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GROUP ACTIONS ON PRODUCT SYSTEMS

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Abstract. We introduce the concept of a crossed product of a product system by a locally compact group. We prove that the crossed product of a row-finite and faithful product system by an amenable group is also a row-finite and faithful product system. We generalize a theorem of Hao and Ng about the crossed product of the Cuntz-Pimsner algebra of a C^*-correspondence by a group action to the context of product systems. We present examples related to group actions on k-graphs and to higher rank Doplicher-Roberts algebras.

1. Introduction

Product systems over various discrete semigroups P were introduced by N. Fowler in [**7**], inspired by work of W. Arveson and studied by several authors (see [**1, 3, 16**], for example). Several interesting examples of product systems already occur over the semigroup $\left(\mathbb{N}^k, +\right)$, where $k \geq 2$, for example product systems associated to k-graphs. A lot of interest was shown for the particular case when the semigroup P embeds in a group Q and the pair (Q, P) is a quasi-lattice ordered group in the sense of Nica.

We first recall the definition of the Toeplitz algebra and of the Cuntz–Pimsner algebra of a product system. We use the covariance condition in Fowler's sense. Next, we introduce the concept of an action of a (locally compact and Hausdorff) group on a product system and then define the associated crossed product product system. We prove that the crossed product of a row-finite and faithful product system by an amenable group is also row-finite and faithful, and, furthermore, we establish a version of the Hao–Ng Theorem (see Theorem 2.10 in [**9**]) for product systems over $\mathbb{N}^k$.

Motivations for introducing group actions on product systems come from at least two sources: (i) group actions on higher rank graphs; (ii) the higher rank Doplicher–Roberts algebra defined from k representations of a compact group. We feel that the concept of crossed product of a product system could be used for other purposes, for example to study group actions on topological k-graphs.

2. C^*-Algebras of Product Systems

Let us first recall the definition of a product system. Let $(P, \cdot)$ be a discrete monoid with identity e, and let A be a C^*-algebra. A P-indexed *product system* of

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C^*-correspondences over A is a semigroup $\mathsf{Y} = \bigsqcup_{p \in P} \mathsf{Y}_p$ (which can be viewed as a surjective map $\mathsf{Y} \to P$) with the following properties:

- For each $p \in P$, the object Y_p is a C^*-correspondence over A, which we call the *fiber* of Y at p. Its inner product is denoted by $\langle \cdot | \cdot \rangle_{\mathsf{Y}_p}$.
- The fiber Y_e of Y at e is ${}_A A_A$, which is A viewed as an A-correspondence over itself.
- For each $p, q \in P$, the semigroup multiplication on Y maps $\mathsf{Y}_p \times \mathsf{Y}_q$ to Y_{pq}, so we have an A-balanced (compatible with the A-module structure) $\mathbb{C}$-bilinear map

$$\mathsf{M}_{p,q} \stackrel{\text{df}}{=} \begin{Bmatrix} \mathsf{Y}_p \times \mathsf{Y}_q & \to & \mathsf{Y}_{pq} \\ (x, y) & \mapsto & x \cdot y \end{Bmatrix}.$$

- For each $p, q \in P \setminus \{e\}$, the map $\mathsf{M}_{p,q} : \mathsf{Y}_p \times \mathsf{Y}_q \to \mathsf{Y}_{pq}$ induces an isomorphism $\overline{\mathsf{M}}_{p,q} : \mathsf{Y}_p \otimes_A \mathsf{Y}_q \to \mathsf{Y}_{pq}$.
- For each $p \in P$, the maps $\mathsf{M}_{e,p}$ and $\mathsf{M}_{p,e}$ implement, respectively, the left and right actions of A on Y_p. Consequently, $\overline{\mathsf{M}}_{p,e} : \mathsf{Y}_p \otimes_A ({}_A A_A) \to \mathsf{Y}_p$ is an isomorphism for all $p \in P$.

For each $p \in P$, let $\phi_p : A \to \mathcal{L}(\mathsf{Y}_p)$ denote the left action of A on Y_p by adjointable operators. We say that Y is *essential* if and only if Y_p is an essential A-correspondence, i.e., $\mathrm{Span}(\phi_p[A][\mathsf{Y}_p])$ is dense in Y_p, for each $p \in P$. The map $\overline{\mathsf{M}}_{e,p} : ({}_A A_A) \otimes_A \mathsf{Y}_p \to \mathsf{Y}_p$ is an isomorphism if and only if Y_p is an essential A-correspondence, see Remark 2.2 in [**7**].

If ϕ_p takes values in the C^*-algebra $\mathcal{K}(\mathsf{Y}_p)$ of compact operators on Y_p for each $p \in P$, then Y is said to be *row-finite* or *proper*, and if ϕ_p is furthermore injective for each $p \in P$, then Y is said to be *faithful*.

There are various C^*-algebras associated to a product system under certain assumptions. For our future reference, let us recall some standard facts.

Let Y be a P-indexed product system over A, and let B be a C^*-algebra. A map $\psi : \mathsf{Y} \to B$ is then called a *Toeplitz representation* of Y if and only if, writing ψ_p for $\psi|_{\mathsf{Y}_p}$, the following properties hold:

- $\psi_p : \mathsf{Y}_p \to B$ is $\mathbb{C}$-linear for all $p \in P$.
- $\psi_e : A \to B$ is a C^*-homomorphism, and $\psi_e\Big(\langle \zeta | \eta \rangle_{\mathsf{Y}_p}\Big) = \psi_p(\zeta)^* \psi_p(\eta)$ for all $p \in P$ and $\zeta, \eta \in \mathsf{Y}_p$.
- $\psi_p(\zeta)\psi_q(\eta) = \psi_{pq}(\zeta\eta)$ for all $p, q \in P$, $\zeta \in \mathsf{Y}_p$, and $\eta \in \mathsf{Y}_q$.

One can construct a C^*-algebra $\mathcal{T}(\mathsf{Y})$ — known as the *Toeplitz algebra* of Y — and a Toeplitz representation $i_{\mathsf{Y}} : \mathsf{Y} \to \mathcal{T}(\mathsf{Y})$ of Y such that the pair $(\mathcal{T}(\mathsf{Y}), i_{\mathsf{Y}})$ is universal in the following sense: $\mathcal{T}(\mathsf{Y})$ is generated by $i_{\mathsf{Y}}[\mathsf{Y}]$, and for any Toeplitz representation $\psi : \mathsf{Y} \to B$, there is a C^*-homomorphism $\psi_* : \mathcal{T}(\mathsf{Y}) \to B$ such that $\psi_* \circ i_{\mathsf{Y}} = \psi$.

Let us denote by $\Theta_{\zeta,\eta}$ the rank-one operator $\xi \mapsto \zeta \langle \eta | \xi \rangle_{\mathsf{Y}_p}$. For each $p \in P$, there exists a C^*-homomorphism $\psi^{(p)} : \mathcal{K}(\mathsf{Y}_p) \to B$ obtained as the continuous extension of the map

$$\forall \zeta_1, \ldots, \zeta_n, \eta_1, \ldots, \eta_n \in \mathsf{Y}_p : \qquad \sum_{i=1}^{n} \Theta_{\zeta_i,\eta_i} \mapsto \sum_{i=1}^{n} \psi_p(\zeta_i)\psi_p(\eta_i)^*.$$

Note that as $\mathcal{K}(A) \cong A$ (via the identification of $\Theta_{a,b}$ with ab^*), we have $\psi^{(e)} = \psi_e$.

A Toeplitz representation $\psi : \mathsf{Y} \to B$ is then called *Cuntz–Pimsner covariant* (in Fowler's sense) if and only if

$$\forall p \in P,\ \forall a \in \phi_p^{-1}[\mathcal{K}(\mathsf{Y}_p)] : \qquad \psi^{(p)}(\phi_p(a)) = \psi_e(a).$$

One can construct a C^*-algebra $\mathcal{O}(\mathsf{Y})$ — known as the *Cuntz–Pimsner algebra* of Y — and a Cuntz–Pimsner covariant Toeplitz representation $j_{\mathsf{Y}} : \mathsf{Y} \to \mathcal{O}(\mathsf{Y})$ of Y such that the pair $(\mathcal{O}(\mathsf{Y}), j_{\mathsf{Y}})$ is universal in the following sense: $\mathcal{O}(\mathsf{Y})$ is generated by $j_{\mathsf{Y}}[\mathsf{Y}]$, and for any Cuntz–Pimsner covariant Toeplitz representation $\psi : \mathsf{Y} \to B$, there is a C^*-homomorphism $\psi_* : \mathcal{O}(\mathsf{Y}) \to B$ such that $\psi_* \circ j_{\mathsf{Y}} = \psi$.

Example 2.1. A C^*-correspondence X over A gives rise to a product system Y over $\mathbb{N}$ with fibers $\mathsf{Y}_n = \mathsf{X}^{\otimes n}$ for $n \geq 1$ and $\mathsf{Y}_0 = A$. In this case, $\mathcal{T}(\mathsf{Y}) = \mathcal{T}(\mathsf{X})$ and $\mathcal{O}(\mathsf{Y}) = \mathcal{O}(\mathsf{X})$.

Example 2.2. For a product system $\mathsf{Y} \to P$ whose fibers Y_p are nonzero finite-dimensional Hilbert spaces, in particular $A = \mathsf{Y}_e = \mathbb{C}$, let us fix an orthonormal basis $\mathcal{B}_p$ in Y_p. Then a Toeplitz representation $\psi : \mathsf{Y} \to B$ gives rise to a P-indexed family $(\psi(\xi) : \xi \in \mathcal{B}_p)_{p \in P}$ of isometries with mutually orthogonal range projections. In this case, $\mathcal{T}(\mathsf{Y})$ is generated by a collection of Cuntz–Toeplitz algebras that interact according to the multiplication maps $\overline{\mathsf{M}}_{p,q}$ in Y.

A representation $\psi : \mathsf{Y} \to B$ is Cuntz-Pimsner covariant if

$$\forall p \in P : \qquad \sum_{\xi \in \mathcal{B}_p} \psi(\xi)\psi(\xi)^* = \psi(1).$$

The Cuntz–Pimsner algebra $\mathcal{O}(\mathsf{Y})$ is generated by a collection of Cuntz algebras, so it could be thought of as a multidimensional Cuntz algebra. N. Fowler proved in [**6**] that if the function $p \mapsto \dim(\mathsf{Y}_p)$ is injective, then the algebra $\mathcal{O}(\mathsf{Y})$ is simple and purely infinite. For other examples of multidimensional Cuntz algebras, see [**2**].

Example 2.3. A row-finite k-graph with no sources Λ (see [**12**]) determines a product system $\mathsf{Y} \to \mathbb{N}^k$, with $\mathsf{Y}_0 = A = C_0(\Lambda^0)$ and $\mathsf{Y}_n = \overline{C_c(\Lambda^n)}$ for $n \neq 0$, that yields an isomorphism $\mathcal{O}(\mathsf{Y}) \cong C^*(\Lambda)$.

3. Group Actions on Product Systems and Crossed Products

Given a locally compact group G and a C^*-correspondence X over A, an action of G on X (see [**9**]) is a pair (α, β) with the following properties:

- α is a strongly continuous action of G on A by C^*-automorphisms.
- β is a strongly continuous action of G on X by surjective $\mathbb{C}$-linear isometries.
- For all $s \in G$, $a \in A$, and $x, y \in \mathsf{X}$,

$$\langle \beta_s(x) | \beta_s(y) \rangle_{\mathsf{X}} = \alpha_s(\langle x | y \rangle_{\mathsf{X}}), \qquad \beta_s(xa) = \beta_s(x)\alpha_s(a), \qquad \beta_s(ax) = \alpha_s(a)\beta_s(x).$$

The crossed product $\mathsf{X} \rtimes_\beta G$ of X by G is defined in [**9**] as the completion of the $C_c(G, A)$-bimodule $C_c(G, \mathsf{X})$, and its $(A \rtimes_\alpha G)$-correspondence structure is uniquely determined by the following operations:

$$\forall f \in C_c(G,A),\ \forall \zeta,\eta \in C_c(G,\mathsf{X}),\ \forall s \in G:$$

$$(f\zeta)(s) = \int_G f(t)\beta_t\big(\zeta\big(t^{-1}s\big)\big)\ \mathrm{d}t, \quad (\zeta f)(s) = \int_G \zeta(t)\alpha_t\big(f\big(t^{-1}s\big)\big)\ \mathrm{d}t,$$

$$\langle \zeta|\eta\rangle_{\mathsf{X}\rtimes_\beta G}(s) = \int_G \alpha_{t^{-1}}(\langle \zeta(t)|\eta(ts)\rangle_{\mathsf{X}})\ \mathrm{d}t.$$

By the universal property of Cuntz–Pimsner algebras (see [**11**]), there is an action γ of G on $\mathcal{O}(\mathsf{X})$ satisfying $\gamma_s(j_A(a)) = j_A(\alpha_s(a))$ and $\gamma_s(j_{\mathsf{X}}(x)) = j_{\mathsf{X}}(\beta_s(x))$, where (j_A, j_{X}) is the universal Cuntz–Pimsner representation of (A,X). For G amenable, it is proven in [**9**] that

$$\mathcal{O}(\mathsf{X}) \rtimes_\gamma G \cong \mathcal{O}(\mathsf{X} \rtimes_\beta G).$$

Definition 3.1. An action β of a locally compact group G on a product system $\mathsf{Y} \to P$ over A is a P-indexed family $(\beta^p)_{p\in P}$ such that (β^e, β^p) is an action of G on Y_p for each $p \in P$, and furthermore,

$$\forall s \in G,\ \forall \zeta \in \mathsf{Y}_p,\ \forall \eta \in \mathsf{Y}_q: \qquad \beta^{pq}_s(\zeta\eta) = \beta^p_s(\zeta)\beta^q_s(\eta).$$

We will usually denote β^e by α.

Example 3.2. For an essential product system Y indexed by $P = \big(\mathbb{N}^k, +\big)$ such that ϕ_p is an injection into $\mathcal{K}(\mathsf{Y}_p)$ for all $p = (p_1,\ldots,p_k) \in \mathbb{N}^k$, universality allows us to define a strongly continuous gauge action $\sigma : \mathbb{T}^k \to \mathrm{Aut}(\mathcal{O}(\mathsf{Y}))$ such that

$$\forall z \in \mathbb{T}^k,\ \forall p \in \mathbb{N}^k,\ \forall a \in A,\ \forall \zeta \in \mathsf{Y}_p: \qquad \sigma_z(a) = a \qquad \text{and} \qquad \sigma_z(j_{\mathsf{Y}}(\zeta)) = z^p j_{\mathsf{Y}}(\zeta).$$

Here, $z^p \overset{\mathrm{df}}{=} \prod_{i=1}^k z_i^{p_i}$. Then the fixed-point algebra $\mathcal{O}(\mathsf{Y})^\sigma$ is C^*-isomorphic to the inductive limit

$$\varinjlim_{p\in\mathbb{N}^k} \mathcal{K}(\mathsf{Y}_p),$$

where the order relation on $\mathbb{N}^k$ is the coordinate-wise order, and for $p \le q$, the map $\mathcal{K}(\mathsf{Y}_p) \to \mathcal{K}(\mathsf{Y}_q)$ is given by $T \mapsto T \otimes I_{q-p}$.

Example 3.3. For a compact group G and k finite-dimensional unitary representations ρ_i of G on Hilbert spaces $\mathcal{H}_i$ for $i \in \{1,\ldots,k\}$, we can construct a product system Y with fibers

$$\mathsf{Y}_n = \mathcal{H}_1^{\otimes n_1} \otimes \cdots \otimes \mathcal{H}_k^{\otimes n_k},$$

for $n = (n_1,\ldots,n_k) \in \mathbb{N}^k$; see [**4**]. Then the group G acts on each fiber Y_n via the representation $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$. This action is compatible with the multiplication maps and commutes with the gauge action of $\mathbb{T}^k$.

Proposition 3.4. *Let β be an action of G on a P-indexed product system Y. Define a multiplication on the disjoint union $\bigsqcup_{p\in P}(\mathsf{Y}_p \rtimes_{\beta^p} G)$ of fibers $\mathsf{Y}_p \rtimes_{\beta^p} G$ (which are C^*-correspondences over $A \rtimes_\alpha G$) as follows: For $\zeta \in C_c(G,\mathsf{Y}_p)$ and $\eta \in C_c(G,\mathsf{Y}_q)$, the product $\zeta\eta \in C_c(G,\mathsf{Y}_{pq})$ is*

$$\forall s \in G: \qquad (\zeta\eta)(s) = \int_G \zeta(t)\beta^q_t\big(\eta\big(t^{-1}s\big)\big)\ \mathrm{d}t.$$

Then the semigroup $\mathsf{Y} \rtimes_\beta G = \bigsqcup_{p\in P}(\mathsf{Y}_p \rtimes_{\beta^p} G)$ with this multiplication law is a product system over $A \rtimes_\alpha G$, called the crossed product $\mathsf{Y} \rtimes_\beta G$. *If Y is essential, then $\mathsf{Y} \rtimes_\beta G$ is also essential.*

Proof. Let us first prove that the multiplication law for $\mathsf{Y} \rtimes_\beta G$ is associative on the function-algebra level. Let $p, q, r \in P$, and let $\zeta \in C_c(G, \mathsf{Y}_p)$, $\eta \in C_c(G, \mathsf{Y}_q)$, and $\xi \in C_c(G, \mathsf{Y}_r)$. Then for all $s \in G$,

$$\begin{aligned}
[(\zeta\eta)\xi](s) &= \int_G (\zeta\eta)(t) \beta^r_t\big(\xi\big(t^{-1}s\big)\big) \, \mathrm{d}t \\
&= \int_G \left[\int_G \zeta(u) \beta^q_u\big(\eta\big(u^{-1}t\big)\big) \, \mathrm{d}u \right] \beta^r_t\big(\xi\big(t^{-1}s\big)\big) \, \mathrm{d}t \\
&= \int_{G \times G} \big[\zeta(u) \beta^q_u\big(\eta\big(u^{-1}t\big)\big)\big] \beta^r_t\big(\xi\big(t^{-1}s\big)\big) \, \mathrm{d}(u \times t) \\
&= \int_{G \times G} \zeta(u) \big[\beta^q_u\big(\eta\big(u^{-1}t\big)\big) \beta^r_t\big(\xi\big(t^{-1}s\big)\big)\big] \, \mathrm{d}(u \times t)
\end{aligned}$$

and

$$\begin{aligned}
[\zeta(\eta\xi)](s) &= \int_G \zeta(u) \beta^{qs}_u\big((\eta\xi)\big(u^{-1}s\big)\big) \, \mathrm{d}u \\
&= \int_G \zeta(u) \beta^{qs}_u \left(\int_G \eta(t) \beta^s_t\big(\xi\big(t^{-1}u^{-1}s\big)\big) \, \mathrm{d}t \right) \mathrm{d}u \\
&= \int_{G \times G} \zeta(u) \beta^{qs}_u\big(\eta(t) \beta^s_t\big(\xi\big(t^{-1}u^{-1}s\big)\big)\big) \, \mathrm{d}(t \times u) \\
&= \int_{G \times G} \zeta(u) \beta^q_u(\eta(t)) \beta^s_{ut}\big(\xi\big(t^{-1}u^{-1}s\big)\big) \, \mathrm{d}(t \times u) \\
&\qquad \text{(By the axioms of a group action.)} \\
&= \int_{G \times G} \zeta(u) \beta^q_u\big(\eta\big(u^{-1}t\big)\big) \beta^s_t\big(\xi\big(t^{-1}s\big)\big) \, \mathrm{d}(t \times u). \\
&\qquad \big(\text{By the change of variables } t \mapsto u^{-1}t.\big)
\end{aligned}$$

It follows that for all $p, q \in P$,

$$\begin{Bmatrix} C_c(G, \mathsf{Y}_p) \times C_c(G, \mathsf{Y}_q) & \to & C_c(G, \mathsf{Y}_{pq}) \\ (\zeta, \eta) & \mapsto & \zeta\eta \end{Bmatrix}$$

is a $C_c(G, A)$-balanced $\mathbb{C}$-bilinear map (take $q = e$ in the associativity calculation), which then induces a $\mathbb{C}$-linear map

$$\Omega_{p,q} = \begin{Bmatrix} C_c(G, \mathsf{Y}_p) \otimes_{C_c(G,A)} C_c(G, \mathsf{Y}_q) & \to & C_c(G, \mathsf{Y}_{pq}) \\ \displaystyle\sum_{i=1}^n \zeta_i \odot \eta_i & \mapsto & \displaystyle\sum_{i=1}^n \zeta_i \eta_i \end{Bmatrix}.$$

Let us show that $\Omega_{p,q}$ extends uniquely to a $\mathbb{C}$-linear isometry

$$\overline{\Omega}_{p,q} : (\mathsf{Y}_p \rtimes_{\beta^p} G) \otimes_{A \rtimes_\alpha G} (\mathsf{Y}_q \rtimes_{\beta^q} G) \to \mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G.$$

Observe that for all $\zeta_1, \ldots, \zeta_n \in C_c(G, \mathsf{Y}_p)$ and $\eta_1, \ldots, \eta_n \in C_c(G, \mathsf{Y}_q)$ we have

$$\begin{aligned}
&\left\| \sum_{i=1}^n \zeta_i \otimes \eta_i \right\|_{\left(\mathsf{Y}_p \rtimes_{\beta^p} G\right) \otimes_{A \rtimes_\alpha G} \left(\mathsf{Y}_q \rtimes_{\beta^q} G\right)} \\
&= \left\| \left\langle \sum_{i=1}^n \zeta_i \otimes \eta_i \middle| \sum_{j=1}^n \zeta_j \otimes \eta_j \right\rangle_{\left(\mathsf{Y}_p \rtimes_{\beta^p} G\right) \otimes_{A \rtimes_\alpha G} \left(\mathsf{Y}_q \rtimes_{\beta^q} G\right)} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}} \\
&= \left\| \sum_{i,j=1}^n \langle \zeta_i \otimes \eta_i | \zeta_j \otimes \eta_j \rangle_{\left(\mathsf{Y}_p \rtimes_{\beta^p} G\right) \otimes_{A \rtimes_\alpha G} \left(\mathsf{Y}_q \rtimes_{\beta^q} G\right)} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}} \\
&= \left\| \sum_{i,j=1}^n \left\langle \eta_i \middle| \langle \zeta_i | \zeta_j \rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G} \eta_j \right\rangle_{\mathsf{Y}_q \rtimes_{\beta^q} G} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
\left\| \sum_{i=1}^n \zeta_i \eta_i \right\|_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G} &= \left\| \left\langle \sum_{i=1}^n \zeta_i \eta_i \middle| \sum_{j=1}^n \zeta_j \eta_j \right\rangle_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}} \\
&= \left\| \sum_{i,j=1}^n \langle \zeta_i \eta_i | \zeta_j \eta_j \rangle_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}}.
\end{aligned}$$

To see that

$$\left\| \sum_{i=1}^n \zeta_i \otimes \eta_i \right\|_{\left(\mathsf{Y}_p \rtimes_{\beta^p} G\right) \otimes_{A \rtimes_\alpha G} \left(\mathsf{Y}_q \rtimes_{\beta^q} G\right)} = \left\| \sum_{i=1}^n \zeta_i \eta_i \right\|_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G},$$

it thus suffices to show that for all $i, j \in \{1, \ldots, n\}$,

$$\left\langle \eta_i \middle| \langle \zeta_i | \zeta_j \rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G} \eta_j \right\rangle_{\mathsf{Y}_q \rtimes_{\beta^q} G} \qquad \text{and} \qquad \langle \zeta_i \eta_i | \zeta_j \eta_j \rangle_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G}$$

are identical elements of $C_c(G,A)$. Indeed, for all $r \in G$,

$$
\begin{aligned}
&\Big\langle \eta_i \Big| \langle \zeta_i|\zeta_j\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\eta_j \Big\rangle_{\mathsf{Y}_q \rtimes_{\beta^q} G}(r)\\
&= \int_G \alpha_{u^{-1}}\Big(\Big\langle \eta_i(u)\Big|\Big(\langle \zeta_i|\zeta_j\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\eta_j\Big)(ur)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}u\\
&= \int_G \alpha_{u^{-1}}\left(\left\langle \eta_i(u)\middle|\int_G \langle \zeta_i|\zeta_j\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}(t)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\ \mathrm{d}t\right\rangle_{\mathsf{Y}_q}\right)\ \mathrm{d}u\\
&= \int_{G\times G} \alpha_{u^{-1}}\Big(\Big\langle \eta_i(u)\Big|\langle \zeta_i|\zeta_j\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}(t)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(t\times u)\\
&= \int_{G\times G} \alpha_{u^{-1}}\left(\left\langle \eta_i(u)\middle|\left[\int_G \alpha_{s^{-1}}\Big(\langle \zeta_i(s)|\zeta_j(st)\rangle_{\mathsf{Y}_p}\Big)\ \mathrm{d}s\right]\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\right\rangle_{\mathsf{Y}_q}\right)\ \mathrm{d}(t\times u)\\
&= \int_{G\times G\times G} \alpha_{u^{-1}}\Big(\Big\langle \eta_i(u)\Big|\alpha_{s^{-1}}\Big(\langle \zeta_i(s)|\zeta_j(st)\rangle_{\mathsf{Y}_p}\Big)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u)
\end{aligned}
$$

and

$$
\begin{aligned}
&\langle \zeta_i\eta_i|\zeta_j\eta_j\rangle_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G}(r)\\
&= \int_G \alpha_{u^{-1}}\Big(\langle (\zeta_i\eta_i)(u)|(\zeta_j\eta_j)(ur)\rangle_{\mathsf{Y}_{pq}}\Big)\ \mathrm{d}u\\
&= \int_G \alpha_{u^{-1}}\left(\left\langle \int_G \zeta_i(s)\beta^q_s\big(\eta_i\big(s^{-1}u\big)\big)\ \mathrm{d}s\middle|\int_G \zeta_j(t)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\ \mathrm{d}t\right\rangle_{\mathsf{Y}_{pq}}\right)\ \mathrm{d}u\\
&= \int_{G\times G\times G} \alpha_{u^{-1}}\Big(\big\langle \zeta_i(s)\beta^q_s\big(\eta_i\big(s^{-1}u\big)\big)\big|\zeta_j(t)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\big\rangle_{\mathsf{Y}_{pq}}\Big)\ \mathrm{d}(s\times t\times u)\\
&= \int_{G\times G\times G} \alpha_{u^{-1}}\Big(\big\langle \zeta_i(s)\otimes\beta^q_s\big(\eta_i\big(s^{-1}u\big)\big)\big|\zeta_j(t)\otimes\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\big\rangle_{\mathsf{Y}_p\otimes_A\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u)\\
&= \int_{G\times G\times G} \alpha_{u^{-1}}\Big(\Big\langle \beta^q_s\big(\eta_i\big(s^{-1}u\big)\big)\Big|\langle \zeta_i(s)|\zeta_j(t)\rangle_{\mathsf{Y}_p}\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u)\\
&= \int_{G\times G\times G} \alpha_{u^{-1}s}\Big(\Big\langle \eta_i\big(s^{-1}u\big)\Big|\alpha_{s^{-1}}\Big(\langle \zeta_i(s)|\zeta_j(t)\rangle_{\mathsf{Y}_p}\Big)\beta^q_{s^{-1}t}\big(\eta_j\big(t^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u)
\end{aligned}
$$

(By the axioms of a group action on a C^*-correspondence.)

$$
= \int_{G\times G\times G} \alpha_{u^{-1}s}\Big(\Big\langle \eta_i\big(s^{-1}u\big)\Big|\alpha_{s^{-1}}\Big(\langle \zeta_i(s)|\zeta_j(st)\rangle_{\mathsf{Y}_p}\Big)\beta^q_t\big(\eta_j\big(t^{-1}s^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u)
$$

(By the change of variables $t \mapsto st$.)

$$
= \int_{G\times G\times G} \alpha_{u^{-1}}\Big(\Big\langle \eta_i(u)\Big|\alpha_{s^{-1}}\Big(\langle \zeta_i(s)|\zeta_j(st)\rangle_{\mathsf{Y}_p}\Big)\beta^q_t\big(\eta_j\big(t^{-1}ur\big)\big)\Big\rangle_{\mathsf{Y}_q}\Big)\ \mathrm{d}(s\times t\times u).
$$

(By the change of variables $u \mapsto su$.)

Hence,

$$
\forall r \in G: \qquad \Big\langle \eta_i \Big| \langle \zeta_i|\zeta_j\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\eta_j \Big\rangle_{\mathsf{Y}_q \rtimes_{\beta^q} G}(r) = \langle \zeta_i\eta_i|\zeta_j\eta_j\rangle_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G}(r)
$$

as claimed, so

$$\left\|\sum_{i=1}^{n}\zeta_i\otimes\eta_i\right\|_{(\mathsf{Y}_p\rtimes_{\beta^p}G)\otimes_{A\rtimes_\alpha G}(\mathsf{Y}_q\rtimes_{\beta^q}G)}=\left\|\Omega_{p,q}\left(\sum_{i=1}^{n}\zeta_i\otimes\eta_i\right)\right\|_{\mathsf{Y}_{pq}\rtimes_{\beta^{pq}}G}.$$

As $C_c(G,\mathsf{Y}_p)\otimes_{C_c(G,A)}C_c(G,\mathsf{Y}_q)$ is dense in $(\mathsf{Y}_p\rtimes_{\beta^p}G)\otimes_{A\rtimes_\alpha G}(\mathsf{Y}_q\rtimes_{\beta^q}G)$, we conclude that $\Omega_{p,q}$ extends uniquely to a $\mathbb{C}$-linear isometry

$$\overline{\Omega}_{p,q}:(\mathsf{Y}_p\rtimes_{\beta^p}G)\otimes_{A\rtimes_\alpha G}(\mathsf{Y}_q\rtimes_{\beta^q}G)\to\mathsf{Y}_{pq}\rtimes_{\beta^{pq}}G.$$

We wish to show that $\overline{\Omega}_{p,q}$ is $(A\rtimes_\alpha G)$-linear for all $p,q\in P$, but this will turn out to be a consequence of the following two facts about these maps:

- For $p\in P$, $f\in A\rtimes_\alpha G$, and $\zeta\in\mathsf{Y}_p\rtimes_{\beta^p}G$,

$$f\zeta=\overline{\Omega}_{e,p}(f\otimes\zeta)\qquad\text{and}\qquad\zeta f=\overline{\Omega}_{p,e}(\zeta\otimes f),$$

  which are true by both the definitions of $\overline{\Omega}_{e,p}$ and $\overline{\Omega}_{p,e}$.
- For $p,q,r\in P$, $\zeta\in\mathsf{Y}_p\rtimes_{\beta^p}G$, $\eta\in\mathsf{Y}_q\rtimes_{\beta^q}G$, and $\xi\in Y_r\rtimes_{\beta^r}G$,

$$\overline{\Omega}_{pq,r}\big(\overline{\Omega}_{p,q}(\zeta\otimes\eta)\otimes\xi\big)=\overline{\Omega}_{p,qr}\big(\zeta\otimes\overline{\Omega}_{q,r}(\eta\otimes\xi)\big),$$

  which holds because the multiplication law of the product system is associative.

Now, to see the $(A\rtimes_\alpha G)$-linearity of $\overline{\Omega}_{p,q}$ for all $p,q\in P$, simply observe for all $f\in A\rtimes_\alpha G$, $\zeta\in\mathsf{Y}_p\rtimes_{\beta^p}G$, and $\eta\in\mathsf{Y}_q\rtimes_{\beta^q}G$ that

$$\begin{aligned}\overline{\Omega}_{p,q}((\zeta\otimes\eta)f)&=\overline{\Omega}_{p,q}(\zeta\otimes\eta f)\\&=\overline{\Omega}_{p,q}\big(\zeta\otimes\overline{\Omega}_{q,e}(\eta\otimes f)\big)\\&=\overline{\Omega}_{pq,e}\big(\overline{\Omega}_{p,q}(\zeta\otimes\eta)\otimes f\big)\\&=\overline{\Omega}_{p,q}(\zeta\otimes\eta)f.\end{aligned}$$

A similar computation gives $\overline{\Omega}_{p,q}(f(\zeta\otimes\eta))=f\overline{\Omega}_{p,q}(\zeta\otimes\eta)$. By linearity and continuity, $\overline{\Omega}_{p,q}$ is therefore $(A\rtimes_\alpha G)$-linear.

Finally, we will prove that $\overline{\Omega}_{p,q}$ is surjective for all $p,q \in P$ such that $p \neq e$. Firstly, note that for all $p \in P$ and $\zeta \in C_c(G,\mathsf{Y}_p)$,

$$\begin{aligned}
\|\zeta\|_{\mathsf{Y}_p \rtimes_{\beta^p} G} &= \left\| \langle \zeta | \zeta \rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G} \right\|_{A \rtimes_\alpha G}^{\frac{1}{2}} \\
&\leq \left\| \langle \zeta | \zeta \rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G} \right\|_{L^1(G,A)}^{\frac{1}{2}} \quad \text{(By Lemma 2.27 of [\textbf{17}].)} \\
&= \left[\int_G \left\| \langle \zeta | \zeta \rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}(t) \right\|_A \,\mathrm{d}t \right]^{\frac{1}{2}} \\
&= \left[\int_G \left\| \int_G \alpha_{s^{-1}} \Big(\langle \zeta(s) | \zeta(st) \rangle_{\mathsf{Y}_p} \Big) \,\mathrm{d}s \right\|_A \,\mathrm{d}t \right]^{\frac{1}{2}} \\
&\leq \left[\int_{G \times G} \left\| \alpha_{s^{-1}} \Big(\langle \zeta(s) | \zeta(st) \rangle_{\mathsf{Y}_p} \Big) \right\|_A \,\mathrm{d}(s \times t) \right]^{\frac{1}{2}} \\
&= \left[\int_{G \times G} \left\| \langle \zeta(s) | \zeta(st) \rangle_{\mathsf{Y}_p} \right\|_A \,\mathrm{d}(s \times t) \right]^{\frac{1}{2}} \\
&\leq \left[\int_{G \times G} \|\zeta(s)\|_{\mathsf{Y}_p} \|\zeta(st)\|_{\mathsf{Y}_p} \,\mathrm{d}(s \times t) \right]^{\frac{1}{2}} \\
&\quad \text{(By the Cauchy–Schwarz Inequality.)} \\
&= \left[\int_G \left(\|\zeta(s)\|_{\mathsf{Y}_p} \int_G \|\zeta(st)\|_{\mathsf{Y}_p} \,\mathrm{d}t \right) \mathrm{d}s \right]^{\frac{1}{2}} \\
&= \left[\int_G \|\zeta(s)\|_{\mathsf{Y}_p} \|\zeta\|_{L^1(G,\mathsf{Y}_p)} \,\mathrm{d}s \right]^{\frac{1}{2}} \\
&= \left[\|\zeta\|_{L^1(G,\mathsf{Y}_p)}^2 \right]^{\frac{1}{2}} \\
&= \|\zeta\|_{L^1(G,\mathsf{Y}_p)}.
\end{aligned}$$

Fix $p,q \in P$ with $p \neq e$. By a routine partition-of-unity argument, we can approximate a function $\zeta \in C_c(G,\mathsf{Y}_{pq})$ with respect to $\|\cdot\|_{L^1(G,\mathsf{Y}_{pq})}$ — and hence with respect to $\|\cdot\|_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G}$ — by a linear combination of functions of the form $f \odot z$, where $f \in C_c(G)$ and $z \in \mathsf{Y}_{pq}$. As $\overline{\mathsf{M}}_{p,q} : \mathsf{Y}_p \otimes_A \mathsf{Y}_q \to \mathsf{Y}_{pq}$ is an isomorphism, we can approximate z itself by a linear combination of elements of Y_{pq} of the form $\overline{\mathsf{M}}_{p,q}(x \otimes y)$, where $x \in \mathsf{Y}_p$ and $y \in \mathsf{Y}_q$. Now, for any $\epsilon > 0$, we can find an open neighborhood U of the identity $e_G \in G$ and a non-negative function $h \in C_c(G,\mathbb{R})$ with $\mathrm{Supp}(h) \subseteq U$ and integral 1 such that

$$\left\| f \odot \overline{\mathsf{M}}_{p,q}(x \otimes y) - \Omega_{p,q}((h \odot x) \otimes (f \odot y)) \right\|_{L^1(G,\mathsf{Y}_{pq})} < \epsilon.$$

This yields, according to the foregoing discussion,

$$\left\| f \odot \overline{\mathsf{M}}_{p,q}(x \otimes y) - \Omega_{p,q}((h \odot x) \otimes (f \odot y)) \right\|_{\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G} < \epsilon.$$

Therefore, $\mathrm{Range}(\overline{\Omega}_{p,q})$ is dense in $\mathsf{Y}_{pq} \rtimes_{\beta^{pq}} G$, and as $\overline{\Omega}_{p,q}$ is an isometry between Banach spaces, it follows that $\overline{\Omega}_{p,q}$ is surjective.

As $\overline{\Omega}_{p,q}$ is a surjective $(A \rtimes_\alpha G)$-linear isometry for all $p, q \in P$ with $p \neq e$, we can apply the main result of [**13**] by Lance to conclude that it is a unitary operator.

If Y is essential, then $\overline{\mathsf{M}}_{e,q}$ is an isomorphism, so $\overline{\Omega}_{e,q}$ is also an isomorphism and $\mathsf{Y} \rtimes_\beta G$ is essential. □

Theorem 3.5. *Suppose that a group G acts on a row-finite and faithful P-indexed product system Y over A via automorphisms β_g^p. Then G acts on $\mathcal{O}(\mathsf{Y})$ via automorphisms denoted by γ_g. Moreover, if G is amenable, then $\mathsf{Y} \rtimes_\beta G$ is row-finite and faithful, and for $P = \mathbb{N}^k$ and Y essential, we even have*

$$\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G \cong \mathcal{O}(\mathsf{Y} \rtimes_\beta G).$$

Proof. Let $p \in P$. Recall that there is a strongly continuous action τ^p of G on $\mathcal{K}(\mathsf{Y}_p)$ given by

$$\forall x, y \in \mathsf{Y}_p : \qquad \tau_g^p(\Theta_{x,y}) = \Theta_{\beta_g^p(x), \beta_g^p(y)}.$$

The left-action $\phi_p : A \to \mathcal{K}(\mathsf{Y}_p)$ is injective by assumption. To see that it is equivariant for α and τ^p, first observe that for all $g \in G$, $a \in A$, and $x \in \mathsf{Y}_p$

$$\beta_g^p([\phi_p(a)](x)) = \beta_g^p(ax) = \alpha_g(a)\beta_g^p(x) = [\phi_p(\alpha_g(a))]\big(\beta_g^p(x)\big),$$

so $\beta_g^p \circ \phi_p(a) = \phi_p(\alpha_g(a)) \circ \beta_g^p$; equivalently, $\beta_g^p \circ \phi_p(a) \circ \beta_{g^{-1}}^p = \phi_p(\alpha_g(a))$. Next, observe for all $g \in G$ and $x, y, z \in \mathsf{Y}_p$ that

$$\begin{aligned}
\Big(\beta_g^p \circ \Theta_{x,y} \circ \beta_{g^{-1}}^p\Big)(z) &= \beta_g^p\left(x\Big\langle y\Big|\beta_{g^{-1}}^p(z)\Big\rangle_{\mathsf{Y}_p}\right) \\
&= \beta_g^p(x)\alpha_g\left(\Big\langle y\Big|\beta_{g^{-1}}^p(z)\Big\rangle_{\mathsf{Y}_p}\right) \\
&= \beta_g^p(x)\big\langle \beta_g^p(y)\big|z\big\rangle_{\mathsf{Y}_p} \\
&= \Theta_{\beta_g^p(x), \beta_g^p(y)}(z),
\end{aligned}$$

so $\tau_g^p(\Theta_{x,y}) = \beta_g^p \circ \Theta_{x,y} \circ \beta_{g^{-1}}^p$. In particular, as $\mathrm{Range}(\phi_p) \subseteq \mathcal{K}(\mathsf{Y}_p)$, we have

$$\forall a \in A : \qquad \tau_g^p(\phi_p(a)) = \beta_g^p \circ \phi_p(a) \circ \beta_{g^{-1}}^p = \phi_p(\alpha_g(a)),$$

which proves the equivariance of ϕ_p for α and τ^p. According to the theory of reduced C^*-crossed products, ϕ_p induces the injective $*$-homomorphism

$$\overline{\phi_p} : A \rtimes_{\alpha,\mathrm{red}} G \to \mathcal{K}(\mathsf{Y}_p) \rtimes_{\tau^p,\mathrm{red}} G,$$

where $\overline{\phi_p}(f) = \phi_p \circ f$ for all $f \in C_c(G, A)$. However, G is amenable, so $\overline{\phi_p}$: $A \rtimes_\alpha G \to \mathcal{K}(\mathsf{Y}_p) \rtimes_{\tau^p} G$ and $\mathcal{K}(\mathsf{Y}_p) \rtimes_{\tau^p} G \xrightarrow{\cong} \mathcal{K}(\mathsf{Y}_p \rtimes_{\beta^p} G)$, where the inverse Λ of this $*$-isomorphism is defined in [**9**] by

$$\forall \zeta, \eta \in C_c(G, \mathsf{Y}_p),\ \forall s \in G : \qquad [\Lambda(\Theta_{\zeta,\eta})](s) = \int_G \Delta\big(s^{-1}r\big)\Theta_{\zeta(r), \beta_s^p(\eta(s^{-1}r))}\, \mathrm{d}r,$$

where Δ is the modular function of G. Therefore, $\mathsf{Y} \rtimes_\beta G$ is also a row-finite and faithful product system, as claimed.

Next, we show that there exists a strongly continuous action γ of G on $\mathcal{O}(\mathsf{Y})$ that satisfies

$$\forall g \in G,\ \forall p \in P,\ \forall y \in \mathsf{Y}_p : \qquad \gamma_g(j_{\mathsf{Y}}(y)) = j_{\mathsf{Y}}\big(\beta_g^p(y)\big), \tag{1}$$

where $j_{\mathsf{Y}} : \mathsf{Y} \to \mathcal{O}(\mathsf{Y})$ denotes the universal Cuntz–Pimsner representation. Let $g \in G$. Then the map $\Psi_g : \mathsf{Y} \to \mathcal{O}(\mathsf{Y})$ defined by $\Psi_g(y) \overset{\mathrm{df}}{=} j_{\mathsf{Y}}\big(\beta^p_g(y)\big)$ for all $p \in P$ and $y \in \mathsf{Y}_p$ is a Cuntz-Pimsner representation of Y on $\mathcal{O}(\mathsf{Y})$:

- For all $p, q \in P$, $x \in \mathsf{Y}_p$, and $y \in \mathsf{Y}_q$, we have
$$\begin{aligned}\Psi_g(xy) &= j_{\mathsf{Y}}\big(\beta^{pq}_g(xy)\big)\\ &= j_{\mathsf{Y}}\big(\beta^p_g(x)\beta^q_g(y)\big)\\ &= j_{\mathsf{Y}}\big(\beta^p_g(x)\big)j_{\mathsf{Y}}\big(\beta^q_g(y)\big)\\ &= \Psi_g(x)\Psi_g(y).\end{aligned}$$
- For all $p \in P$ and $x, y \in \mathsf{Y}_p$, we have
$$\begin{aligned}\Psi_g\Big(\langle x|y\rangle_{\mathsf{Y}_p}\Big) &= j_{\mathsf{Y}}\Big(\alpha_g\Big(\langle x|y\rangle_{\mathsf{Y}_p}\Big)\Big)\\ &= j_{\mathsf{Y}}\Big(\big\langle \beta^p_g(x)\big|\beta^p_g(y)\big\rangle_{\mathsf{Y}_p}\Big)\\ &= j_{\mathsf{Y}}\big(\beta^p_g(x)\big)^* j_{\mathsf{Y}}\big(\beta^p_g(y)\big)\\ &= \Psi_g(x)^*\Psi_g(y).\end{aligned}$$
- Let $p \in P$. The foregoing argument tells us that Ψ_g is a Toeplitz representation of Y on $\mathcal{O}(\mathsf{Y})$, so there exists an extension $\Psi^{(p)}_g : \mathcal{K}(\mathsf{Y}_p) \to \mathcal{O}(\mathsf{Y})$ such that
$$\begin{aligned}\forall x, y \in \mathsf{Y}_P : \qquad \Psi^{(p)}_g(\Theta_{x,y}) &= \Psi_g(x)\Psi_g(y)^*\\ &= j_{\mathsf{Y}}\big(\beta^p_g(x)\big)j_{\mathsf{Y}}\big(\beta^p_g(y)\big)^*\\ &= j^{(p)}_{\mathsf{Y}}\Big(\Theta_{\beta^p_g(x),\beta^p_g(y)}\Big)\\ &= j^{(p)}_{\mathsf{Y}}\big(\tau^p_g(\Theta_{x,y})\big),\end{aligned}$$
which implies by linearity and continuity that $\Psi^{(p)}_g = j^{(p)}_{\mathsf{Y}} \circ \tau^p_g$. As we have shown that ϕ_p is equivariant for α and τ^p and since j_{Y} is Cuntz–Pimsner-covariant, we have
$$\forall a \in A : \qquad \Psi^{(p)}_g(\phi_p(a)) = j^{(p)}_{\mathsf{Y}}\big(\tau^p_g(\phi_p(a))\big) = j^{(p)}_{\mathsf{Y}}(\phi_p(\alpha_g(a))) = j_{\mathsf{Y}}(\alpha_g(a)) = \Psi_g(a),$$
proving that Ψ_g is a Cuntz–Pimsner representation of Y.

By universality, there is thus a C^*-endomorphism S on $\mathcal{O}(\mathsf{Y})$ such that
$$\forall p \in P,\ \forall y \in \mathsf{Y}_p : \qquad S(j_{\mathsf{Y}}(y)) = j_{\mathsf{Y}}\big(\beta^p_g(y)\big).$$
Similarly, there is a C^*-endomorphism T on $\mathcal{O}(\mathsf{Y})$ such that
$$\forall p \in P,\ \forall y \in \mathsf{Y}_p : \qquad T(j_{\mathsf{Y}}(y)) = j_{\mathsf{Y}}\Big(\beta^p_{g^{-1}}(y)\Big).$$
As $ST = \mathrm{Id}_{\mathcal{O}(\mathsf{Y})} = TS$, we see that S is a C^*-isomorphism, and as g is arbitrary and β is an action of G on Y, there is an action γ of G on $\mathcal{O}(\mathsf{Y})$ that satisfies (1). The strong continuity of γ immediately follows from the continuity of j_{Y} and the strong continuity of each β^p.

We now show that a Cuntz–Pimsner representation $\psi : \mathsf{Y} \rtimes_\beta G \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$ exists and that it satisfies
$$\forall p \in P,\ \forall \zeta \in C_c(G, \mathsf{Y}_p) : \qquad \psi_p(\zeta) = j_{\mathsf{Y}} \circ \zeta.$$

As $j_{\mathsf{Y}}|_A : A \to \mathcal{O}(\mathsf{Y})$ is a $*$-homomorphism, and as $\gamma_g(j_{\mathsf{Y}}(a)) = j_{\mathsf{Y}}(\alpha_g(a))$ for all $a \in A$, we find that $j_{\mathsf{Y}}|_A$ is equivariant for α and γ. Hence, $j_{\mathsf{Y}}|_A$ induces a $*$-homomorphism

$$\psi_e : A \rtimes_\alpha G \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$$

such that $\psi_e(f) = j_{\mathsf{Y}} \circ f$ for all $f \in C_c(G, A)$. Let $p \in P$ and $\zeta, \eta \in C_c(G, \mathsf{Y}_p)$. Then for all $s \in G$,

$$\begin{aligned}
\big[(j_{\mathsf{Y}} \circ \zeta)^*(j_{\mathsf{Y}} \circ \zeta)\big](s) &= \int_G (j_{\mathsf{Y}} \circ \zeta)^*(r)\gamma_r\big((j_{\mathsf{Y}} \circ \zeta)\big(r^{-1}s\big)\big) \,\mathrm{d}r \\
&= \int_G \Delta\big(r^{-1}\big) \cdot \gamma_r\Big(j_{\mathsf{Y}}\big(\zeta\big(r^{-1}\big)\big)^*\Big)\gamma_r\big(j_{\mathsf{Y}}\big(\zeta\big(r^{-1}s\big)\big)\big) \,\mathrm{d}r \\
&= \int_G \gamma_{r^{-1}}\big(j_{\mathsf{Y}}(\zeta(r))^*\big)\gamma_{r^{-1}}(j_{\mathsf{Y}}(\zeta(rs))) \,\mathrm{d}r \\
&= \int_G \gamma_{r^{-1}}\big(j_{\mathsf{Y}}(\zeta(r))^* j_{\mathsf{Y}}(\zeta(rs))\big) \,\mathrm{d}r \\
&= \int_G \gamma_{r^{-1}}\Big(j_{\mathsf{Y}}\Big(\langle\zeta(r)|\zeta(rs)\rangle_{\mathsf{Y}_p}\Big)\Big) \,\mathrm{d}r \\
&= \int_G j_{\mathsf{Y}}\Big(\alpha_{r^{-1}}\Big(\langle\zeta(r)|\zeta(rs)\rangle_{\mathsf{Y}_p}\Big)\Big) \,\mathrm{d}r \\
&= j_{\mathsf{Y}}\bigg(\int_G \alpha_{r^{-1}}\Big(\langle\zeta(r)|\zeta(rs)\rangle_{\mathsf{Y}_p}\Big) \,\mathrm{d}r\bigg) \quad \text{(By the continuity of } j_{\mathsf{Y}}.) \\
&= j_{\mathsf{Y}}\Big(\langle\zeta|\zeta\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}(s)\Big) \\
&= \Big[\psi\Big(\langle\zeta|\zeta\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\Big)\Big](s),
\end{aligned}$$

so

$$\begin{aligned}
\|j_{\mathsf{Y}} \circ \zeta\|_{\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G} &= \big\|(j_{\mathsf{Y}} \circ \zeta)^*(j_{\mathsf{Y}} \circ \zeta)\big\|_{\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G}^{\frac{1}{2}} \\
&= \Big\|\psi\Big(\langle\zeta|\zeta\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\Big)\Big\|_{\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G}^{\frac{1}{2}} \\
&\leq \Big\|\langle\zeta|\zeta\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\Big\|_{A \rtimes_\alpha G}^{\frac{1}{2}} \\
&= \|\zeta\|_{\mathsf{Y}_p \rtimes_{\beta^p} G}.
\end{aligned}$$

In light of this norm-inequality, there exists a continuous linear map

$$\psi_p : \mathsf{Y}_p \rtimes_{\beta^p} G \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$$

such that $\psi_p(\zeta) = j_{\mathsf{Y}} \circ \zeta$ for all $\zeta \in C_c(G, \mathsf{Y}_p)$. By combining the various ψ_p's, we get a map $\psi : \mathsf{Y} \rtimes_\beta G \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$. The following show that ψ is a Toeplitz representation:

- As seen above, $\psi_e\Big(\langle\zeta|\zeta\rangle_{\mathsf{Y}_p \rtimes_{\beta^p} G}\Big) = \psi_p(\zeta)^*\psi_p(\zeta)$ for all $p \in P$ and $\zeta \in C_c(G, \mathsf{Y}_p)$.

- For all $p, q \in P$, $\zeta \in \mathsf{Y}_p \rtimes_{\beta^p} G$, $\eta \in \mathsf{Y}_q \rtimes_{\beta^q} G$, and $s \in G$,

$$\begin{aligned}[\psi_p(\zeta)\psi_q(\eta)](s) &= \int_G [\psi_p(\zeta)](r)\gamma_r\big([\psi_q(\eta)](r^{-1}s)\big)\,\mathrm{d}r \\ &= \int_G j_{\mathsf{Y}}(\zeta(r))\gamma_r\big(j_{\mathsf{Y}}\big(\eta\big(r^{-1}s\big)\big)\big)\,\mathrm{d}r \\ &= \int_G j_{\mathsf{Y}}(\zeta(r))j_{\mathsf{Y}}\big(\beta_r^q\big(\eta\big(r^{-1}s\big)\big)\big)\,\mathrm{d}r \\ &= j_{\mathsf{Y}}\bigg(\int_G \zeta(r)\beta_r^q\big(\eta\big(r^{-1}s\big)\big)\,\mathrm{d}r\bigg) \\ &= j_{\mathsf{Y}}((\zeta\eta)(s)) \\ &= [\psi_{pq}(\zeta\eta)](s),\end{aligned}$$

  so $\psi_p(\zeta)\psi_q(\eta) = \psi_{pq}(\zeta\eta)$.

It thus remains to check Cuntz–Pimsner covariance. If

$$\psi^{(p)} : \mathcal{K}(\mathsf{Y}_p \rtimes_{\beta^p} G) \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$$

denotes the extension of ψ_p, then letting $p \in P$, $\zeta, \eta \in C_c(G, \mathsf{Y}_p)$, and $s \in G$, we obtain that

$$\begin{aligned}\Big[\psi^{(p)}(\Theta_{\zeta,\eta})\Big](s) &= \big[\psi_p(\zeta)\psi_p(\eta)^*\big](s) \\ &= \int_G [\psi_p(\zeta)](r)\gamma_r\big(\big[\psi_p(\eta)^*\big]\big(r^{-1}s\big)\big)\,\mathrm{d}r \\ &= \int_G j_{\mathsf{Y}}(\zeta(r))\gamma_r\Big(\Delta\big(s^{-1}r\big)\cdot\gamma_{r^{-1}s}\Big(j_{\mathsf{Y}}\big(\eta\big(s^{-1}r\big)\big)^*\Big)\Big)\,\mathrm{d}r \\ &= \int_G \Delta\big(s^{-1}r\big)\cdot j_{\mathsf{Y}}(\zeta(r))\gamma_s\Big(j_{\mathsf{Y}}\big(\eta\big(s^{-1}r\big)\big)^*\Big)\,\mathrm{d}r \\ &= \int_G \Delta\big(s^{-1}r\big)\cdot j_{\mathsf{Y}}(\zeta(r))j_{\mathsf{Y}}\big(\beta_s^p\big(\eta\big(s^{-1}r\big)\big)\big)^*\,\mathrm{d}r \\ &= \int_G \Delta\big(s^{-1}r\big)\cdot j_{\mathsf{Y}}^{(p)}\big(\Theta_{\zeta(r),\beta_s^p(\eta(s^{-1}r))}\big)\,\mathrm{d}r \\ &= j_{\mathsf{Y}}^{(p)}\bigg(\int_G \Delta\big(s^{-1}r\big)\cdot\Theta_{\zeta(r),\beta_s^p(\eta(s^{-1}r))}\,\mathrm{d}r\bigg) \\ &= \Big[j_{\mathsf{Y}}^{(p)}\circ\Lambda(\Theta_{\zeta,\eta})\Big](s).\end{aligned}$$

Hence, $\psi^{(p)}(\Theta_{\zeta,\eta}) = j_{\mathsf{Y}}^{(p)} \circ \Lambda(\Theta_{\zeta,\eta})$, which means that $\psi^{(p)}(T) = j_{\mathsf{Y}}^{(p)} \circ \Lambda(T)$ for all $T \in \mathcal{K}(\mathsf{Y} \rtimes_{\beta^p} G)$. In particular, we have for all $f \in C_c(G, A)$ that

$$\begin{aligned}\psi^{(p)}\big(\Lambda^{-1}\big(\overline{\phi_p}(f)\big)\big) &= j_{\mathsf{Y}}^{(p)}\circ\Lambda\big(\Lambda^{-1}(\phi_p\circ f)\big) \\ &= j_{\mathsf{Y}}^{(p)}\circ\phi_p\circ f \\ &= j_{\mathsf{Y}}\circ f \\ &= \psi_e(f).\end{aligned}$$

Therefore, $\psi^{(p)} \circ \big(\Lambda^{-1} \circ \overline{\phi_p}\big) = \psi_e$ for all $p \in P$, which proves that ψ is Cuntz–Pimsner covariant. By universality, the representation $\psi : \mathsf{Y} \rtimes_\beta G \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$

determines a unique $*$-homomorphism

$$\psi_* : \mathcal{O}(\mathsf{Y} \rtimes_\beta G) \to \mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$$

such that $\psi_*\big(j_{\mathsf{Y}\rtimes_\beta G}(f)\big) = \psi_p(f)$ for $f \in C_c(G, \mathsf{Y}_p)$. The image of ψ_* generates $\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$, so ψ_* is surjective.

For $P = \mathbb{N}^k$ and Y essential, recall that there is a gauge action σ of $\mathbb{T}^k$ on $\mathcal{O}(\mathsf{Y})$ such that $\sigma_z(a) = a$ and $\sigma_z(j_{\mathsf{Y}}(\zeta)) = z^p j_{\mathsf{Y}}(\zeta)$. As the action γ of G on $\mathcal{O}(\mathsf{Y})$ is equivariant, we get a gauge action of $\mathbb{T}^k$ on $\mathcal{O}(\mathsf{Y}) \rtimes_\gamma G$. The injectivity of ψ_* now follows from the injectivity of ψ_e (note that j_{Y} is injective); see Lemma 3.3.2 in [**5**] or Corollary 4.14 in [**3**]. □

Remark 3.6. *Katsoulis obtained similar results for the so-called generalized gauge action on a product system over a semigroup P that is the positive cone of an abelian group, see Theorem 3.8 in* [**10**]. *Moreover, using a Fourier transform, he proved a Takai-duality result and generalized some results of Schafhauser from* [**15**].

Remark 3.7. *Suppose Y is a row-finite, faithful, and essential product system indexed by $P = \mathbb{N}^k$. If A is AF and each C^*-correspondence Y_n is full and separable, then there is a gauge action σ of $\mathbb{T}^k$ on $\mathcal{O}(\mathsf{Y})$ and $\mathcal{O}(\mathsf{Y}) \rtimes_\sigma \mathbb{T}^k$ is AF.*

Proof. Like in Example 3.2, there is a gauge action of $\mathbb{T}^k$ on $\mathcal{O}(\mathsf{Y})$. In this case, $\mathcal{O}(\mathsf{Y}) \rtimes_\sigma \mathbb{T}^k$ is Morita–Rieffel equivalent to the core $\mathcal{O}(Y)^\sigma \cong \varinjlim_{n\in\mathbb{N}^k} \mathcal{K}(\mathsf{Y}_n)$, and each $\mathcal{K}(\mathsf{Y}_n)$ is Morita–Rieffel equivalent to A as Y_n is full. It follows that $\mathcal{O}(\mathsf{Y}) \rtimes_\sigma \mathbb{T}^k$ is AF. □

Example 3.8. In the setting of Example 3.3, the compact group G acts on each fiber Y_n of the product system Y via the representation $\rho^n = \rho_1^{\otimes n_1} \otimes \cdots \otimes \rho_k^{\otimes n_k}$. This action is compatible with the multiplication maps and commutes with the gauge action of $\mathbb{T}^k$. The crossed product $\mathsf{Y} \rtimes G$ is a row-finite and faithful product system indexed by $\mathbb{N}^k$ over the group C^*-algebra $C^*(G)$. Moreover,

$$\mathcal{O}(\mathsf{Y}) \rtimes G \cong \mathcal{O}(\mathsf{Y} \rtimes G).$$

The Doplicher–Roberts algebra $\mathcal{O}_{\rho_1,\ldots,\rho_k}$ constructed in [**4**] from intertwiners $\mathrm{Hom}(\rho^n, \rho^m)$ is isomorphic to the fixed point algebra $\mathcal{O}(\mathsf{Y})^G$ and is Morita–Rieffel equivalent to $\mathcal{O}(\mathsf{Y}) \rtimes G$.

Example 3.9. If a locally compact group G acts on a k-graph Λ by automorphisms, then G acts on the product system Y constructed from Λ as in Example 2.3 and the C^*-algebra of the product system $\mathsf{Y} \rtimes G$ is isomorphic to $C^*(\Lambda) \rtimes G$. In [**8**], the authors consider the particular case when $G = \mathbb{Z}^\ell$ and they construct a $(k+\ell)$-graph $\Lambda \times \mathbb{Z}^\ell$ such that $C^*\big(\Lambda \times \mathbb{Z}^\ell\big) \cong C^*(\Lambda) \rtimes \mathbb{Z}^\ell$. Our result gives a new perspective on this situation.

References

[1] S. Albandik and R. Meyer, *Product systems over Ore monoids*, Doc. Math. **20** (2015), 1331–1402.

[2] B. Burgstaller, *Some multidimensional Cuntz algebras*, Aequationes Math. **76** (2008), 19–32. Doi: 10.1007/s00010-007-2924-4

[3] T. M. Carlsen, N. S. Larsen, A. Sims and S. T. Vittadello, *Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems*, Proc. Lond. Math. Soc. (3) **103** (2011), 563–600. Doi: 10.1112/plms/pdq028

[4] V. Deaconu, *C^*-algebras from k group representations*, J Aust. Math. Soc. **113** (2022), 318–338. Doi: 10.1017/S1446788721000392

[5] V. Deaconu, A. Kumjian, D. Pask and A. Sims, *Graphs of C^*-correspondences and Fell bundles*, Indiana Univ. Math. J. **59** (2010), 1687–1735. Doi: 10.1512/iumj.2010.59.3893

[6] N. J. Fowler, *Discrete product systems of finite-dimensional Hilbert spaces, and generalized Cuntz algebras*, preprint 1999, arXiv:math/9904116 [math.OA]

[7] N. J. Fowler, *Discrete product systems of Hilbert bimodules*, Pac. J. Math. **204** (2002), 335–375. Doi: 10.2140/pjm.2002.204.335

[8] C. Farthing, D. Pask, and A. Sims, *Crossed products of k-graph C^*-algebras by $\mathbb{Z}^l$*, Houston J. Math. **35** (2009), 903–933.

[9] G. Hao and C.-K. Ng, *Crossed products of C^*-correspondences by amenable group actions*, J. Math. Anal. Appl. **345** (2008), 702–707. Doi: 10.1016/j.jmaa.2008.04.058

[10] E. Katsoulis, *Product systems of* C**-correspondences and Takai duality*, Isr. J. Math. **240** (2020), 223–251. Doi: 10.1007/s11856-020-2063-3

[11] T. Katsura, *On C^*-algebras associated with C^*-correspondences*, J. Funct. Anal. **217** (2004), 366–401. Doi: 10.1016/j.jfa.2004.03.010

[12] A. Kumjian and D. Pask, *Actions of $\mathbb{Z}^k$ associated to higher rank graphs*, Ergodic Theory Dyn. Syst. **23** (2003), 1153–1172. Doi: 10.1017/S0143385702001670

[13] E. C. Lance, *Unitary operators on Hilbert C^*-modules*, Bull. Lond. Math. Soc., **26** (1994), 363–366. Doi: 10.1112/blms/26.4.363

[14] M. V. Pimsner, *A Class of C^*-Algebras Generalizing Both Cuntz–Krieger Algebras and Crossed Products by $\mathbb{Z}$*, Free probability theory (Waterloo, ON, 1995), Fields Institute Communications 12, 189–212, American Mathematical Society, Providence, RI, 1997.

[15] C. P. Schafhauser, *Cuntz-Pimsner algebras, crossed products, and K-theory*, J. Funct. Anal. **269** (2015), 2927–2946. Doi: 10.1016/j.jfa.2015.08.008

[16] A. Sims and T. Yeend, *C^*-algebras associated to product systems of Hilbert bimodules*, J. Oper. Theory **64** (2010), 349–376.

[17] D. P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs 134, American Mathematical Society, Providence, RI, 2007.

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