(Q_r) > 0$ for all $r \geq r_0$.

Using (e)-(f) of Proposition 4.2, the usual Hölder inequality and Fubini's theorem it follows that

$$\begin{aligned}
&\int_{Q_r} \inf\left\{\lambda > 0 : \int_0^1 \left\|\frac{\varphi(x+t)}{\lambda}\right\|^{q(x+t)} dx \leq 1\right\} d\mu(t)\\
&\leq \int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\,dx\right)^{\gamma} d\mu(t)\\
&\leq (\mu(Q_r))^{1-\gamma}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|^{q(t+x)}\,dx\right) d\mu(t)\right]^{\gamma}\\
&\leq (\mu(Q_r))^{1-\gamma}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|.\,\|\varphi\|_\infty^{q(t+x)-1}\,dx\right) d\mu(t)\right]^{\gamma}\\
&\leq (\mu(Q_r))^{1-\gamma}\Big(\|\varphi\|_\infty + 1\Big)^{\gamma(q^+-1)}\left[\int_{Q_r}\left(\int_0^1 \|\varphi(t+x)\|\,dx\right) d\mu(t)\right]^{\gamma}\\
&= (\mu(Q_r))^{1-\gamma}\Big(\|\varphi\|_\infty + 1\Big)^{\gamma(q^+-1)}\left[\int_0^1\left(\int_{Q_r} \|\varphi(t+x)\|\,d\mu(t)\right) dx\right]^{\gamma}\\
&= (\mu(Q_r))\Big(\|\varphi\|_\infty + 1\Big)^{\gamma(q^+-1)}\left[\int_0^1\left(\frac{1}{\mu(Q_r)}\int_{Q_r} \|\varphi(t+x)\|\,d\mu(t)\right) dx\right]^{\gamma},
\end{aligned}$$

where

$$\gamma = \begin{cases} \frac{1}{q^+} & \text{if } \|\varphi\| < 1, \\ \\ \frac{1}{q^-} & \text{if } \|\varphi\| \geq 1. \end{cases}$$

Using the fact that $\mathcal{E}(\mathbb{X}, \mu)$ is translation invariant and the (usual) Dominated Convergence Theorem, it follows that

$$\lim_{r\to+\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \inf \left\{ \lambda > 0 : \int_0^1 \left\| \frac{\varphi(x+t)}{\lambda} \right\|^{q(x+t)} dx \leq 1 \right\} d\mu(t)$$
$$\leq \Big(\|\varphi\|_\infty + 1 \Big)^{\gamma(q^+-1)} \left[\int_0^1 \left(\lim_{r\to+\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|\varphi(t+x)\| \, d\mu(t) \right) dx \right]^\gamma = 0.$$

□

Theorem 4.19. Let $p, q \geq 1$ be constants, $\mu \in \mathcal{N}_2$ and $f \in S^{p,q}_{paa}(\mathbb{X}, \mu)$ be such that

$$f = h + \varphi$$

where $h^b \in AA\big(L^p((0,1), \mathbb{X})\big)$ and $\varphi^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$. Then

$$\Big\{ h(t+.) : t \in \mathbb{R} \Big\} \subset \overline{\Big\{ f(t+.) : t \in \mathbb{R} \Big\}}, \qquad \text{in} \qquad BS^{p,q}(\mathbb{X}).$$

Proof. We prove it by contradiction. Indeed, if this is not true, then there exists $t_0 \in \mathbb{R}$ and an $\varepsilon > 0$ such that

$$\|h(t_0 + \cdot) - f(t + \cdot)\|_{S^{p,q}} \geq 2\varepsilon, \quad t \in \mathbb{R}.$$

Since $h^b \in AA\big(L^p((0,1), \mathbb{X})\big)$ and $\big(BS^p(\mathbb{X}), \|\cdot\|_{S^p}\big) \hookrightarrow \big(BS^{p,q}(\mathbb{X}), \|\cdot\|_{S^{p,q}}\big)$, fix $t_0 \in \mathbb{R}, \varepsilon > 0$ and write, $B_\varepsilon := \{\tau \in \mathbb{R}; \|h(t_0 + \tau + \cdot) - g(t_0 + \cdot)\|_{S^{p,q}} < \varepsilon\}$. By [**23**, Lemma 2.1], there exist $s_1, \ldots, s_m \in \mathbb{R}$ such that

$$\cup_{i=1}^m (s_i + B_\varepsilon) = \mathbb{R}.$$

Write

$$\hat{s}_i = s_i - t_0 \quad (1 \leq i \leq m), \quad \eta = \max_{1\leq i\leq m} |\hat{s}_i|.$$

For $r \in \mathbb{R}$ with $|r| > \eta$; we put

$$B^{(i)}_{\varepsilon,r} = [-r + \eta - \hat{s}_i, r - \eta - \hat{s}_i] \cap (t_0 + B_\varepsilon), \quad 1 \leq i \leq m,$$

one has $\cup_{i=1}^m (\hat{s}_i + B^{(i)}_{\varepsilon,r}) = [-r + \eta, r - \eta]$.

Letting $\mu_\tau(\{a + \tau : a \in A\})$ for $A \in \mathcal{B}$, from $\mu \in \mathcal{N}_2$ it follows that μ and μ_τ are equivalent (see Definitions 4.22). Using the fact that $B^{(i)}_{\varepsilon,r} \subset [-r, r] \cap (t_0 + B_\varepsilon)$,

$i = 1, \dots, m$, we obtain

$$\begin{aligned}\mu(Q_{r-\eta}) &= \mu([-r+\eta, r-\eta]) \\ &\leq \sum_{i=1}^{m} \mu(\hat{s}_i + B^{(i)}_{\varepsilon,r}) \\ &\leq \beta \sum_{i=1}^{m} \mu(B^{(i)}_{\varepsilon,r}) \\ &\leq m\beta \max_{1\leq i\leq m} \{\mu(B^{(i)}_{\varepsilon,r})\} \\ &\leq m\beta\, \mu([-r,r] \cap (t_0 + B_\varepsilon)),\end{aligned}$$

On the other hand, by using the Minkowski inequality, for any $t \in t_0 + B_\varepsilon$, one has

$$\begin{aligned}\|\varphi(t+\cdot)\|_{S^q} &= \|\varphi(t+\cdot)\|_{S^{p,q}} \\ &= \|f(t+\cdot) - h(t+\cdot)\|_{S^{p,q}} \\ &\geq \|h(t_0+\cdot) - f(t+\cdot)\|_{S^{p,q}} - \|h(t+\cdot) - h(t_0+\cdot)\|_{S^{p,q}} > \varepsilon.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\mu(Q_r)} \int_{Q_r} \|\varphi(t+\cdot)\|_{S^q}\, d\mu(t) &\geq \frac{1}{\mu(Q_r)} \int_{Q_r \cap (t_0+B_\varepsilon)} \|\varphi(t+\cdot)\|_{S^q}\, d\mu(t) \\ &\geq \frac{\varepsilon}{\mu(Q_r)} \int_{Q_r \cap (t_0+B_\varepsilon)} d\mu(t) \\ &= \frac{\varepsilon}{\mu(Q_r)} \mu\big(Q_r \cap (t_0 + B_\varepsilon)\big) \\ &\geq \varepsilon \frac{\mu(Q_{r-\eta})}{\mu(Q_r)} (m\beta)^{-1} \to \varepsilon (m\beta)^{-1}, \quad \text{as } r \to \infty.\end{aligned}$$

This is a contradiction, since $\varphi^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$. □

Corollary 4.20. Let $p, q \geq 1$ be constants and $\mu \in \mathcal{N}_1$. Then the decomposition of a $S^{p,q}$-μ-pseudo-almost automorphic function in the form $f = h + \varphi$ where $h^b \in AA\big(L^p((0,1), \mathbb{X})\big)$ and $\varphi^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$, is unique.

Proof. Suppose that $f = h_1 + \varphi_1 = h_2 + \varphi_2$ where $h_1^b, h_2^b \in AA\big(L^p((0,1), \mathbb{X})\big)$ and $\varphi_1^b, \varphi_1^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$. Then $0 = (h_1 - h_2) + (\varphi_1 - \varphi_2) \in S^{p,q}_{paa}(\mathbb{X}, \mu)$ where $h_1^b - h_2^b \in AA\big(L^p((0,1), \mathbb{X})\big)$ and $\varphi_1^b - \varphi_1^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$. From Theorem 4.19 we obtain $(h_1 - h_2)(\mathbb{R}) \subset \{0\}$, therefore one has $h_1 = h_2$ and $\varphi_1 = \varphi_2$. □

Theorem 4.21. Let $p, q \geq 1$ be constants and $\mu \in \mathcal{N}_1$. The space $S^{p,q}_{paa}(\mathbb{X}, \mu)$ equipped with the norm $\|\cdot\|_{S^{p,q}}$ is a Banach space.

Proof. It suffices to prove that $S^{p,q}_{paa}(\mathbb{X}, \mu)$ is a closed subspace of $BS^{p,q}(\mathbb{X})$. Let $f_n = h_n + \varphi_n$ be a sequence in $S^{p,q}_{paa}(\mathbb{X}, \mu)$ with $(h_n^b)_{n\in\mathbb{N}} \subset AA\big(L^p((0,1), \mathbb{X})\big)$ and $(\varphi_n^b)_{n\in\mathbb{N}} \subset \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right)$ such that $\|f_n - f\|_{S^{p,q}} \to 0$ as $n \to \infty$. By Theorem 4.19, one has

$$\{h_n(t+.) : t \in \mathbb{R}\} \subset \overline{\{f_n(t+.) : t \in \mathbb{R}\}},$$

and hence

$$\|h_n\|_{S^p} = \|h_n\|_{S^{p,q}} \leq \|f_n\|_{S^{p,q}} \quad \text{for all } n \in \mathbb{N}.$$

Consequently, there exists a function $h \in S^p_{aa}(\mathbb{X})$ such that $\|h_n - h\|_{S^p} \to 0$ as $n \to \infty$. Using the previous fact, it easily follows that the function $\varphi := f - h \in BS^q(\mathbb{X})$ and that $\|\varphi_n - \varphi\|_{S^q} = \|(f_n - h_n) - (f - h)\|_{S^q} \to 0$ as $n \to \infty$. From $\mu(\mathbb{R}) = \infty$, we deduce the existence of $r_0 \geq 0$ such that $\mu(Q_r) > 0$ for all $r \geq r_0$. Using the fact that $\varphi = (\varphi - \varphi_n) + \varphi_n$ and the triangle inequality, it follows that

$$\begin{aligned}
&\frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^1 \left\| \varphi(\tau + t) \right\|^q d\tau \right)^{\frac{1}{q}} d\mu(t) \\
&\quad \leq \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^1 \left\| \varphi(\tau + t) - \varphi_n(\tau + t) \right\|^q d\tau \right)^{\frac{1}{q}} d\mu(t) \\
&\qquad + \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^1 \left\| \varphi_n(\tau + t) \right\|^q d\tau \right)^{\frac{1}{q}} d\mu(t) \\
&\quad \leq \|\varphi_n - \varphi\|_{S^q} + \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^1 \left\| \varphi_n(\tau + t) \right\|^q d\tau \right)^{\frac{1}{q}} d\mu(t).
\end{aligned}$$

Letting $r \to +\infty$ and then $n \to \infty$ in the previous inequality yields

$$\varphi^b \in \mathcal{E}\left(L^q((0,1), \mathbb{X}), \mu\right),$$

that is, $f = h + \varphi \in S^{p,q}_{paa}(\mathbb{X}, \mu)$. □

Definition 4.22 ([**4**])**.** Let $\mu_1, \mu_2 \in \mathcal{M}$. μ_1 is said to be equivalent to μ_2 ($\mu_1 \sim \mu_2$) if there exist constants $\alpha, \beta > 0$ and a bounded interval I (eventually $I \neq \emptyset$) such that

$$\alpha \mu_1(A) \leq \mu_2(A) \leq \beta \mu_1(A), \quad \text{for all } \ A \in \mathcal{B} \ \text{ such that } \ A \cap I = \emptyset.$$

Theorem 4.23. Let $p \geq 1$ be a constant, $q \in C_+(\mathbb{R})$ and $\mu_1, \mu_2 \in \mathcal{M}$. If μ_1 and μ_2 are equivalent, then $S^{p,q(x)}_{paa}(\mathbb{X}, \mu_1) = S^{p,q(x)}_{paa}(\mathbb{X}, \mu_2)$.

Proof. The proof is similar to that of [**4**, Theorem 2.21]. Since $\mu_1 \sim \mu_2$, and $\mathcal{B}$ is the Lebesgue σ-field of $\mathbb{R}$, we obtain for r sufficiently large

$$\begin{aligned}
\frac{\alpha}{\beta} \frac{\mu_1\big(\{t \in Q_r \setminus I : \, \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\}\big)}{\mu(Q_r \setminus I)} &\leq \frac{\mu_2\big(\{t \in Q_r \setminus I : \, \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\}\big)}{\mu(Q_r \setminus I)} \\
&\leq \frac{\beta}{\alpha} \frac{\mu_1\big(\{t \in Q_r \setminus I : \, \|f(t)\|_{S^{p,q(\cdot)}} > \varepsilon\}\big)}{\mu(Q_r \setminus I)}.
\end{aligned}$$

By using Theorem 2.5, we deduce that

$$\mathcal{E}\left(L^{q^b(x)}((0,1), \mathbb{X}), \mu_1\right) = \mathcal{E}\left(L^{q^b(x)}((0,1), \mathbb{X}), \mu_1\right).$$

From the definition of a weighted $S^{p,q(x)}$-pseudo-almost automorphic function, we deduce that $S^{p,q(x)}_{paa}(\mathbb{X}, \mu_1) = S^{p,q(x)}_{paa}(\mathbb{X}, \mu_2)$. □

Definition 4.24. A function $F : \mathbb{R} \times \mathbb{Y} \to \mathbb{X}$ with $F(.,u) \in BS^{p,q(x)}(\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be $S^{p,q(x)}$-μ-pseudo-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \mapsto F(t,u)$ is $S^{p,q(x)}$-μ-pseudo-almost automorphic for each $u \in B$ where $B \subset \mathbb{Y}$ is an arbitrary bounded set.

This means, there exist two functions $G, H : \mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ such that $F = G + H$, where $G^b \in AA(\mathbb{Y}, L^p((0,1),\mathbb{X}))$ and $H^b \in \mathcal{E}\big(\mathbb{Y}, L^{q^b(x)}((0,1),\mathbb{X}),\mu\big)$, that is,

$$\lim_{r\to+\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\inf\left\{\lambda>0 : \int_0^1\Big\|\frac{H(x+t,u)}{\lambda}\Big\|^{q(x+t)}dx\le 1\right\}d\mu(t)=0,$$

uniformly in $u\in B$ where $B\subset\mathbb{Y}$ is an arbitrary bounded set.

The collection of such functions will be denoted by $S^{p,q(x)}_{paa}(\mathbb{Y},\mathbb{X},\mu)$.

Let $Lip^r(\mathbb{Y},\mathbb{X})$ denote the collection of functions $f : \mathbb{R}\times\mathbb{Y}\to\mathbb{X}$ satisfying: there exists a nonnegative function $L^b_f \in L^r(\mathbb{R})$ such that

$$\|f(t,u)-f(t,v)\|\le L_f(t)\|u-v\|_{\mathbb{Y}}\quad\text{for all}\quad u,v\in\mathbb{Y},\quad t\in\mathbb{R}. \tag{4.1}$$

Now, we recall the following composition theorem for S^p_{aa} functions.

Theorem 4.25 ([**16**])**.** Let $p>1$ be a constant. We suppose that the following conditions hold:
(a) $f\in S^p_{aa}(\mathbb{Y},\mathbb{X})\cap Lip^r(\mathbb{Y},\mathbb{X})$ with $r\ge\max\{p,\frac{p}{p-1}\}$.
(b) $\phi\in S^p_{aa}(\mathbb{X})$ and there exists a set $E\subset\mathbb{R}$ such that $K:=\overline{\{\phi(t): t\in\mathbb{R}\setminus E\}}$ is compact in $\mathbb{X}$.
Then there exists $m\in[1,p)$ such that $f(\cdot,\phi(\cdot))\in S^m_{aa}(\mathbb{X})$.

To obtain the composition theorem for weighted $S^{p,q}$ functions, we need the following lemma.

Lemma 4.26. Let $p,q>1$ be constants and let $\mu\in\mathcal{N}_2$. Assume that $f=g+h\in S^{p,q}_{paa}(\mathbb{Y},\mathbb{X},\mu)$ with $g^b\in AA\big(\mathbb{Y},L^p((0,1),\mathbb{X})\big)$ and $h^b\in\mathcal{E}\big(\mathbb{Y},L^q((0,1),\mathbb{X}),\mu\big)$. If $f\in Lip^p(\mathbb{Y},\mathbb{X})$, then g satisfies

$$\Big(\int_0^1\|g(t+s,u(s))-g(t+s,v(s))\|^p\,ds\Big)^{1/p}\le c\|L_f\|_{S^p}\|u-v\|_{\mathbb{Y}}.$$

for all $u,v\in\mathbb{Y}$ and $t\in\mathbb{R}$, where c is a nonnegative constant.

Proof. The proof is similar to that of [**13**, Lemma 4.19]. So we omit it. □

Theorem 4.27. Let $p,q>1$ be constants such that $p\le q$ and $\mu\in\mathcal{N}_2$. Suppose that the following conditions hold:
(a) $f=g+h\in S^{p,q}_{paa}(\mathbb{Y},\mathbb{X},\mu)$ with $g^b\in AA\big(\mathbb{Y},L^p((0,1),\mathbb{X})\big)$ and $h^b\in\mathcal{E}\big(\mathbb{Y},L^q((0,1),\mathbb{X}),\mu\big)$. Further, $f,g\in Lip^r(\mathbb{Y},\mathbb{X})$ with $r\ge\max\{p,\frac{p}{p-1}\}$.
(b) $\phi=\alpha+\beta\in S^{p,q}_{paa}(\mathbb{Y})$ with $\alpha^b\in AA\big(L^p((0,1),\mathbb{Y})\big)$ and $\beta^b\in\mathcal{E}\big(L^q((0,1),\mathbb{Y}),\mu\big)$, and there exists a set $E\subset\mathbb{R}$ with $mes\ (E)=0$ such that

$$K:=\overline{\{\alpha(t): t\in\mathbb{R}\setminus E\}}$$

is compact in $\mathbb{Y}$.
Then, there exists $m\in[1,p)$ such that $f(\cdot,\phi(\cdot))\in S^{m,m}_{paa}\big(\mathbb{Y},\mathbb{X},\mu\big)$.

Proof. We will make use of ideas of [**13**, Theorem 4.20]. Indeed, decompose f^b as follows:

$$f^b(\cdot,\phi^b(\cdot))=g^b(\cdot,\alpha^b(\cdot))+f^b(\cdot,\phi^b(\cdot))-f^b(\cdot,\alpha^b(\cdot))+h^b(\cdot,\alpha^b(\cdot)).$$

From Lemma 4.26, one has $g\in S^p_{aa}(\mathbb{R}\times\mathbb{X})$. Now using the theorem of composition of S^p-almost automorphic functions (Theorem 4.25), it is easy to see that there exists $m \in [1,p)$ with $\frac{1}{m} = \frac{1}{p} + \frac{1}{r}$ such that $g^b(\cdot, \alpha^b(\cdot)) \in AA(\mathbb{Y}, L^m((0,1),\mathbb{X}))$. Set $\Phi^b(\cdot) = f^b(\cdot, \phi^b(\cdot)) - f^b(\cdot, \alpha^b(\cdot))$. Clearly, $\Phi^b \in \mathcal{E}(\mathbb{R} \times L^m((0,1),\mathbb{X}), \mu)$. Indeed, from $\mu(\mathbb{R}) = \infty$, there exists $r_0 > 0$ such that, for all $r > r_0$, one has

$$\begin{aligned}
&\frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_t^{t+1} \|\Phi^b(s)\|^m ds \right)^{1/m} d\mu(t) \\
&= \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_t^{t+1} \|f^b(s, \phi^b(s)) - f^b(s, \alpha^b(s))\|^m ds \right)^{1/m} d\mu(t) \\
&\leq \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_t^{t+1} \left(L_f^b(s) \|\beta^b(s)\|_{\mathbb{Y}} \right)^m ds \right)^{1/m} d\mu(t) \\
&\leq \|L_f^b\|_{S^r} \Big[\frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_t^{t+1} \|\beta^b(s)\|_{\mathbb{Y}}^p ds \right)^{1/p} d\mu(t) \Big] \\
&\leq \|L_f^b\|_{S^r} \Big[\frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_t^{t+1} \|\beta^b(s)\|_{\mathbb{Y}}^q ds \right)^{1/q} d\mu(t) \Big].
\end{aligned}$$

Using the fact that $\beta^b \in \mathcal{E}\big(L^q((0,1),\mathbb{Y})\big)$, it follows that $\Phi^b \in \mathcal{E}(\mathbb{R} \times L^m((0,1),\mathbb{X}))$.

On the other hand, since $f, g \in Lip^r(\mathbb{R}, \mathbb{X}) \subset Lip^p(\mathbb{R}, \mathbb{X})$, one has

$$\begin{aligned}
&\Big(\int_0^1 \|h(t+s, u(s)) - h(t+s, v(s))\|^m ds \Big)^{1/m} \\
&\leq \Big(\int_0^1 \|f(t+s, u(s)) - f(t+s, v(s))\|^m ds \Big)^{1/m} \\
&\quad + \Big(\int_0^1 \|g(t+s, u(s)) - g(t+s, v(s))\|^m ds \Big)^{1/m} \\
&\leq \Big(\int_0^1 \Big(L_f(t+s) \|u(s) - v(s)\|_{\mathbb{Y}} \Big)^m ds \Big)^{1/m} \\
&\quad + \Big(\int_0^1 \Big(L_g(t+s) \|u(s) - v(s)\|_{\mathbb{Y}} \Big)^m ds \Big)^{1/m} \\
&\leq \Big(\|L_f\|_{S^r} + \|L_g\|_{S^r} \Big) \|u(s) - v(s)\|_p.
\end{aligned}$$

Since $K := \overline{\{\alpha(t) : t \in \mathbb{R}\}}$ is compact in $\mathbb{Y}$, then for each $\varepsilon > 0$, there exists a finite number of open balls $B_k = B(x_k, \varepsilon)$, centered at $x_k \in K$ with radius ε such that

$$\{\alpha(t) : t \in \mathbb{R}\} \subset \cup_{k=1}^m B_k.$$

Therefore, for $1 \leq k \leq m$, the set $U_k = \{t \in \mathbb{R} : \alpha \in B_k\}$ is open and $\mathbb{R} = \cup_{k=1}^m U_k$. Now, for $2 \leq k \leq m$, set $V_k = U_k - \cup_{i=1}^{k-1} U_i$ and $V_1 = U_1$. Clearly, $V_i \cap V_j = \emptyset$ for all $i \neq j$. Define the step function $\overline{x} : \mathbb{R} \to \mathbb{Y}$ by $\overline{x}(t) = x_k, t \in V_k, k = 1, 2, \ldots, m$. It easy to see that

$$\|\alpha(s) - \overline{x}(s)\|_{\mathbb{Y}} \leq \varepsilon, \quad \text{for all } s \in \mathbb{R}.$$

which yields

$$\begin{aligned}
&\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_t^{t+1}\|h(s,\alpha(s))\|^m ds\right)^{1/m} d\mu(t)\\
&\leq \frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_t^{t+1}\|h(s,\alpha(s))-h(s,\overline{x}(s))\|^m ds\right)^{1/m} d\mu(t)\\
&\quad+\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\int_t^{t+1}\|h(s,\overline{x}(s))\|^m ds\right)^{1/m} d\mu(t)\\
&\leq \Big(\|L_f\|_{S^r}+\|L_g\|_{S^r}\Big)\varepsilon+\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\sum_{k=1}^m\int_{V_k\cap[t,t+1]}\|h(s,\overline{x}(s))\|^m ds\right)^{1/m} d\mu(t)\\
&\leq \Big(\|L_f\|_{S^r}+\|L_g\|_{S^r}\Big)\varepsilon+\frac{1}{\mu(Q_r)}\int_{Q_r}\left(\sum_{k=1}^m\int_{V_k\cap[t,t+1]}\|h(s,\overline{x}(s))\|^q ds\right)^{1/q} d\mu(t).
\end{aligned}$$

Since ε is arbitrary and $h^b \in \mathcal{E}\left(\mathbb{R}\times L^q((0,1),\mathbb{X})\right)$, it follows that the function $h^b(\cdot,\alpha^b(\cdot))$ belongs to $\mathcal{E}\left(\mathbb{R}\times L^m((0,1),\mathbb{X})\right)$. This completes the proof. □

Remark 4.28. A general composition theorem in $S^{p,q(x)}_{paa}(\mathbb{R}\times\mathbb{X})$ is unlikely as compositions of elements of $S^{p,q(x)}_{paa}(\mathbb{R}\times\mathbb{X},\mu)$ may not be well-defined unless $q(\cdot)$ is the constant function.

5. Application to Abstract Evolution Equations

Fix $\mu\in\mathcal{N}_2$, $p,q>1$, and $\vartheta\in C_+(\mathbb{R})$. To study the existence of a weighted pseudo-almost automorphic solution to Eq. (1.1) with weighted $S^{p,q}_{paa}$ coefficients we will assume that the following assumptions hold:

(H1) The family of closed linear operators $A(t)$ for $t\in\mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the Acquistapace and Terreni conditions, the evolution family of operators $\mathcal{U}=\{U(t,s)\}_{t\geq s}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N,\delta>0$ and dichotomy projections $P(t)$ $(t\in\mathbb{R})$. Moreover, $0\in\rho(A)$ for each $t\in\mathbb{R}$ and the following hold

$$\sup_{t,s\in\mathbb{R}}\|A(s)A^{-1}(t)\|_{B(\mathbb{X},\mathbb{X}_\beta)}<c_1 \tag{5.1}$$

(H2) There exists $0\leq\alpha<\beta<1$ such that $\mathbb{X}^t_\alpha=\mathbb{X}_\alpha$ and $\mathbb{X}^t_\beta=\mathbb{X}_\beta$ for all $t\in\mathbb{R}$, with uniform equivalent norms. Let $c_2(\alpha),c_3,c_4$ be the bounds of the continuous injections $\mathbb{X}_\beta\hookrightarrow\mathbb{X}_\alpha,\mathbb{X}_\alpha\hookrightarrow\mathbb{X},\mathbb{X}_\beta\hookrightarrow\mathbb{X}$.

(H3) The function $\mathbb{R}\times\mathbb{R}\to\mathbb{X}$, $(t,s)\to A(s)\Gamma(t,s)y\in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y\in\mathbb{X}_\beta$.

(H4) The function $\mathbb{R}\times\mathbb{R}\to\mathbb{X}$, $(t,s)\to\Gamma(t,s)y\in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y\in\mathbb{X}$.

(H5) The linear operators $B(t),C(t):\mathbb{X}_\alpha\to\mathbb{X}$ are bounded uniformly in $t\in\mathbb{R}$. Moreover, both $t\mapsto B(t)$ and $t\mapsto C(t)$ belong to $AA(B(\mathbb{X}_\alpha,\mathbb{X}))$. We then set

$$c_5:=\max\big(\sup_{t\in\mathbb{R}}\|B\|_{B(\mathbb{X}_\alpha,\mathbb{X})},\sup_{t\in\mathbb{R}}\|C\|_{B(\mathbb{X}_\alpha,\mathbb{X})}\big)$$

(H6) The function $f = h + \varphi \in S^{p,q}_{paa}(\mathbb{X}, \mathbb{X}_\beta, \mu)$ while $g = h' + \varphi' \in S^{p,q}_{paa}(\mathbb{X}, \mathbb{X}, \mu)$. Moreover; $f, h \in Lip^r(\mathbb{R}, \mathbb{X}_\beta)$ and $g, h' \in Lip^r(\mathbb{R}, \mathbb{X})$. with

$$r \geq \max\left\{p, \frac{p}{p-1}\right\}.$$

Definition 5.1. A continuous function $u : \mathbb{R} \to \mathbb{X}_\alpha$ is said to be a mild solution to (1.1) provided that the functions $s \to A(s)U(t,s)P(s)f(s, B(s)u(s))$ and $s \to A(s)U(t,s)Q(s)f(s, B(s)u(s))$ are integrable on (t,s) and

$$\begin{aligned}
&u(t) = -f(t, B(t)u(t)) + U(t,s)(u(s) + f(s, B(s)u(s)))\\
&- \int_s^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds + \int_t^s A(s)U(t,s)Q(s)f(s, B(s)u(s))ds\\
&+ \int_s^t U(t,s)P(s)g(s, C(s)u(s))ds - \int_t^s U(t,s)Q(s)g(s, C(s)u(s))ds
\end{aligned}$$

for $t \geq s$ and for all $t, s \in \mathbb{R}$.

Under previous assumptions (H1)–(H6), it can be easily shown that (1.1) has a unique mild solution given by

$$\begin{aligned}
&u(t) = -f(t, B(t)u(t)) - \int_{-\infty}^t A(s)U(t,s)P(s)f(s, B(s)u(s))ds\\
&+ \int_t^\infty A(s)U_Q(t,s)Q(s)f(s, B(s)u(s))ds + \int_{-\infty}^t U(t,s)P(s)g(s, C(s)u(s))ds\\
&- \int_t^\infty U_Q(t,s)Q(s)g(s, C(s)u(s))ds
\end{aligned}$$

for each $t \in \mathbb{R}$.

The proof of our main result requires the next technical lemmas:

Lemma 5.2. Under assumption (H5), if $u \in PAA(\mathbb{X}_\alpha, \mu)$, then $B(\cdot)u(\cdot)$ and $C(\cdot)u(\cdot)$ belong to $PAA(\mathbb{X}, \mu)$.

Proof. We will make use of ideas of [**8**, Lemma 3.2]. Let $u = h + \varphi \in PAA(\mathbb{X}_\alpha, \mu)$ where $h \in AA(\mathbb{X}_\alpha)$ and $\varphi \in \mathcal{E}\left(\mathbb{X}_\alpha, \mu\right)$, then $B(\cdot)u(\cdot) = B(\cdot)h(\cdot) + B(\cdot)\varphi(\cdot)$. First, it is easy to see that $B(\cdot)u(\cdot) \in BC(\mathbb{R}, \mathbb{X}_\alpha)$. Since $h \in AA(\mathbb{X}_\alpha)$, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a measurable function g_1 such that

$$\lim_{n\to\infty} \|h(s_n + s) - g_1(s)\|_\alpha = 0,$$

and

$$\lim_{n\to\infty} \|g_1(s - s_n) - h(s)\|_\alpha = 0$$

for each $t \in \mathbb{R}$.
Since $B(\cdot) \in AA(B(\mathbb{X}_\alpha, \mathbb{X}))$, there exists a subsequence $(s_{n_k})_{k\in\mathbb{N}}$ of $(s_n)_{n\in\mathbb{N}}$ and a measurable function g_2 such that

$$\|B(s_{n_k} + s) - g_2(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})} \to 0,$$

and

$$\|g_2(s - s_{n_k}) - B(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})} \to 0$$

as $k \to \infty$ for each $t \in \mathbb{R}$.

By using the triangle inequality, one has

$$\|B(s_{n_k}+s)h(s_{n_k}+s)-g_2(s)g_1(s)\| \le \|B(s_{n_k}+s)h(s_{n_k}+s)-B(s_{n_k}+s)g_1(s)\|$$
$$+\|B(s_{n_k}+s)g_1(s)-g_2(s)g_1(s)\|$$
$$\le c_5\|h(s_{n_k}+s)-g_1(s)\|_{X_\alpha}+\|g_1\|_\infty\|B(s_{n_k}+s)-g_2(s)\|_{B(\mathbb{X}_\alpha,\mathbb{X})}.$$

Then,

$$\lim_{n\to\infty}\|B(s_{n_k}+s)h(s_{n_k}+s)-g_2(s)g_1(s)\|=0,$$

Analogously, one can prove that

$$\lim_{n\to\infty}\|g_2(s-s_{n_k})g_1(s-s_{n_k})-B(s)h(s)\|=0.$$

Hence, $B(\cdot)h(\cdot)\in AA(\mathbb{X})$.

To complete the proof, it suffices to notice that for r sufficiently large

$$\frac{1}{\mu(Q_r)}\int_{Q_r}\|B(s)\varphi(s)\|\,d\mu(s)\le\frac{c_5}{\mu(Q_r)}\int_{Q_r}\|\varphi(s)\|_{X_\alpha}\,d\mu(s)$$

and hence,

$$\lim_{r\to\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|B(s)\varphi(s)\|\,d\mu(s)=0.$$

□

Lemma 5.3 ([**11**])**.** For each $x\in\mathbb{X}$, suppose that Assumptions (H1)–(H2) hold and let α,β be real numbers such that $0<\alpha<\beta<1$ with $2\beta>\alpha+1$, then there are constants $r(\alpha,\beta),r'(\alpha,\beta),d(\beta)>0$ such that

$$\|A(t)U(t,s)P(s)x\|_\beta\le r'(\alpha,\beta)e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|,\quad t>s \tag{5.2}$$

$$\|A(s)U(t,s)P(s)x\|_\beta\le r(\alpha,\beta)e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|,\quad t>s \tag{5.3}$$

and

$$\|A(s)\widetilde{U}_Q(s,t)Q(t)x\|_\beta\le d(\beta)e^{-\delta(s-t)}\|x\|,\quad t\le s \tag{5.4}$$

Lemma 5.4. Under assumptions (H1)–(H6), the integral operators Γ_1 and Γ_2 defined by

$$(\Gamma_1u)(t):=\int_{-\infty}^t A(s)U(t,s)P(s)f(s,B(s)u(s))ds$$

and

$$(\Gamma_2u)(t):=\int_t^{\infty} A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds$$

map $PAA(\mathbb{X}_\alpha,\mu)$ into itself.

Proof. Let $u\in PAA(\mathbb{X}_\alpha,\mu)$. By Lemma (5.2) one has $B(\cdot)u(\cdot)\in PAA(\mathbb{X},\mu)\subset S^{p,q}_{paa}(\mathbb{X},\mu)$. Using the composition theorem for weighted $S^{p,q}_{paa}$ functions, we deduce that $F(t):=f(t,B(t)u(t))\in S^{p,q}_{paa}(\mathbb{X}_\beta,\mu)$. Now write $F=\phi+\psi$, where $\phi^b\in AA\big(L^p((0,1),\mathbb{X}_\beta)\big)$ and $\psi^b\in\mathcal{E}\left(L^q((0,1),\mathbb{X}_\beta),\mu\right)$. Then Γ_1 can be decomposed as

$$(\Gamma_1u)(t)=\Phi(t)+\Psi(t)$$

where

$$\Phi(t)=\int_{-\infty}^t A(s)U(t,s)P(s)\phi(s)ds\quad\text{and}\quad\Psi(t)=\int_{-\infty}^t A(s)U(t,s)P(s)\psi(s)ds,$$

Clearly $\Phi \in AA(\mathbb{X}_\alpha)$. Indeed; for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$; we set

$$\Phi_k(t) := \int_{k-1}^{k} A(t-s)U(t,t-s)P(t-s)\phi(t-s)ds = \int_{t-k}^{t-k+1} A(s)U(t,s)P(s)\phi(s)ds,$$

Let $d > 1$ such that $\frac{1}{p} + \frac{1}{d} = 1$, where $p > 1$. Using Eq. (5.3) and the Hölder's inequality, it follows that

$$\begin{aligned}
\|\Phi_k(t)\|_\alpha \le c_2(\alpha)\|\Phi_k(t)\|_\beta &\le c_2(\alpha)r(\alpha,\beta)\int_{t-k}^{t-k+1} e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|\phi(s)\|_\beta ds \\
&\le c_2(\alpha)r(\alpha,\beta)\Big[\int_{t-k}^{t-k+1} e^{\frac{-d\delta}{4}(t-s)}(t-s)^{-d\beta}ds\Big]^{1/d} \\
&\quad \times \Big[\int_{t-k}^{t-k+1} \|\phi(s)\|_\beta^p ds\Big]^{1/p} \\
&\le c_2(\alpha)r(\alpha,\beta)\Big[\int_{k-1}^{k} e^{\frac{-d\delta}{4}s}s^{-d\beta}ds\Big]^{1/d}\|\phi\|_{S^p(\mathbb{X}_\beta)} \\
&\le c_2(\alpha)r(\alpha,\beta)\sqrt[d]{\frac{1+e^{\frac{d\delta}{4}}}{\frac{d\delta}{4}}}(k-1)^{-\beta}e^{\frac{-\delta}{4}k}\|\phi\|_{S^p(\mathbb{X}_\beta)} \\
&:= C_d(\alpha,\beta,\delta)\|\phi\|_{S^p(\mathbb{X}_\beta)}.
\end{aligned}$$

Since the series $\sum_{k=1}^{\infty}\Big((k-1)^{-\beta}e^{\frac{-\delta}{4}k}\Big)$ is convergent, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}\Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\phi(s)ds = \sum_{k=1}^{\infty}\Phi_k(t),$$

$\Phi \in C(\mathbb{R},\mathbb{X}_\alpha)$ and

$$\|\Phi(t)\|_\alpha \le \sum_{k=1}^{\infty}\|\Phi_k(t)\|_\alpha \le \sum_{k=1}^{\infty} C_d(\alpha,\beta,\delta)\|\phi\|_{S^p(\mathbb{X}_\beta)}.$$

Fix $k \in \mathbb{N}$, let us take a sequence $(s'_n)_n$ of real numbers. Since $\phi^b \in AA\big(L^p((0,1),\mathbb{X}_\beta)\big)$ and $A(s)U(t,s)P(s)y \in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y \in X_\beta$, then for every sequence $(s'_n)_n$ there exists a subsequence $(s_n)_n$ and functions θ, h such that

$$\lim_{n\to\infty} A(s+s_n)U(t+s_n,s+s_n)P(s+s_n)x = \theta(t,s)x \quad \text{for each} \quad t,s \in \mathbb{R}, x \in \mathbb{X}_\beta. \tag{5.5}$$

$$\lim_{n\to\infty}\theta(t-s_n,s-s_n)x = A(s)U(t,s)P(s)x \quad \text{for each} \quad t,s \in \mathbb{R}, x \in \mathbb{X}_\beta. \tag{5.6}$$

$$\lim_{n\to\infty}\|\phi(t+s_n+\cdot)-h(t+\cdot)\|_{S^p(\mathbb{X}_\beta)} = 0, \quad \text{for each} \quad t \in \mathbb{R}. \tag{5.7}$$

$$\lim_{n\to\infty}\|h(t-s_n+\cdot)-\phi(t+\cdot)\|_{S^p(\mathbb{X}_\beta)} = 0 \quad \text{for each} \quad t \in \mathbb{R}. \tag{5.8}$$

We set

$$G_k(t) := \int_{k-1}^{k}\theta(t,t-s)h(t-s)ds.$$

Using triangle inequality, we obtain that

$$\|\Phi_k(t+s_n) - G_k(t)\|_\alpha \le a_n^k(t) + b_n^k(t),$$

where

$$a_n^k(t) := \int_{k-1}^{k} \big\|A(t+s_n-s)U(t+s_n,t+s_n-s)P(t+s_n-s)\left(\phi(t+s_n-s) - h(t-s)\right)\big\|_\alpha ds,$$

and

$$b_n^k(t) := \int_{k-1}^{k} \big\|\left[A(t+s_n-s)U(t+s_n,t+s_n-s)P(t+s_n-s) - \theta(t,t-s)\right]h(t-s)\big\|_\alpha ds$$

Using Eq. (5.3) and the Hölder's inequality it follows that

$$a_n^k(t) \le C_d(\alpha,\beta,\delta)\|\phi(t+s_n-s) - h(t-s)\|_{S^p(\mathbb{X}_\beta)}.$$

Then, by (5.7), $\lim_{n\to\infty} a_n^k(t) = 0$. Again, using the Lebesgue dominated convergence theorem and (5.5), one can get $\lim_{n\to\infty} b_n^k(t) = 0$. Thus,

$$\lim_{n\to\infty} \Phi_k(t+s_n) = \int_{k-1}^{k} \theta(t,t-\sigma)h(t-\sigma)d\sigma, \quad \text{for each} \ \ t\in\mathbb{R}.$$

Analogously, one can prove that

$$\lim_{n\to\infty} \int_{k-1}^{k} \theta(t-s_n,t-s_n-s)h(t-s_n-s)ds = \Phi_k(t), \quad \text{for each} \ \ t\in\mathbb{R}.$$

Therefore, $\Phi_k \in AA(\mathbb{X}_\alpha)$. Applying Proposition (2.2), we deduce that the uniform limit

$$\Phi(\cdot) = \sum_{k=1}^{\infty} \Phi_k(\cdot) \in AA(\mathbb{X}_\alpha).$$

Now, we prove that $\Psi \in \mathcal{E}\left(\mathbb{X}_\alpha,\mu\right)$. For this; for each $t\in\mathbb{R}$ and $k\in\mathbb{N}$; we set

$$\Psi_k(t) := \int_{k-1}^{k} A(t-s)U(t,t-s)P(t-s)\psi(t-s)ds = \int_{t-k}^{t-k+1} A(s)U(t,s)P(s)\psi(s)ds.$$

By carrying similar arguments as above, we deduce that $\Psi_k(t) \in BC(\mathbb{R},\mathbb{X}_\alpha)$, $\sum_{k=1}^{\infty}\Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \in BC(\mathbb{R},\mathbb{X}_\alpha).$$

To complete the proof, it remains to show that

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|\Psi(t)\|_\alpha \, d\mu(t) = 0.$$

In fact, the estimate in Eq. (5.3) yields

$$\begin{aligned}\|\Psi_k(t)\|_\alpha &\le c_2(\alpha)r(\alpha,\beta)\left(\int_{t-k}^{t-k+1} e^{\frac{-\delta}{4}(t-s)}(t-s)^{-\beta}\|\psi(s)\|_\beta ds\right)\\ &\le c_2(\alpha)r(\alpha,\beta)\sqrt[d']{\frac{1+e^{\frac{d'\delta}{4}}}{\frac{d'\delta}{4}}}(k-1)^{-\beta}e^{\frac{-\delta}{4}k}\left(\int_{t-k}^{t-k+1}\|\psi(s)\|_\beta^q ds\right)^{1/q}\\ &= C_{d'}(\alpha,\beta,\delta)\left(\int_{t-k}^{t-k+1}\|\psi(s)\|_\beta^q ds\right)^{1/q},\end{aligned}$$

where $d' > 1$ such that $\frac{1}{q}+\frac{1}{d'}=1$. Then, one has

$$\begin{aligned}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi_k(t)\|_\alpha\, d\mu(t) &\le \frac{C_{d'}(\alpha,\beta,\delta)}{\mu(Q_r)}\int_{Q_r}\left(\int_{t-k}^{t-k+1}\|\psi(s)\|_\beta^q ds\right)^{1/q} d\mu(t)\\ &\le \frac{C_{d'}(\alpha,\beta,\delta)}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1\|\psi(s+t-k)\|_\beta^q ds\right)^{\frac{1}{q}} d\mu(t).\end{aligned}$$

Since $\psi^b \in \mathcal{E}\left(L^q((0,1),\mathbb{X}_\beta),\mu\right)$, the above inequality leads to $\Psi_k \in \mathcal{E}\left(\mathbb{X}_\alpha,\mu\right)$. Then by the following inequality

$$\begin{aligned}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi(t)\|_\alpha\, d\mu(t) \le \frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi(t) - &\sum_{k=1}^{\infty}\Psi_k(t)\|_\alpha\, d\mu(t)\\ &+\sum_{k=1}^{\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi_k(t)\|_\alpha\, d\mu(t),\end{aligned}$$

we deduce that the uniform limit $\Psi(\cdot)=\sum_{k=1}^{\infty}\Psi_k(\cdot)\in\mathcal{E}\left(\mathbb{X}_\alpha,\mu\right)$, which ends the proof.

Of course, the proof for $(\Gamma_2 u)(\cdot)$ is similar to that for $(\Gamma_1 u)(\cdot)$. However, one makes use of Eq. (5.4) rather than Eq. (5.3). □

Lemma 5.5. Under assumptions (H1)–(H6), the integral operators Γ_3 and Γ_4 defined by

$$(\Gamma_3 u)(t) := \int_{-\infty}^{t} U(t,s)P(s)g(s,C(s)u(s))ds$$

and

$$(\Gamma_4 u)(t) := \int_{t}^{\infty} U_Q(t,s)Q(s)g(s,C(s)u(s))ds$$

map $PAA(\mathbb{X}_\alpha,\mu)$ into itself.

Proof. Let $u \in PAA(\mathbb{X}_\alpha,\mu)$, since $C(\cdot)\in AA(B(\mathbb{X}_\alpha,\mathbb{X}))$; by Lemma (5.2); it follows that $C(\cdot)u(\cdot)\in PAA(\mathbb{X},\mu)\subset S^{p,q}_{paa}(\mathbb{X},\mu)$ Using the composition theorem for weighted $S^{p,q}_{paa}$ functions (Theorem (4.27)), we deduce that $G(t):=g(t,C(t)u(t))\in S^{p,q}_{paa}(\mathbb{X},\mu)$. Now write $G=\phi+\psi$, where $\phi^b\in AA\big(L^p((0,1),\mathbb{X})\big)$ and $\psi^b\in\mathcal{E}\left(L^q((0,1),\mathbb{X}),\mu\right)$. Thus Γ_3 can be rewritten as

$$(\Gamma_3 u)(t) = \Phi(t)+\Psi(t),$$

where

$$\Phi(t) = \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds \quad \text{and} \quad \Psi(t) = \int_{-\infty}^{t} U(t,s)P(s)\psi(s)ds,$$

Now we will show that $\Phi \in AA(\mathbb{X}_\alpha)$. For each $t \in \mathbb{R}$ and $k \in \mathbb{N}$ we set

$$\Phi_k(t) := \int_{k-1}^{k} U(t,t-s)P(t-s)\phi(t-s)ds = \int_{t-k}^{t-k+1} U(t,s)P(s)\phi(s)ds.$$

Let $d > 1$ such that $\frac{1}{p} + \frac{1}{d} = 1$, where $p > 1$. Using Eq. (3.7) and the Hölder's inequality, it follows that

$$\begin{aligned}
\|\Phi_k(t)\|_\alpha &\leq \int_{t-k}^{t-k+1} \|U(t,s)P(s)\phi(s)\|_\alpha ds \\
&\leq n(\alpha) \int_{t-k}^{t-k+1} e^{\frac{-\delta}{2}(t-s)}(t-s)^{-\alpha}\|\phi(s)\|ds \\
&\leq n(\alpha)\Big[\int_{t-k}^{t-k+1} e^{\frac{-d\delta}{2}(t-s)}(t-s)^{-d\alpha}ds\Big]^{1/d} \times \Big[\int_{t-k}^{t-k+1} \|\phi(s)\|^p ds\Big]^{1/p} \\
&\leq n(\alpha)\Big[\int_{k-1}^{k} e^{\frac{-d\delta}{2}s}s^{-d\alpha}ds\Big]^{1/d} \|\phi\|_{S^p(\mathbb{X})} \\
&\leq n(\alpha)\sqrt[d]{\frac{1+e^{\frac{d\delta}{2}}}{\frac{d\delta}{2}}}(k-1)^{-\alpha}e^{\frac{-\delta}{2}k}\|\phi\|_{S^p(\mathbb{X})} \\
&:= C_d(\alpha,\delta)\|\phi\|_{S^p(\mathbb{X})}.
\end{aligned}$$

Since the series $\displaystyle\sum_{k=1}^{\infty}\Big((k-1)^{-\alpha}e^{\frac{-\delta}{2}k}\Big)$ is convergent, we deduce from the well-known Weierstrass test that the series $\sum_{k=1}^{\infty}\Phi_k(t)$ is uniformly convergent on $\mathbb{R}$. Furthermore

$$\Phi(t) = \int_{-\infty}^{t} U(t,s)P(s)\phi(s)ds = \sum_{k=1}^{\infty}\Phi_k(t),$$

$\Phi \in C(\mathbb{R},\mathbb{X}_\alpha)$ and

$$\|\Phi(t)\|_\alpha \leq \sum_{k=1}^{\infty}\|\Phi_k(t)\| \leq \sum_{k=1}^{\infty} C_d(\alpha,\delta)\|\phi\|_{S^p(\mathbb{X})}.$$

Fix $k \in \mathbb{N}$, let us take a sequence $(s'_n)_n$ of real numbers. Since $\phi^b \in AA\big(L^p((0,1),\mathbb{X})\big)$ and $U(t,s)y \in bAA(\mathbb{T},\mathbb{X}_\alpha)$ uniformly for $y \in \mathbb{X}$, then for every sequence $(s'_n)_n$ there exists a subsequence $(s_n)_n$ and functions θ, h such that

$$\lim_{n\to\infty} U(t+s_n, s+s_n)P(s+s_n)x = \theta(t,s)x \quad \text{for each} \quad t,s \in \mathbb{R}, x \in \mathbb{X}. \tag{5.9}$$

$$\lim_{n\to\infty} \theta(t-s_n, s-s_n)x = U(t,s)P(s)x \quad \text{for each} \quad t,s \in \mathbb{R}, x \in \mathbb{X}. \tag{5.10}$$

$$\lim_{n\to\infty} \|\phi(t+s_n+\cdot) - h(t+\cdot)\|_{S^p(\mathbb{X})} = 0, \quad \text{for each} \quad t \in \mathbb{R}. \tag{5.11}$$

$$\lim_{n\to\infty} \|h(t-s_n+\cdot) - \phi(t+\cdot)\|_{S^p(\mathbb{X})} = 0 \quad \text{for each} \quad t \in \mathbb{R}. \tag{5.12}$$

We set

$$H_k(t) := \int_{k-1}^{k} \theta(t, t-s)h(t-s)ds.$$

Using triangle inequality, Eq. (3.7) and the Hölder's inequality, we obtain that

$$\|\Phi_k(t+s_n) - H_k(t)\|_\alpha \leq c_n^k(t) + d_n^k(t),$$

where

$$\begin{aligned} c_n^k(t) &:= \Big\| \int_{k-1}^{k} U(t+s_n, t+s_n-s)P(t+s_n-s)\left(\phi(t+s_n-s) - h(t-s)\right) ds \Big\|_\alpha \\ &\leq n(\alpha) \left(\int_{k-1}^{k} e^{\frac{-\delta}{2}s} s^{-\alpha} \|\phi(t+s_n-s) - h(t-s)\| ds \right) \\ &\leq C_d(\alpha,\delta) \|\phi(t+s_n-s) - h(t-s)\|_{S^p(\mathbb{X})}, \end{aligned}$$

and

$$\begin{aligned} d_n^k(t) &:= \Big\| \int_{k-1}^{k} \left[U(t+s_n, t+s_n-s)P(t+s_n-s) - \theta(t, t-s)\right] h(t-s) ds \Big\|_\alpha \\ &\leq \int_{k-1}^{k} \|U(t+s_n, t+s_n-s)P(t+s_n-s) - \theta(t,t-s)h(t-s)\|_\alpha ds. \end{aligned}$$

By (5.11), $\lim_{n\to\infty} c_n^k(t) = 0$ and by using the Lebesgue dominated convergence theorem and (5.9), one can get $\lim_{n\to\infty} c_n^k(t) = 0$. Thus,

$$\lim_{n\to\infty} \Phi_k(t+s_n) = \int_{k-1}^{k} \theta(t, t-\sigma)h(t-\sigma)d\sigma, \quad \text{for each } \ t \in \mathbb{R}.$$

Analogously, one can prove that

$$\lim_{n\to\infty} \int_{k-1}^{k} \theta(t-s_n, t-s_n-s)h(t-s_n-s)ds = \Phi_k(t), \quad \text{for each } \ t \in \mathbb{R}.$$

Therefore, $\Phi_k \in AA(\mathbb{X}_\alpha)$. Applying Proposition (2.2), we deduce that the uniform limit

$$\Phi(\cdot) = \sum_{k=1}^{\infty} \Phi_k(\cdot) \in AA(\mathbb{X}_\alpha).$$

Now, we prove that $\Psi \in PAP_0(\mathbb{X}_\alpha)$. For this; for each $t \in \mathbb{R}$ and $k \in \mathbb{N}$; we set

$$\Psi_k(t) := \int_{k-1}^{k} U(t, t-s)P(t-s)\psi(t-s)ds = \int_{t-k}^{t-k+1} U(t,s)P(s)\psi(s)ds.$$

By carrying similar arguments as above, we deduce that $\Psi_k(t) \in BC(\mathbb{R}, \mathbb{X}_\alpha)$, $\sum_{k=1}^{\infty} \Psi_k(t)$ is uniformly convergent on $\mathbb{R}$ and

$$\Psi(t) = \sum_{k=1}^{\infty} \Psi_k(t) = \int_{-\infty}^{t} A(s)U(t,s)P(s)\psi(s)ds \in BC(\mathbb{R}, \mathbb{X}_\alpha).$$

To complete the proof, it remains to show that

$$\lim_{r\to\infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|\Psi(t)\|_\alpha \, d\mu(t) = 0.$$

In fact, the estimate in Eq. (3.7) yields

$$\begin{aligned}\|\Psi_k(t)\|_\alpha &\leq n(\alpha)\left(\int_{t-k}^{t-k+1} e^{\frac{-\delta}{2}(t-s)}(t-s)^{-\alpha}\|\psi(s)\|ds\right)\\ &\leq n(\alpha)\sqrt[d']{\frac{1+e^{\frac{d'\delta}{2}}}{\frac{d'\delta}{2}}}(k-1)^{-\alpha}e^{\frac{-\delta}{2}k}\left(\int_{t-k}^{t-k+1}\|\psi(s)\|^q ds\right)^{1/q}\\ &= C_{d'}(\alpha,\delta)\|\psi\|_{S^q(\mathbb{X})},\end{aligned}$$

where $d' > 1$ such that $\frac{1}{q}+\frac{1}{d'}=1$. Then, one has

$$\begin{aligned}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi_k(t)\|_\alpha\, d\mu(t) &\leq \frac{C_{d'}(\alpha,\delta)}{\mu(Q_r)}\int_{Q_r}\left(\int_{t-k}^{t-k+1}\|\psi(s)\|^q ds\right)^{\frac{1}{q}} d\mu(t)\\ &\leq \frac{C_{d'}(\alpha,\delta)}{\mu(Q_r)}\int_{Q_r}\left(\int_0^1\|\psi(s+t-k)\|^q ds\right)^{\frac{1}{q}} d\mu(t).\end{aligned}$$

Since $\psi^b \in \mathcal{E}\left(L^q((0,1),\mathbb{X}),\mu\right)$, the above inequality leads to $\Psi_k \in \mathcal{E}\left(\mathbb{X}_\alpha,\mu\right)$. Then, by the following inequality

$$\begin{aligned}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi(t)\|_\alpha\, d\mu(t) \leq \frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi(t)-&\sum_{k=1}^{\infty}\Psi_k(t)\|_\alpha\, d\mu(t)\\ &+\sum_{k=1}^{\infty}\frac{1}{\mu(Q_r)}\int_{Q_r}\|\Psi_k(t)\|_\alpha\, d\mu(t),\end{aligned}$$

we deduce that the uniform limit $\Psi(\cdot)=\sum_{k=1}^{\infty}\Psi_k(\cdot)\in\mathcal{E}\left(\mathbb{X}_\alpha,\mu\right)$, which ends the proof.

Of course, the proof for $(\Gamma_4 u)(\cdot)$ is similar to that for $(\Gamma_3 u)(\cdot)$. However, one makes use of Eq. (3.8) rather than Eq. (3.7). □

Theorem 5.6. Under the assumptions (H1)–(H6), the evolution equation (1.1) has a unique μ-pseudo-almost automorphic mild solution whenever $L = \max\left(\|L_f\|_{S^r};\|L_g\|_{S^r}\right)$ is small enough.

Proof. Consider the nonlinear operator Π defined on $PAA(\mathbb{X}_\alpha,\mu)$ by

$$\begin{aligned}&\Pi u(t) = -f(t,B(t)u(t)) - \int_{-\infty}^{t} A(s)U(t,s)P(s)f(s,B(s)u(s))ds\\ &+\int_t^{\infty} A(s)U_Q(t,s)Q(s)f(s,B(s)u(s))ds + \int_{-\infty}^{t} U(t,s)P(s)g(s,C(s)u(s))ds\\ &-\int_t^{\infty} U_Q(t,s)Q(s)g(s,C(s)u(s))ds\end{aligned}$$

for each $t\in\mathbb{R}$. As we have previously seen, for every $u\in PAA(\mathbb{X}_\alpha,\mu)$, $f(\cdot,Bu(\cdot))\in PAA(\mathbb{X}_\beta,\mu)\subset PAA(\mathbb{X}_\alpha,\mu)$. In view of Lemmas (5.4) and (5.5), it follows that Π maps $PAA(\mathbb{X}_\alpha,\mu)$ into its self. To complete the proof one has to show that Π has a unique fixed point.

Let $u, v \in PAA(\mathbb{X}_\alpha, \mu)$. For Γ_1 and Γ_2, we have the following approximations:

$$\begin{aligned}
\|(\Gamma_1 u)(t) - (\Gamma_1 v)(t)\|_\alpha &\le \int_{-\infty}^{t} \|A(s)U(t,s)P(s)\left[f(s,B(s)u(s)) - f(s,B(s)v(s))\right]\|_\alpha ds \\
&\le c_2(\alpha)c_4 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}\|f(s,B(s)u(s)) - f(s,B(s)v(s))\|_\beta ds \\
&\le c_2(\alpha)c_4 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}L_f(s)\|B(s)u(s) - B(s)v(s)\|ds \\
&\le c_2(\alpha)c_4c_5 r(\alpha,\beta)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}L_f(s)\|u(s) - v(s)\|_\alpha ds \\
&\le c_2(\alpha)c_4c_5 r(\alpha,\beta)\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1}(t-s)^{-\alpha}e^{\frac{-\delta}{4}(t-s)}L_f(s)\|u - v\|_{\alpha,\infty}ds \\
&\le c_2(\alpha)c_4c_5 r(\alpha,\beta)\sum_{n=1}^{\infty}\Big(\int_{t-n}^{t-n+1}(t-s)^{-\alpha r_0}e^{\frac{-r_0\delta}{4}(t-s)}\,ds\Big)^{\frac{1}{r_0}}\|L_f\|_{S^r}\|u - v\|_{\alpha,\infty} \\
&\le c_2(\alpha)c_4c_5 r(\alpha,\beta)\sum_{n=1}^{\infty}(n-1)^{-\alpha}\Big(\int_{t-n}^{t-n+1}e^{\frac{-r_0\delta}{4}(t-s)}\,ds\Big)^{\frac{1}{r_0}}\|L_f\|_{S^r}\|u - v\|_{\alpha,\infty} \\
&\le c_2(\alpha)c_4c_5 r(\alpha,\beta)\sqrt[r_0]{\frac{4\big(1+e^{\frac{r_0\delta}{4}}\big)}{r_0\delta}}\sum_{n=1}^{\infty}(n-1)^{-\alpha}e^{\frac{-n\delta}{4}}\|L_f\|_{S^r}\|u - v\|_{\alpha,\infty} \\
&= c_2(\alpha)c_4c_5 r(\alpha,\beta)S(r_0,\frac{\delta}{4})\|L_f\|_{S^r}\|u - v\|_{\alpha,\infty},
\end{aligned}$$

where r_0 is such that $\frac{1}{r} + \frac{1}{r_0} = 1$ and $S(r_0,\delta) = \sqrt[r_0]{\frac{1+e^{r_0\delta}}{r_0\delta}}\sum_{n=1}^{\infty}(n-1)^{-\alpha}e^{-n\delta}$.

$$\begin{aligned}
\|(\Gamma_2 u)(t) - (\Gamma_2 v)(t)\|_\alpha &\le \int_{t}^{\infty}\|A(s)U_Q(t,s)Q(s)\left[f(s,B(s)u(s)) - f(s,B(s)v(s))\right]\|_\alpha ds \\
&\le c_2(\alpha)c_4 d(\beta)\int_{t}^{\infty}e^{-\delta(s-t)}\|f(s,B(s)u(s)) - f(s,B(s)v(s))\|_\beta ds \\
&\le Lc_2(\alpha)c_4 d(\beta)\int_{t}^{\infty}e^{-\delta(s-t)}\|B(s)u(s) - B(s)v(s)\|ds \\
&\le Lc_2(\alpha)c_4c_5 d(\beta)\int_{t}^{\infty}e^{-\delta(s-t)}\|u(s) - v(s)\|_\alpha ds \\
&\le Lc_2(\alpha)c_4c_5 d(\beta)\delta^{-1}\|u - v\|_{\alpha,\infty}.
\end{aligned}$$

Similarly, For Γ_3 and Γ_4, we have the following approximations

$$\begin{aligned}
\|(\Gamma_3 u)(t) - (\Gamma_3 v)(t)\|_\alpha &\leq \int_{-\infty}^{t} \|U(t,s)P(s)\left[g(s,C(s)u(s)) - f(s,C(s)v(s))\right]\|_\alpha ds \\
&\leq n(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}\|g(s,C(s)u(s)) - g(s,C(s)v(s))\|_\beta ds \\
&\leq n(\alpha)\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}L_g\|C(s)u(s) - C(s)v(s)\|ds \\
&\leq n(\alpha)c_5\int_{-\infty}^{t}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}L_g\|u(s)-v(s)\|_\alpha ds \\
&\leq n(\alpha)c_5\sum_{n=1}^{\infty}\int_{t-n}^{t-n+1}(t-s)^{-\alpha}e^{\frac{-\delta}{2}(t-s)}L_g(s)\|u-v\|_{\alpha,\infty}ds \\
&\leq n(\alpha)c_5\sum_{n=1}^{\infty}\Big(\int_{t-n}^{t-n+1}(t-s)^{-\alpha r_0}e^{\frac{-r_0\delta}{2}(t-s)}\,ds\Big)^{\frac{1}{r_0}}\|L_g\|_{S^r}\|u-v\|_{\alpha,\infty} \\
&\leq n(\alpha)c_5\sum_{n=1}^{\infty}(n-1)^{-\alpha}\Big(\int_{t-n}^{t-n+1}e^{\frac{-r_0\delta}{2}(t-s)}\,ds\Big)^{\frac{1}{r_0}}\|L_g\|_{S^r}\|u-v\|_{\alpha,\infty} \\
&\leq n(\alpha)c_5\sqrt[r_0]{\frac{2\big(1+e^{\frac{r_0\delta}{2}}\big)}{r_0\delta}}\sum_{n=1}^{\infty}(n-1)^{-\alpha}e^{\frac{-n\delta}{2}}\|L_g\|_{S^r}\|u-v\|_{\alpha,\infty} \\
&= n(\alpha)c_5 S(r_0,\frac{\delta}{2})\|L_g\|_{S^r}\|u-v\|_{\alpha,\infty},
\end{aligned}$$

and

$$\begin{aligned}
\|(\Gamma_4 u)(t) - (\Gamma_4 v)(t)\|_\alpha &\leq \int_{t}^{\infty}\|U_Q(t,s)Q(s)\left[g(s,C(s)u(s)) - g(s,C(s)v(s))\right]\|_\alpha ds \\
&\leq m(\alpha)\int_{t}^{\infty}e^{-\delta(s-t)}\|g(s,C(s)u(s)) - g(s,C(s)v(s))\|ds \\
&\leq Lm(\alpha)\int_{t}^{\infty}e^{-\delta(s-t)}\|C(s)u(s) - C(s)v(s)\|ds \\
&\leq Lm(\alpha)c_5\int_{t}^{\infty}e^{-\delta(s-t)}\|u(s)-v(s)\|_\alpha ds \\
&\leq Lm(\alpha)c_5\delta^{-1}\|u-v\|_{\alpha,\infty}.
\end{aligned}$$

Consequently,

$$\|\Pi u - \Pi v\|_{\alpha,\infty} \leq L\Theta\|u-v\|_{\alpha,\infty},$$

where

$$\Theta := c_5\bigg(c_2(\alpha)c_4 r(\alpha,\beta)S\big(r_0,\frac{\delta}{4}\big) + c_2(\alpha)c_4 d(\beta)\delta^{-1} + n(\alpha)S\big(r_0,\frac{\delta}{2}\big) + m(\alpha)\delta^{-1}\bigg).$$

By taking L small enough, that is, $L < \Theta^{-1}$, the operator Π becomes a contraction on $PAA(\mathbb{X}_\alpha,\mu)$ and hence has a unique fixed point in $PAA(\mathbb{X}_\alpha,\mu)$, which obviously is the unique μ-pseudo-almost automorphic mild solution to (1.1). □

References

[1] P. Acquistapace, *Evolution operators and strong solutions of abstract linear parabolic equations,* Differential Integral Equations, **1** (4) (1988), 433-457.

[2] P. Acquistapace and B. Terreni, *A unified approach to abstract linear nonautonomous parabolic equations,* Rend. Sem. Mat. Univ. Padova, **78** (1987), 47-107.

[3] H. Amann, *Linear and Quasilinear Parabolic Problems*, Birkhäuser, Berlin 1995.

[4] J. Blot, P. Cieutat, and K. Ezzinbi, *New approach for weighted pseudo almost periodic functions under the light of measure theory, basic results and applications*, Applicable Analysis: An International Journal, **92** (3) (2013), 493-526.

[5] J. Blot, P. Cieutat, and K. Ezzinbi, *Measure theory and pseudo almost automorphic functions: New developments and applications,* Nonlinear Anal. (TMA), **75** (2012), 2426-2447.

[6] S. Bochner, *A new approach to almost periodicity*, *Proc. Nat. Acad. Sci. U.S.A.*, **48** (1962), 2039-2043.

[7] M. Baroun, S. Boulite, T. Diagana, and L. Maniar, *Almost periodic solutions to some semilinear non-autonomous thermoelastic plate equations*, *J. Math. Anal. Appl.*, **349** (1) (2009), 74-84.

[8] C. Bensouda, and M. Zitane, *Generalized Stepanov-like pseudo almost automorphic solutions to some classes of nonautonomous partial evolution equations*, *J. Math. Comput. Sci.*, **3** (1) (2013), 278-303.

[9] K. J. Engel and R. Nagel; *One Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics, Springer Verlag, 1999.

[10] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces,* Springer, New York, 2013.

[11] T. Diagana, *Pseudo-almost periodic solutions for some classes of nonautonomous partial evolution equations,* J. of The Franklin Institute, **348** (2011), 2082-2098.

[12] T. Diagana and M. Zitane, *Stepanov-like pseudo-almost periodic functions in Lebesgue space with variable exponents* $L^{p(x)}$, New Frontiers of Multidisciplinary Research in STEAM-H, **90** (2014), 295-314. In *Springer Proceedings in Mathematics and Statistics*, editor B. Toni.

[13] T. Diagana and M. Zitane, *Stepanov-like pseudo-almost automorphic functions in Lebesgue spaces with variable exponents* $L^{p(x)}$, Electron. J. Diff. Equ., **2013** (188) (2013), 1-20.

[14] T. Diagana and M. Zitane, *Weighted Stepanov-like pseudo-almost periodic funcitons in Lebesgue spaces with variable exponents* $L^{p(x)}$*,* Afr. Diaspora J. Math. **15** (2) (2013), 56–75.

[15] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents,* Lecture Notes in Mathematics, Springer, Heidelberg, 2011.

[16] W. Long, and H. S. Ding, *Composition theorems of Stepanov almost periodic functions and Stepanov-like pseudo-almost periodic functions,* Advance in Difference Equations, Article ID 654695 (2011), 12 pages.

[17] X.L. Fan and D. Zhao, *On the spaces* $L^{p(x)}(O)$ *and* $W^{m,p(x)}(O)$, *J. Math. Anal. Appl.*, **263** (2001), 424-446.

[18] Z. Hu and Z. Jin, *Stepanov-like pseudo-almost automorphic mild solutions to nonautonomous evolution equations,* Nonlinear Anal., **71** (2009), 2349-2360.

[19] J. Liang, J. Zhang, and T.J. Xiao, *Composition of pseudo-almost automorphic and asymptotically almost automorphic functions*, J. Math. Anal. Appl., **340** (2008), 1493-1499.

[20] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications, 16. Birkhäuser Verlag, Basel, 1995.

[21] G.M. N'Guérékata, *Topics in Almost Automorphy.* Springer-Verlag, New York, 2005.

[22] P.Q.H. Nguyen, *On Variable Lebesgue Spaces.* Ph.D. Thesis, Kansas State University. ProQuest LLC, Ann Arbor, MI, 2011.

[23] T. J. Xiao, J. Liang, and J. Zhang, *Pseudo-almost automorphic solutions to semilinear differential equations in Banach space,* Semigroup Forum, **76** (2008), 518-524.

Mohamed ZITANE
Moulay Ismaïl University,
Faculty of Sciences,
Department of Mathematics,
Research team: Algebra,
Operator Theory and
Applications (AOTA),
Meknès, Morocco.
zitanem@gmail.com