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# AN ALTERNATIVE DEFINITION OF THE HENSTOCK-KURZWEIL INTEGRAL USING PRIMITIVES

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Abstract. We introduce the notion of an $H$-primitive being the limit of a sequence of absolutely continuous functions satisfying certain conditions and use it to formulate an alternative definition of the Henstock-Kurzweil integral on a closed bounded interval. Furthermore, the definition provides a characterisation of the primitive of a Henstock-Kurzweil integrable function.

## 1. Introduction

The Henstock-Kurzweil integral has several alternative definitions. In particular, it has a Riesz-type definition. More precisely, a function $f$ defined on a closed bounded interval $[a, b]$ is Henstock-Kurzweil integrable there if and only if there exists a control-convergent sequence $\{\phi_n\}$ of step functions such that $\phi_n(x) \to f(x)$ for almost all $x$ in $[a, b]$ as $n \to \infty$ (see [**1**, Definition 10.1]). Note that in this definition we begin with a sequence of elementary functions, namely the step functions $\phi_n$, converging to the function $f$ which will be Henstock-Kurzweil integrable if the sequence satisfies certain condition, in this case control convergence. Note that step functions are Lebesgue integrable and primitives of Lebesgue integrable functions are absolutely continuous.

In this short paper, we shall show that we can also define the Henstock-Kurzweil integral if we begin with a sequence of primitives, of Lebesgue integrable functions, converging to a function which will be the primitive of a Henstock-Kurzweil integrable function if the sequence has certain properties which will be made precise in due course.

## 2. Preliminaries

Throughout this paper we shall let a closed bounded interval $[a, b]$ be fixed and consider real-valued point functions $f$ defined on $[a, b]$.

A set $\{(I_i, x_i) : i = 1, 2, \ldots, n\}$ or simply $\{(I_i, x_i)\}_{i=1}^n$ of interval-point pairs is called a *partial division* of $[a, b]$ if $I_1, I_2, \ldots, I_n$ are non-overlapping closed subintervals of $[a, b]$ such that $\bigcup_{i=1}^n I_i \subseteq [a, b]$ and $x_i \in I_i$ for each $i$. We call $x_i$ the *associated point* of $I_i$ and the collection of the intervals $I_i$ a *partial partition* of $[a, b]$. A partial division $D^*$ of $[a, b]$ *refines* or is a *refinement* of another partial division $D$ of $[a, b]$ if for each $(I, x) \in D^*$, we have $I \subseteq J$ for some $(J, y) \in D$. If $P^*$ denotes the collection of intervals $I$ in $D^*$ and $P$ denotes the collection of intervals

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$J$ in $D$, then $P^*$ and $P$ are partial divisions, and we also say that $P^*$ is a refinement of $P$.

A *division* of $[a,b]$ is a partial division $\{(I_i,x_i)\}_{i=1}^n$ such that the union of $I_i$ is $[a,b]$. Note that a division $D=\{(I_i,x_i)\}_{i=1}^n$ of $[a,b]$ is essentially a selection of points in $[a,b]$ of the form

$$a=u_1\le x_1\le v_1=u_2\le x_2\le v_2\le\cdots\le u_n\le x_n\le v_n=b$$

where $I_i=[u_i,v_i]$ for $i=1,2,...,n$. We call the collection of the intervals $I_i$ of a division $\{(I_i,x_i)\}_{i=1}^n$ of $[a,b]$ a *partition* of $[a,b]$.

Let $\delta:[a,b]\to(0,\infty)$ be a positive function. We call $\delta$ a *gauge* on $[a,b]$. An interval-point pair $(I,x)$ is $\delta$*-fine* if $I\subseteq(x-\delta(x),x+\delta(x))$. A partial division $\{(I_i,x_i)\}_{i=1}^n$ of $[a,b]$ is $\delta$*-fine* if $(I_i,x_i)$ is $\delta$-fine for each $i=1,2,\ldots,n$. Since divisions are themselves partial divisions, $\delta$-fine divisions of $[a,b]$ are similarly defined. A gauge $\delta_1$ is said to be *finer* than a gauge $\delta_2$ on $[a,b]$ if for every $x\in[a,b]$ we have $\delta_1(x)\le\delta_2(x)$.

We shall next define the Henstock-Kurzweil integral. For brevity and where there is no ambiguity, $D=\{(I,x)\}$ shall denote a finite collection of interval-point pairs $(I,x)$ and have its corresponding Riemann sum denoted by $(D)\sum f(x)\,|I|$ where $|I|$ denotes the length of the interval $I$.

**Definition 2.1.** A function $f$ is said to be Henstock-Kurzweil integrable, or briefly $HK$-integrable, on $[a,b]$ to a real number $A$ if for every $\varepsilon>0$, there exists a gauge $\delta$ on $[a,b]$ such that for every $\delta$-fine division $D=\{(I,x)\}$ of $[a,b]$, we have

$$\left|(D)\sum f(x)\,|I|-A\right|<\varepsilon.$$

We write $(H)\int_a^b f(x)dx=A$. The $HK$-integrability of $f$ on any subinterval of $[a,b]$ is similarly defined.

If a real-valued function $F$ is defined on $[a,b]$, for each $[u,v]\subseteq[a,b]$, we will write $F(u,v)=F(v)-F(u)$. If $P=\{[u_i,v_i]\}_{i=1}^n$ is a partial partition of $[a,b]$, for brevity we will sometimes write $(P)\sum|F(u,v)|$ for $\sum_{i=1}^n|F(v_i)-F(u_i)|$ where $[u,v]$ represents the intervals $[u_i,v_i]$.

We will provide an alternative definition of the $HK$-integral using primitives in the next section. To this end, we need the following definitions.

**Definition 2.2.** A function $F$ is said to be absolutely continuous, or briefly $AC$, on $[a,b]$ if for every $\varepsilon>0$ there exists $\eta>0$ such that for any partial partition $P=\{[u,v]\}$ of $[a,b]$ satisfying the condition $(P)\sum(v-u)<\eta$, we have

$$(P)\sum|F(u,v)|<\varepsilon.$$

**Definition 2.3.** Let $X\subseteq[a,b]$. A function $F$ defined on $[a,b]$ is said to be $AC^*(X)$ if for every $\varepsilon>0$ there exists $\eta>0$ such that for every partial partition $P=\{[u,v]\}$ of $[a,b]$ with end points $u$ or $v$ belonging to $Y$ satisfying the condition that $(P)\sum(v-u)<\eta$, we have

$$(P)\sum|F(u,v)|<\varepsilon.$$

**Definition 2.4.** If $[a,b]$ is the union of a sequence $\{X_n\}$ of closed sets such that $F$ is $AC^*(X_n)$ for each $n=1,2,...$, then $F$ is said to be $ACG^*$ on $[a,b]$.

## 3. Main Results

We shall first prove the following convergence theorem which serves as a motivation for the alternative definition of the $HK$-integral that we will formulate in this paper. In what follows, a property is said to hold almost everywhere in $[a, b]$, or equivalently for almost all $x$ in $[a, b]$, if it holds everywhere in $[a, b]$ except perhaps in a set of measure zero.

**Theorem 3.1.** *Let $f_n$ be $HK$-integrable on $[a, b]$ with primitives $F_n$ for $n = 1, 2, \ldots$. Suppose that $f_n(x) \to f(x)$ for almost all $x$ in $[a, b]$ and $F_n(x) \to F(x)$ as $n \to \infty$ for all $x \in [a, b]$. Then the function $f$ is $HK$-integrable on $[a, b]$ with primitive $F$ if and only if the following condition is satisfied.*

(+) *For every $\varepsilon > 0$ there is $M(x)$ taking positive integer values such that for infinitely many positive integers $m(x) \geq M(x)$ there is $\delta(x) > 0$ and for any $\delta$-fine division $D = \{[u, v], \xi\}$ of $[a, b]$ we have*

$$(D) \sum \left| F_{m(\xi)}(u, v) - F(u, v) \right| < \varepsilon.$$

**Proof.** We may assume that $f_n(x) \to f(x)$ everywhere as $n \to \infty$. Suppose $f$ is $HK$-integrable on $[a, b]$ with primitive $F$. Given $\varepsilon > 0$ and $x \in [a, b]$, there is a positive integer $M(x)$ such that whenever $m(x) \geq M(x)$, we have

$$|f_{m(x)}(x) - f(x)| < \varepsilon.$$

Since each $f_n$ is $HK$-integrable on $[a, b]$, there is a gauge $\delta_n$ on $[a, b]$ such that for any $\delta_n$-fine division $D = \{(I, x)\}$ of $[a, b]$, we have

$$(D) \sum |F_n(I) - f_n(x)\,|I|\,| < \varepsilon(2^{-n}).$$

Also, there is a gauge $\delta_0$ on $[a, b]$ such that for any $\delta_0$-fine division $D = \{(I, x)\}$ of $[a, b]$, we have

$$(D) \sum |F(I) - f(x)\,|I|\,| < \varepsilon.$$

Now for every $m(x) \geq M(x)$, we define a gauge $\delta$ on $[a, b]$ given by

$$\delta(x) = \min(\delta_{m(x)}(x), \delta_0(x))$$

for all $x \in [a, b]$. Then for any $\delta$-fine division $D = \{(I, x)\}$ of $[a, b]$, we have

$$\begin{aligned}
(D) \sum |F_{m(x)}(I) - F(I)| &\leq (D) \sum |F_{m(x)}(I) - f_{m(x)}(x)\,|I|\,| \\
&\quad + (D) \sum |f_{m(x)}(x) - f(x)|\,|I| \\
&\quad + (D) \sum |f(x)\,|I| - F(I)| \\
&< \sum_{n=1}^{\infty} \varepsilon\left(2^{-n}\right) + \varepsilon(b - a) + \varepsilon \\
&= 2\varepsilon + \varepsilon(b - a).
\end{aligned}$$

Hence we have proved the result for every positive integer $m(x) \geq M(x)$. Conversely, suppose the hypotheses are satisfied. Using the same notations as above, for each $x \in [a, b]$ we choose a positive integer $m(x) \geq M(x)$ such that

$$|f_{m(x)}(x) - f(x)| < \varepsilon.$$

Then we modify the gauge $\delta$ so that $\delta(x) \le \delta_{m(x)}(x)$ for each $x \in [a,b]$. Thus for any $\delta$-fine division $D = \{(I,x)\}$ of $[a,b]$, we obtain

$$\begin{aligned}(D)\sum |f(x)\,|I| - F(I)| &\le (D)\sum \left|f(x) - f_{m(x)}(x)\right| |I| \\ &\quad +(D)\sum \left|f_{m(x)}(x)\,|I| - F_{m(x)}(I)\right| \\ &\quad +(D)\sum \left|F_{m(x)}(I) - F(I)\right| \\ &< \varepsilon(b-a) + \sum_{n=1}^{\infty} \varepsilon\left(2^{-n}\right) + \varepsilon \\ &= 2\varepsilon + \varepsilon(b-a).\end{aligned}$$

Hence $f$ is $HK$-integrable with primitive $F$ on $[a,b]$. □

In Theorem 3.1, it is immaterial whether we require the condition $f_n(x) \to f(x)$ as $n \to \infty$ to hold for all, or almost all, $x$ in $[a,b]$ because in the latter case we can always assume $f_n(x) = f(x) = 0$ for $x$ in the set of measure zero where the condition does not hold, and write $f_n(x) \to f(x)$ as $n \to \infty$ for all $x$ in $[a,b]$. It changes nothing as far as the primitives are concerned because if $f(x) = g(x)$ for almost all $x$ in $[a,b]$, then $f$ is $HK$-integrable on $[a,b]$ if and only if $g$ is $HK$-integrable there and the integral values are equal [**2**, Proposition 1.1]. Also note that in the theorem, it is necessary that condition (+) holds for all $m(x) \ge M(x)$. However, it is sufficient to have only infinitely many $m(x) \ge M(x)$ in order that $f$ is $HK$-integrable with primitive $F$.

We shall call condition (+) in Theorem 3.1 the *H-condition* on $[a,b]$ and say that $\{F_n\}$ satisfies the $H$-condition on $[a,b]$.

**Definition 3.2.** A continuous function $F$ is called an $H$-primitive on $[a,b]$ if there is a sequence $\{F_n\}$ of functions which are $AC$ on $[a,b]$ such that $F_n(x) \to F(x)$ as $n \to \infty$ for all $x \in [a,b]$ and $\{F_n\}$ satisfies the $H$-condition on $[a,b]$. We shall call $\{F_n\}$ an $H$-sequence of $F$ on $[a,b]$.

Obviously the set of all $H$-primitives on $[a,b]$ is closed under addition and scalar multiplication.

It is well known that functions which are $AC$ on $[a,b]$ are primitives of Lebesgue integrable functions there. Thus an $H$-sequence is necessarily a sequence of primitives of Lebesgue integrable functions.

The following result shows that the $H$-primitives include all primitives of $HK$-integrable functions.

**Theorem 3.3.** *If a function $F$ is the primitive of an $HK$-integrable function on $[a,b]$, then it is an $H$-primitive on $[a,b]$.*

**Proof.** Since $F$ is the primitive of an $HK$-integrable function on $[a,b]$, it is continuous and $ACG^*$ on $[a,b]$ (see for example, [**1**, Theorem 6.13]). Hence there is a sequence $\{X_n\}$ of closed sets with union $[a,b]$ such that $F$ is $AC^*(X_n)$ for each $n$. We may assume that for each $n$ we have $X_n \subseteq X_{n+1}$ and $a,b \in X_n$. Now for each $n = 1, 2, ...$, let $(a,b) \setminus X_n = \bigcup_{i=1}^{\infty}(a_i^{(n)}, b_i^{(n)})$, where $b_i^{(n)} \le a_{i+1}^{(n)}$ for each $i$, and define $F_n(x) = F(x)$ when $x \in X_n$, in particular $F_n(a) = F(a)$ and $F_n(b) = F(b)$, and

$$F_n(x) = F(a_i^{(n)}) + \frac{F(b_i^{(n)}) - F(a_i^{(n)})}{b_i^{(n)} - a_i^{(n)}}(x - a_i^{(n)})$$

when $x \in (a_i^{(n)}, b_i^{(n)})$. It is easy to see that $F_n(x) \to F(x)$ as $n \to \infty$ for all $x \in [a,b]$. We shall prove that each $F_n$ is $AC$ on $[a,b]$. To this end we first let $\varepsilon > 0$ be given and let $n$ be fixed. Since $F$ is $AC^*(X_n)$, there exists $\eta_1 > 0$ such that for every partial partition $P = \{[u,v]\}$ of $[a,b]$ with end points $u$ or $v$ belonging to $X_n$ satisfying the condition that $(P)\sum(v-u) < \eta_1$, we have

$$(P)\sum|F(u,v)| < \frac{\varepsilon}{3}. \tag{3.1}$$

Choose a positive integer $N$ such that $\sum_{i=N+1}^{\infty}(b_i^{(n)} - a_i^{(n)}) < \eta_1$. Since $a_i^{(n)}, b_i^{(n)} \in X_n$ by (3.1),

$$\sum_{i=N+1}^{\infty}\left|F(a_i^{(n)}, b_i^{(n)})\right| < \frac{\varepsilon}{3}. \tag{3.2}$$

Since on each $[a_i^{(n)}, b_i^{(n)}]$ the function $F_n$ is defined linearly, and is thus $AC$ there, we choose $\eta_2 > 0$ such that for every partial partition $P = \{[u,v]\}$ of $\bigcup_{i=1}^{N}[a_i^{(n)}, b_i^{(n)}]$ satisfying the condition that $(P)\sum(v-u) < \eta_2$, we have

$$(P)\sum|F_n(u,v)| < \frac{\varepsilon}{3}. \tag{3.3}$$

Take any partial partition $P = \{[u,v]\}$ of $[a,b]$ such that $(P)\sum(v-u) < \eta$ where $\eta := \min(\eta_1, \eta_2)$. Let $Y_n = \bigcup_{i=1}^{N}(a_i^{(n)}, b_i^{(n)})$ and $Z_n = \bigcup_{i=N+1}^{\infty}(a_i^{(n)}, b_i^{(n)})$. Consider the following cases. (1) $u, v \in X_n$; (2) $u, v \in Y_n$; (3) $u, v \in Z_n$; (4a) $u \in X_n, v \in Y_n$; (4b) $u \in Y_n, v \in X_n$; (5a) $u \in X_n, v \in Z_n$; (5b) $u \in Z_n, v \in X_n$; (6) $u \in Y_n, v \in Z_n$. Note that it is not possible to have $u \in Z_n, v \in Y_n$. In (4a), $v \in (a_q^{(n)}, b_q^{(n)})$ for some positive integer $q \le N$. We write $[u,v] = [u, a_q^{(n)}] \cup [a_q^{(n)}, v]$ and note that $F(u,v) = F(u, a_q^{(n)}) + F(a_q^{(n)}, v)$. In (4b), $u \in (a_p^{(n)}, b_p^{(n)})$ for some positive integer $p \le N$. We write $[u,v] = [u, b_p^{(n)}] \cup [b_p^{(n)}, v]$ and note that $F(u,v) = F(u, b_p^{(n)}) + F(b_p^{(n)}, v)$. Likewise, in (5a), $v \in (a_s^{(n)}, b_s^{(n)})$ for some positive integer $s > N$ and we write $[u,v] = [u, a_s^{(n)}] \cup [a_s^{(n)}, v]$, while in (5b), $u \in (a_r^{(n)}, b_r^{(n)})$ for some positive integer $r > N$ and we write $[u,v] = [u, b_r^{(n)}] \cup [b_r^{(n)}, v]$. In (6), $u \in (a_p^{(n)}, b_p^{(n)})$ for some positive integer $p \le N$ and $v \in (a_s^{(n)}, b_s^{(n)})$ for some positive integer $s > N$. We write $[u,v] = [u, b_p^{(n)}] \cup [b_p^{(n)}, a_s^{(n)}] \cup [a_s^{(n)}, v]$ and note that

$$F_n(u,v) = F_n(u, b_p^{(n)}) + F_n(b_p^{(n)}, a_s^{(n)}) + F_n(a_s^{(n)}, v).$$

Let $P_1$ be the collection of all intervals $[u,v]$ from (1), all intervals $[u, a_q^{(n)}]$ from (4a), all intervals $[b_p^{(n)}, v]$ from (4b), all intervals $[u, a_s^{(n)}]$ from (5a), all intervals $[b_r^{(n)}, v]$ from (5b), and all intervals $[b_p^{(n)}, a_s^{(n)}]$ from (6). Let $P_2$ be the collection of all intervals $[u,v]$ from (2), all intervals $[a_q^{(n)}, v]$ from (4a), all intervals $[u, b_p^{(n)}]$ from (4b), and all intervals $[u, b_p^{(n)}]$ from (6). Let $P_3$ be the collection of all intervals

$[u,v]$ from (3), all intervals $[a_s^{(n)},v]$ from (5a), all intervals $[u,b_r^{(n)}]$ from (5b), and all intervals $[a_s^{(n)},v]$ from (6). Clearly $P_1 \cup P_2 \cup P_3$ is a refinement of $P$ and

$$(P_i)\sum|I| \leq (P_1 \cup P_2 \cup P_3)\sum|I| = (P)\sum|I| < \eta$$

for $i = 1,2,3$ where $I = [u,v]$ denotes a typical interval in the partitions $P_i$ and $P$. Since for every $[u,v] \in P_1$ we have $u,v \in X_n$, applying (3.1) yields

$$\begin{aligned}(P_1)\sum|F_n(u,v)| &= (P_1)\sum|F(u,v)| \\ &< \frac{\varepsilon}{3}.\end{aligned}$$

On the other hand, since $P_2$ is a partial partition of $Y_n$ and $(P_2)\sum|I| < \eta \leq \eta_2$, by (3.3) we obtain

$$(P_2)\sum|F_n(u,v)| < \frac{\varepsilon}{3}.$$

Next, if $[u,v] \in P_3$ where $u,v \in Z_n$ then $u \in (a_p^{(n)},b_p^{(n)})$ and $v \in (a_q^{(n)},b_q^{(n)})$ for some positive integers $p$ and $q$ where $p \leq q$. If $p = q$, then

$$\begin{aligned}|F_n(u,v)| &= \left|\frac{F(b_p^{(n)}) - F(a_p^{(n)})}{b_p^{(n)} - a_p^{(n)}}(v-u)\right| \\ &\leq \left|F(a_p^{(n)},b_p^{(n)})\right|.\end{aligned}$$

Otherwise,

$$\begin{aligned}|F_n(u,v)| &= \left|\frac{F(b_q^{(n)}) - F(a_q^{(n)})}{b_q^{(n)} - a_q^{(n)}}(v-a_q^{(n)}) - \frac{F(b_p^{(n)}) - F(a_p^{(n)})}{b_p^{(n)} - a_p^{(n)}}(u-a_p^{(n)})\right| \\ &\leq \left|F(a_q^{(n)},b_q^{(n)})\right| + \left|F(a_p^{(n)},b_p^{(n)})\right|.\end{aligned}$$

For intervals in $P_3$ of the form $[a_s^{(n)},v]$, we have

$$\begin{aligned}\left|F_n(a_s^{(n)},v)\right| &= \left|\frac{F(b_s^{(n)}) - F(a_s^{(n)})}{b_s^{(n)} - a_s^{(n)}}(v-a_s^{(n)})\right| \\ &\leq \left|F(a_s^{(n)},b_s^{(n)})\right|.\end{aligned}$$

Similarly, for intervals in $P_3$ of the form $[u,b_r^{(n)}]$, we have

$$\begin{aligned}\left|F_n(u,b_r^{(n)})\right| &= \left|\frac{F(b_r^{(n)}) - F(a_r^{(n)})}{b_r^{(n)} - a_r^{(n)}}(b_r^{(n)}-u)\right| \\ &\leq \left|F(a_r^{(n)},b_r^{(n)})\right|.\end{aligned}$$

Consequently, by virtue of (3.2),

$$\begin{aligned}(P_3)\sum|F_n(u,v)| &\leq \sum_{i=N+1}^{\infty}\left|F(a_i^{(n)},b_i^{(n)})\right| \\ &< \frac{\varepsilon}{3}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}(P)\sum|F_n(u,v)| &\le (P_1)\sum|F_n(u,v)| + (P_2)\sum|F_n(u,v)| + (P_3)\sum|F_n(u,v)| \\ &< \varepsilon.\end{aligned}$$

We have thus proved that $F_n$ is $AC$ on $[a,b]$ for each $n$. Note that the derivative $F_n'$ of $F_n$ exists almost everywhere in $[a,b]$. More specifically, $F_n'(x) = F'(x)$ for almost all $x$ in $X_n$, and

$$F_n'(x) = \frac{F(b_i^{(n)}) - F(a_i^{(n)})}{b_i^{(n)} - a_i^{(n)}}$$

when $x \in (a_i^{(n)}, b_i^{(n)})$. Let $f$ be the $HK$-integrable function on $[a,b]$ of which $F$ is the primitive, and for each $n$ let $f_n = F_n'$. Evidently, since $F' = f$ almost everywhere in $[a,b]$, $f_n$ converges pointwise to $f$ almost everywhere in $[a,b]$. By Theorem 3.1, $\{F_n\}$ is an $H$-sequence of $F$ on $[a,b]$. We have therefore proved that $F$ is an $H$-primitive on $[a,b]$ as desired. □

The next result shows the conditions for an $H$-primitive to be a primitive of an $HK$-integrable function. Note that a function which is $AC$ on $[a,b]$ is differentiable almost everywhere in $[a,b]$.

**Theorem 3.4.** *Let $F$ be an $H$-primitive on $[a,b]$ and $\{F_n\}$ an $H$-sequence of $F$ on $[a,b]$ such that $F_n'(x) \to f(x)$ as $n \to \infty$ for almost all $x$ in $[a,b]$. Then $f$ is $HK$-integrable on $[a,b]$ and $F$ is the primitive of $f$.*

**Proof.** Since $\{F_n\}$ is an $H$-sequence of $F$ on $[a,b]$, each $F_n$ is $AC$ on $[a,b]$. Hence the derivative $F_n'$ of $F_n$ exists almost everywhere in $[a,b]$ and is Lebesgue integrable, and thus $HK$-integrable on $[a,b]$. Furthermore, since $F_n(x) \to F(x)$ as $n \to \infty$ for all $x$ in $[a,b]$ and $F_n'(x) \to f(x)$ as $n \to \infty$ for almost all $x$ in $[a,b]$ where $\{F_n\}$ satisfies condition (+), by Theorem 3.1, the function $f$ is $HK$-integrable on $[a,b]$ with primitive $F$. □

With Theorems 3.3 and 3.4, we have thus obtained the following theorem which provides an alternative definition of the $HK$-integral.

**Theorem 3.5.** *A function $f$ is $HK$-integrable on $[a,b]$ with primitive $F$ if and only if $F$ is an $H$-primitive on $[a,b]$ and has an $H$-sequence $\{F_n\}$ on $[a,b]$ such that $F_n'(x) \to f(x)$ as $n \to \infty$ for almost all $x$ in $[a,b]$.*

In fact Theorem 3.5 provides a characterisation of the primitive of an $HK$-integrable function.

It is easy to verify the uniqueness of the $HK$-integral using this alternative definition. If $F$ and $G$ are $H$-primitives which have respectively $H$-sequences $\{F_n\}$ and $\{G_n\}$ on $[a,b]$ such that $F_n'(x) \to f(x)$ and $G_n'(x) \to f(x)$ as $n \to \infty$ for almost all $x$ in $[a,b]$, then for every subinterval $[c,d] \subseteq [a,b]$ and for every $\varepsilon > 0$ there is $M(x)$ taking positive integer values such that for infinitely many positive integers $m(x) \ge M(x)$ there is $\delta(x) > 0$ and for any $\delta$-fine division $D = \{[u,v],\xi\}$ of $[c,d]$

we have

$$
\begin{aligned}
|F(c,d) - G(c,d)| \le\ & (D)\sum \left|F_{m(\xi)}(u,v) - F(u,v)\right| \\
& + (D)\sum \left|F'_{m(\xi)}(\xi)(v-u) - F_{m(\xi)}(u,v)\right| \\
& + (D)\sum \left|F'_{m(\xi)}(\xi) - f(\xi)\right|(v-u) \\
& + (D)\sum \left|G'_{m(\xi)}(\xi) - f(\xi)\right|(v-u) \\
& + (D)\sum \left|G'_{m(\xi)}(\xi)(v-u) - G_{m(\xi)}(u,v)\right| \\
& + (D)\sum \left|G_{m(\xi)}(u,v) - G(u,v)\right| \\
& < \varepsilon.
\end{aligned}
$$

Hence $F(c,d) = G(c,d)$ for every subinterval $[c,d] \subseteq [a,b]$.

Using this alternative definition of the $HK$-integral many basic properties of the $HK$-integral can be verified easily. For instance, we can prove readily that if $f$ is $HK$-integrable on $[a,b]$ then it is $HK$-integrable on every subinterval $[c,d]$ of $[a,b]$ and that the Denjoy space, which is the space of all $HK$-integrable functions, is a vector space.

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