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A NOTE ON THE REGULARITY CRITERION FOR THE MICROPOLAR FLUID EQUATIONS IN HOMOGENEOUS BESOV SPACES

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Abstract. This paper gives a further investigation on the regularity criteria for three-dimensional micropolar equations in Besov spaces. More precisely, it is proved that the weak solution (u, ω) is regular if the velocity u satisfies

$$\int_0^T \|\nabla_h u_h\|^q_{\dot{B}^0_{p,\frac{2p}{3}}} \mathrm{d}t < \infty, \ \textit{with} \ \ \frac{3}{p} + \frac{2}{q} = 2, \ \frac{3}{2} < p \leq \infty,$$

or

$$\int_0^T \|\nabla_h u\|^{\frac{8}{3}}_{\dot{B}^{-1}_{\infty,\infty}} \mathrm{d}t < \infty,$$

or

$$\int_0^T \|\nabla_h u_h\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}} \mathrm{d}t < \infty, \ \textit{with} \ 0 < \alpha < 1.$$

1. Introduction

Micropolar fluids are proposed to characterise a class of complicated fluids that cannot be described by the classical Navier-Stokes equations, such as blood flows, bubbly liquids, liquid crystals and so on, for more details see [**18**]. In this paper, the regularity problem for the following three-dimensional incompressible micropolar fluid equations will be investigated

$$\begin{cases} \partial_t u + u \cdot \nabla u - (\mu + \kappa)\Delta u + \nabla p = 2\kappa \nabla \times \omega, \\ \partial_t \omega + u \cdot \nabla \omega - \gamma \Delta \omega - \nu \nabla \nabla \cdot \omega + 4\kappa\omega = 2\kappa \nabla \times u, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \omega(x,0) = \omega_0(x), \end{cases} \tag{1.1}$$

here $u = (u_1, u_2, u_3) \in \mathbb{R}^3$ is the velocity field, $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$ is the micro-rotational velocity, and $p \in \mathbb{R}$ is the scalar pressure. And the parameters μ , κ, γ and ν are positive constants. For simplicity, we shall assume that $\mu = \kappa = \frac{1}{2}, \nu = \gamma = 1$ on account of their values playing no parts in our discussions.

The classical micropolar fluid equatons were first studied by Eringen [**10**] in 1966, and then Galdi and Rionero [**11**] and Lukaszewicz [**16**],[**17**] studied the well-posedness of solutions to the micropolar fluid equatons. Because the system (1.1)

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contains the Navier-Stokes equations as a subsystem, the well-posedness theory for micropolar fluid equations can't be better than Navier-Stokes equations. Like Navier-Stokes equations, it is returned to study the regularity criterion for the weak solution of micropolar fluid equations and until now different criteria of solutions have arisen, for instance, the interesting and famous Prodi-Serrin type regularity criterion in terms of the velocity, micro-rotational velocity and pressure. For regularity criteria for Navier-Stokes equations and micropolar fluid equations, readers may refer to [**2, 3, 4, 6, 7, 8, 9, 14, 15, 19, 21, 22, 12, 20, 23, 24**] and references therein.

In [**22**], Yuan and Li showed two Prodi-Serrin type regularity criteria in inhomogeneous Besov space. That is if the partial derivatives of velocity and micro-rotational velocity satisfy

$$\int_0^T (\|\nabla_h u\|_{\dot{B}^0_{\infty,\infty}} + \|\nabla_h \omega\|_{\dot{B}^0_{\infty,\infty}})\mathrm{d}t < \infty, \tag{1.2}$$

or

$$\int_0^T (\|\nabla_h u\|^{\frac{8}{3}}_{\dot{B}^{-1}_{\infty,\infty}} + \|\nabla_h \omega\|^{\frac{8}{3}}_{\dot{B}^{-1}_{\infty,\infty}})\mathrm{d}t < \infty, \tag{1.3}$$

then the solution can be extended smoothly beyond T. Here $\nabla_h u = (\partial_1 u, \partial_2 u)$, $\nabla_h \omega = (\partial_1 \omega, \partial_2 \omega)$.

Motivated by the above references and therein, we try to reduce the condition on ω in (1.2) and (1.3). Our result is stated as follows.

Theorem 1.1. Suppose $T > 0$, the initial data $(u_0, \omega_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, and let (u, ω) be a weak solution of (1.1). If one of the following conditions holds:
(i) the weak solution u satisfies

$$\int_0^T \|\nabla_h u_h\|^q_{\dot{B}^0_{p,\frac{2p}{3}}} \mathrm{d}t < \infty, \ \textit{with} \ \frac{3}{p} + \frac{2}{q} = 2, \ \frac{3}{2} < p \le \infty, \tag{1.4}$$

(ii) u satisfies

$$\int_0^T \|\nabla_h u\|^{\frac{8}{3}}_{\dot{B}^{-1}_{\infty,\infty}} \mathrm{d}t < \infty, \tag{1.5}$$

(iii) u satisfies

$$\int_0^T \|\nabla_h u_h\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}} \mathrm{d}t < \infty, \textit{with} \ 0 < \alpha < 1, \tag{1.6}$$

then the solution (u, ω) is regular on $(0, T]$.

Remark 1.1. The condition on micro-rotational velocity is removed in (1.4), (1.5) and (1.6) and $\nabla_h u$ is replaced by $\nabla_h u_h$ in (1.4) and (1.6). Therefore, the (1.4), (1.5) and (1.6) are obviously improved results. What's more, for (1.4), when $p = \infty, q = 1$, the condition (1.4) can reduce being a part of (1.2). In addition, it should be pointed out that (1.4) and (1.5) like the supplementary cases when $\alpha = 0$ and $\alpha = 1$ for (1.6).

Remark 1.2. Noting that $\|\nabla u_h\|^{\frac{2}{2-\alpha}}_{\dot{B}^{-\alpha}_{\infty,\infty}} \approx \|u_h\|^{\frac{2}{2-\alpha}}_{\dot{B}^{1-\alpha}_{\infty,\infty}}$ and letting $1-\alpha = r$, the condition (1.6) can be substituted by

$$\int_0^T \|u_h\|^{\frac{2}{1+r}}_{\dot{B}^{r}_{\infty,\infty}} \mathrm{d}t < \infty,\ 0 < r < 1.$$

2. Preliminaries

In this section, we will review some preliminaries on the Littlewood-Paley decomposition theory and the definition of the homogeneous Besov spaces and introduce some useful inequalities, which will play an important role in the later proof. Let $S(\mathbb{R}^n)$ be the Schwartz class of rapidly decreasing functions, and $f \in S(\mathbb{R}^n)$. The Fourier transform $\mathcal{F}f = \hat{f}$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)e^{-ix\cdot\xi}\mathrm{d}x.$$

and the inverse Fourier transform $\mathcal{F}^{-1}f = \check{f}$ is defined by

$$\check{f}(\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^3} f(x)e^{ix\cdot\xi}\mathrm{d}x.$$

Let us choose a nonnegative radial function $\chi(\xi) \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \chi(\xi) \le 1$ and

$$\begin{cases} \chi(\xi) = 1,\ for\ |\xi| \le \frac{3}{4}, \\ \chi(\xi) = 0,\ for\ |\xi| > \frac{4}{3}, \end{cases}$$

and let $\hat{\varphi}(\xi) = \chi(\xi/2) - \chi(\xi)$, $\chi_j(\xi) = \chi(\frac{\xi}{2^j})$ and $\hat{\varphi}_j(\xi) = \hat{\varphi}(\frac{\xi}{2^j})$ for $j \in \mathbb{Z}$. Write

$$h(x) = \mathcal{F}^{-1}\chi, h_j(x) = 2^{nj}h(2^jx);$$
$$\varphi_j(x) = 2^{nj}\varphi(2^jx).$$

The Littlewood-Paley projection operators S_j and $\triangle_j$ are defined respectively as follows,

$$S_jf(x) = h_j * f(x) = 2^{3j}\int_{\mathbb{R}^3} h(2^jy)f(x-y)\mathrm{d}y,$$
$$\triangle_jf(x) = \varphi_j * f(x) = S_{j+1}f(x) - S_jf(x) = 2^{3j}\int_{\mathbb{R}^3} \varphi(2^jy)f(x-y)\mathrm{d}y.$$

Formally, $\triangle_j$ is a frequency projection to the annulus $|\xi| \sim 2^j$, while S_j is a frequency projection to the ball $|\xi| \sim 2^j$ for $j \in \mathbb{Z}$. For any $f \in L^2(\mathbb{R}^n)$, we have the Littlewood-Paley decomposition

$$f = h * f + \sum_{j\ge 0}\varphi_j * f(x),$$

$$f = \sum_{-\infty}^{+\infty}\varphi_j * f(x),$$

and it is not difficult to verify

$$suup\chi(\xi) \cap suup\hat{\varphi}_j(\xi) = \emptyset\ with\ j \ge 1,$$

$$suup\hat{\varphi}_i(\xi) \cap suup\hat{\varphi}_j(\xi) = \emptyset \ with \ |i-j| \geq 2.$$

For details of Littlewood-Paley theory, one may refer to [**1**, Chapter 2]. With the help of the definitions above, we now give the definition of the homogeneous Besov space.

Definition 2.1. Let $s \in \mathbb{R}$, $p, q \in [1, \infty]$, the homogeneous Besov space is defined by

$$\dot{B}^s_{p,q} = \{f \in \mathcal{Z}'(\mathbb{R}^n); \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\dot{B}^s_{p,q}} = (\sum_{-\infty}^{+\infty} 2^{jsq} \|\varphi_j * f\|^q_{L^p})^{1/q},$$

and $\mathcal{Z}'(\mathbb{R}^n)$ represents the dual space of $\mathcal{Z}(\mathbb{R}^n)$

$$\mathcal{Z}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n); \mathcal{D}^\alpha \hat{f} = 0, \forall \alpha \in \mathbb{N}^n \ multi-index\}.$$

Next, we state some lemmas used in the proof of our main results. The first one is the classical Bernstein inequality (see [**1**, page 52]).

Lemma 2.1. Assume k is a nonnegative integer, $1 \leq p \leq q \leq \infty$, then the following holds

$$\sup_{\alpha=k} \|\partial^\alpha \triangle_j f\|_{L^q} \leq C 2^{jk+3j(\frac{1}{p}-\frac{1}{q})} \|\triangle_j f\|_{L^p}, \tag{2.1}$$

here C is a positive constant independent of f, j.

Lemma 2.2. (Page 2 in [**5**]) Let $f \in H^1(\mathbb{R}^3)$,$2 \leq p \leq 6$, then exists a constant C such that

$$\|f\|_{L^p} \leq C \|f\|^{\frac{6-p}{2p}}_{L^2} \|\partial_1 f\|^{\frac{p-2}{2p}}_{L^2} \|\partial_2 f\|^{\frac{p-2}{2p}}_{L^2} \|\partial_3 f\|^{\frac{p-2}{2p}}_{L^2}. \tag{2.2}$$

Lemma 2.3. (Page 82 in [**1**]). Let $1 < q < p < \infty$ and α be a positive real number. Then there exists a constant C such that

$$\|f\|_{L^p} \leq C \|f\|^{1-\theta}_{\dot{B}^{-\alpha}_{\infty,\infty}} \|f\|^{\theta}_{\dot{B}^{\beta}_{q,q}}, with \ \beta = \alpha(\frac{p}{q} - 1), \theta = \frac{q}{p}. \tag{2.3}$$

In particular, when $\beta = 1$, $q = 2$ and $p = 4$, we have $\alpha = 1$ and

$$\|f\|_{L^4} \leq C \|f\|^{\frac{1}{2}}_{\dot{B}^{-1}_{\infty,\infty}} \|\nabla f\|^{\frac{1}{2}}_{L^2}. \tag{2.4}$$

Lemma 2.4. ([**13**, Lemma 5]) Let $f \in \dot{B}^{-r}_{\infty,\infty}(R^3)$, $g, h \in H^1(R^3)$ and for any $\epsilon > 0$, $0 < r < 1$, we have

$$\int_{\mathbb{R}^3} fgh\mathrm{d}x \leq C \|f\|^{\frac{2}{2-r}}_{\dot{B}^{-r}_{\infty,\infty}} (\|g\|^2_{L^2} + \|h\|^2_{L^2}) + \epsilon(\|\nabla g\|^2_{L^2} + \|\nabla h\|^2_{L^2}).$$

Lemma 2.5. Assume the initial data $(u_0, \omega_0) \in H^1(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$, u and θ is a pair weak solution of the system (1.1). If the following H^1 estimate of u is satisfied

$$\|\nabla u\|^2_{L^2} + \int_0^T \|\Delta u\|^2_{L^2} \mathrm{d}t < \infty, \tag{2.5}$$

then we have

$$\|\nabla \omega\|^2_{L^2} + \int_0^T \|\Delta \omega\|^2_{L^2} \mathrm{d}t \leq C.$$

Proof Firstly, it is not difficult to get the L^2 estimates of u and ω. Multiplying u and ω to the equations $(1.1)_1$ and $(1.1)_2$ respectively, then integrating it on $\mathbb{R}^3$, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)+\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+2\|\omega\|_{L^2}^2+\|\nabla\cdot\omega\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(\nabla\times\omega)u\mathrm{d}x+\int_{\mathbb{R}^3}(\nabla\times u)\omega\mathrm{d}x\le C\int_{\mathbb{R}^3}|\nabla\omega||u|\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla u||\omega|\mathrm{d}x\\
&\le C(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)+\frac{1}{2}(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2).
\end{aligned}$$

Hence it can be inferred by Gronwall's inequality

$$\begin{aligned}
&(\|u\|_{L^2}^2+\|\omega\|_{L^2}^2)+\int_0^T(\|\nabla u\|_{L^2}^2+\|\nabla\omega\|_{L^2}^2+4\|\omega\|_{L^2}^2+2\|\nabla\cdot\omega\|_{L^2}^2)\mathrm{d}t\\
&\le C(\|u_0\|_{L^2}^2+\|\omega_0\|_{L^2}^2).
\end{aligned}$$

Next, multiplying the second equation of (1.1) by $-\Delta\omega$ and integrating it on $\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+2\|\nabla\omega\|_{L^2}^2+\|\nabla\nabla\cdot\omega\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(u\cdot\nabla\omega)\cdot\Delta\omega\mathrm{d}x-\int_{\mathbb{R}^3}\nabla\times u\cdot\Delta\omega\mathrm{d}x.
\end{aligned}\tag{2.6}$$

By Hölder's inequality and the Gagliardo-Nirenberg inequality, we have that

$$\begin{aligned}
&\int_{\mathbb{R}^3}(u\cdot\nabla\omega)\cdot\Delta\omega\mathrm{d}x\le C\|u\|_{L^\infty}\|\nabla\omega\|_{L^2}\|\Delta\omega\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\Delta u\|_{L^2}^{\frac{1}{2}}\|\nabla\omega\|_{L^2}\|\Delta\omega\|_{L^2}\\
&\le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\|\nabla\omega\|_{L^2}^2+\frac{1}{8}\|\Delta\omega\|_{L^2}^2\\
&\le C\|\nabla\omega\|_{L^2}^4\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta\omega\|_{L^2}^2.
\end{aligned}\tag{2.7}$$

For the second term in right of (2.6), it is clear that by Hölder's inequality and Young's inequality that

$$-\int_{\mathbb{R}^3}\nabla\times u\cdot\Delta\omega\mathrm{d}x\le C\|\nabla u\|_{L^2}\|\Delta\omega\|_{L^2}\le C\|\nabla u\|_{L^2}^2+\frac{1}{4}\|\Delta\omega\|_{L^2}^2.\tag{2.8}$$

Then plugging the above estimates (2.7) and (2.8) into (2.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\omega\|_{L^2}^2+\|\Delta\omega\|_{L^2}^2+4\|\nabla\omega\|_{L^2}^2+2\|\nabla\nabla\cdot\omega\|_{L^2}^2\le C(1+\|\nabla\omega\|_{L^2}^4)\|\nabla u\|_{L^2}^2.\tag{2.9}$$

Finally applying the L^2 norm of ∇u in (2.5) and Gronwall's inequality, it yields

$$\begin{aligned}
&\|\nabla\omega\|_{L^2}^2+\int_0^T(\|\Delta\omega\|_{L^2}^2+4\|\nabla\omega\|_{L^2}^2+2\|\nabla\nabla\cdot\omega\|_{L^2}^2)\mathrm{d}t\\
&\le C(1+\|\omega_0\|_{L^2}^2)\|\nabla\omega_0\|_{L^2}^2\le C.
\end{aligned}\tag{2.10}$$

Thus the proof of Lemma 2.4 is completed.

3. Proof of the Main Theorem

In this section, we are dedicated to proving Theorem 1.1. We first deal with (i), showing the validness of condition (1.4). At first, applying ∇ to the first equation in system (1.1), and multiplying the resulting equation with ∇u, then integrating it on $\mathbb{R}^3$, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\|\Delta u\|_{L^2}^2=-\int_{\mathbb{R}^3}\nabla(u\cdot\nabla u)\cdot\nabla u\mathrm{d}x+\int_{\mathbb{R}^3}\nabla(\nabla\times\omega)\cdot\nabla u\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\sum_{i=1}^{2}\partial_k u_i\partial_i u_j\partial_k u_j\mathrm{d}x+\int_{\mathbb{R}^3}\sum_{j,k=1}^{3}\partial_k u_3\partial_3 u_j\partial_k u_j\mathrm{d}x-\int_{\mathbb{R}^3}\nabla(\nabla\times\omega)\cdot\nabla u\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\sum_{k=1}^{3}\sum_{i,j=1}^{2}\partial_k u_i\partial_i u_j\partial_k u_j\mathrm{d}x+\int_{\mathbb{R}^3}\sum_{k=1}^{3}\sum_{i=1}^{2}\partial_k u_i\partial_i u_3\partial_k u_3\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\sum_{k=1}^{3}\sum_{j=1}^{2}(\partial_k u_3\partial_3 u_j\partial_k u_j+\partial_k u_3\partial_3 u_3\partial_k u_3)\mathrm{d}x-\int_{\mathbb{R}^3}\nabla(\nabla\times\omega)\cdot\nabla u\mathrm{d}x\\
&=\int_{\mathbb{R}^3}\sum_{k=1}^{3}\sum_{i,j=1}^{2}\partial_k u_i\partial_i u_j\partial_k u_j\mathrm{d}x+\int_{\mathbb{R}^3}\sum_{k=1}^{2}\sum_{i=1}^{2}(\partial_k u_i\partial_i u_3\partial_k u_3+\partial_3 u_i\partial_i u_3\partial_3 u_3)\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\sum_{k=1}^{2}\sum_{j=1}^{2}(\partial_k u_3\partial_3 u_j\partial_k u_j+\partial_3 u_3\partial_3 u_j\partial_3 u_j+\partial_k u_3\partial_3 u_3\partial_k u_3)\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^3}\nabla(\nabla\times\omega)\cdot\nabla u\mathrm{d}x\\
&\leq C\int_{\mathbb{R}^3}|\nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x+\int_{\mathbb{R}^3}|\nabla\omega||\Delta u|\mathrm{d}x=I_1+I_2.
\end{aligned}\tag{3.1}$$

Bringing the Littlewood-Paley formal decomposition to $\nabla_h u_h$ yields

$$\nabla_h u_h=\sum_{j<-N}\triangle_j\nabla_h u_h+\sum_{j=-N}^{j=N}\triangle_j\nabla_h u_h+\sum_{>N}\triangle_j\nabla_h u_h.$$

Therefore, the term I_1 can be rewritten as follows

$$\begin{aligned}
I_1&\leq C\sum_{j<-N}\int_{\mathbb{R}^3}|\triangle_j\nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x+C\sum_{j=-N}^{j=N}\int_{\mathbb{R}^3}|\triangle_j\nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x\\
&\quad+C\sum_{j>N}\int_{\mathbb{R}^3}|\triangle_j\nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x\\
&=I_{11}+I_{12}+I_{13}.
\end{aligned}$$

In the following, the terms I_{11}, I_{12}, I_{13} will be bounded one by one. Firstly for I_{11}, by the virtue of Hölder's inequality and Bernstein's inequality, it can be deduced that

$$\begin{aligned} I_{11} &= C\sum_{j<-N}\int_{\mathbb{R}^3} |\triangle_j \nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x \le C\|\nabla u\|_{L^2}^2 \sum_{j<-N}\|\triangle_j \nabla_h u_h\|_{L^\infty} \\ &\le C\|\nabla u\|_{L^2}^2 \sum_{j<-N} 2^{\frac{3}{2}j}\|\triangle_j \nabla_h u_h\|_{L^2} \\ &\le C2^{-\frac{3}{2}N}\|\nabla_h u_h\|_{L^2}\|\nabla u\|_{L^2}^2. \end{aligned} \tag{3.2}$$

For I_{12}, making use of Hölder's inequality and the Definition 2.1, we arrive at

$$\begin{aligned} I_{12} &= C\sum_{j=-N}^{j=N}\int_{\mathbb{R}^3} |\triangle_j \nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x \le C\sum_{j=-N}^{j=N}\|\triangle_j \nabla_h u_h\|_{L^p}\|\nabla u\|^2_{L^{\frac{2p}{p-1}}}, \\ &\le C\left(\sum_{j=-N}^{j=N} 1^{\frac{2p}{2p-3}}\right)^{\frac{2p-3}{2p}}\left(\sum_{j=-N}^{j=N}\|\triangle_j \nabla_h u_h\|_{L^p}^{\frac{2p}{3}}\right)^{\frac{3}{2p}}\|\nabla u\|^2_{L^{\frac{2p}{p-1}}} \\ &\le C\|\nabla_h u_h\|_{\dot{B}^0_{p,\frac{2p}{3}}}\|\nabla u\|_{L_2}^{\frac{2p-3}{p}}\|\Delta u\|_{L_2}^{\frac{3}{p}} \\ &\le C\|\nabla_h u_h\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}}\|\nabla u\|_{L_2}^2 + \frac{1}{8}\|\Delta u\|_{L_2}^2. \end{aligned} \tag{3.3}$$

Similarly as above, by Gagliardo-Nirenberg's inequality and Hölder's inequality, we have

$$\begin{aligned} I_{13} &= C\sum_{j>N}\int_{\mathbb{R}^3} |\triangle_j \nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x \le C\|\nabla u\|_{L^3}^2 \sum_{j<-N}\|\triangle_j \nabla_h u_h\|_{L^3} \\ &\le C\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}\sum_{j>N} 2^{\frac{1}{2}j}\|\triangle_j \nabla_h u_h\|_{L^2} \\ &\le C\left(\sum_{j>N} 2^{-j}\right)^{\frac{1}{2}}\left(\sum_{j>N} 2^{2j}\|\triangle_j \nabla_h u_h\|_{L^2}\right)^{\frac{1}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2} \\ &\le C2^{\frac{-N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2. \end{aligned} \tag{3.4}$$

Now, collecting the estimates (3.2)-(3.4), we get

$$I_1 \le C2^{-\frac{3}{2}N}\|\nabla u\|_{L^2}^3 + C\|\nabla_h u_h\|_{\dot{B}^0_{p,\frac{2p}{3}}}^{\frac{2p}{2p-3}}\|\nabla u\|_{L_2}^2 + C2^{-\frac{N}{2}}\|\nabla u\|_{L^2}\|\Delta u\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L_2}^2. \tag{3.5}$$

And it is easy to check that by Hölder's inequality and Young's inequality

$$I_2 = \int_{\mathbb{R}^3} |\nabla\omega||\Delta u|\mathrm{d}x \le C\|\nabla\omega\|_{L^2}\|\Delta u\|_{L^2} \le C\|\nabla\omega\|_{L^2}^2 + \frac{1}{8}\|\Delta u\|_{L^2}^2. \tag{3.6}$$

For I_1, We can choose N such that

$$C2^{-\frac{N}{2}}\|\nabla u\|_{L^2} \le \frac{1}{4},$$

and it can be gotten

$$N \ge \frac{2\log(1 + C\|\nabla u\|_{L^2})}{\log 2} + 4.$$

Therefore, adding the estimates (3.5) and (3.6) into (3.1), combining condition (1.4) and applying Gronwall's inequality, it yields

$$\|\nabla u\|_{L^2}^2 + \int_0^T \|\Delta u\|_{L^2}^2 \mathrm{d}t < C. \tag{3.7}$$

Finally, employing the Lemma 2.5 with above (3.7), we can derive that

$$\|\nabla u\|_{L^2}^2 + \|\nabla \omega\|_{L^2}^2 + \int_0^T (\|\Delta u\|_{L^2}^2 + \|\Delta \omega\|_{L^2}^2)\mathrm{d}t < C. \tag{3.8}$$

In the following part, we will show (ii) and (iii). Firstly, we prove the (ii) that the solution is regular under condition (1.5). Taking ∇_h on the first equation in system (1.1), multiplying $\nabla_h u$ to the resulting equation and integrating it on $\mathbb{R}^3$, one has

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h u\|_{L^2}^2 + \|\nabla_h \nabla u\|_{L^2}^2 = &-\int_{\mathbb{R}^3} \nabla_h(u\cdot\nabla u)\cdot\nabla_h u \mathrm{d}x \\ &+ \int_{\mathbb{R}^3} \nabla_h(\nabla\times\omega)\cdot\nabla_h u\mathrm{d}x := J_1 + J_2.\end{aligned} \tag{3.9}$$

By Hölder's inequality and (2.4), J_1 can be bounded as:

$$\begin{aligned}J_1 = &-\int_{\mathbb{R}^3} \nabla_h u\cdot\nabla u\cdot\nabla_h u\mathrm{d}x \le \int_{\mathbb{R}^3} |\nabla_h u||\nabla_h u||\nabla u|\mathrm{d}x \\ &\le C\|\nabla_h u\|_{L^4}^2\|\nabla u\|_{L^2} \le C\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}\|\nabla\nabla_h u\|_{L^2}\|\nabla u\|_{L^2} \\ &\le C\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2 + \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2.\end{aligned} \tag{3.10}$$

For J_2, we have using Hölder's inequality and Young's inequality that

$$J_2 \le C\|\nabla\omega\|_{L^2}\|\nabla\nabla_h u\|_{L^2} \le C\|\nabla\omega\|_{L^2}^2 + \frac{1}{4}\|\nabla\nabla_h u\|_{L^2}^2. \tag{3.11}$$

Adding (3.10) and (3.11) into (3.9), then integrating on $[0,t), 0<t<T$, we have

$$\|\nabla_h u\|_{L^2}^2 + \int_0^t \|\nabla\nabla_h u\|_{L^2}^2\mathrm{d}t \le \|\nabla_h u_0\|_{L^2}^2 + \int_0^t \|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2\mathrm{d}t + C. \tag{3.12}$$

With this result above, we re-estimate I_1 in (3.1). Employing Lemma 2.2, it can be arrived that

$$\begin{aligned}I_1 \le &C\int_{\mathbb{R}^3} |\nabla_h u||\nabla u||\nabla u|\mathrm{d}x \le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^4}^2 \\ &\le C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}}.\end{aligned} \tag{3.13}$$

Combining (3.1), (3.6) and (3.13), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2 + \int_0^t \|\Delta u\|_{L^2}^2\mathrm{d}t < C\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla\nabla_h u\|_{L^2}\|\Delta u\|_{L^2}^{\frac{1}{2}} + C\|\nabla\omega\|_{L^2}^2. \tag{3.14}$$

Integrating above (3.14) on time, and making use of Hölder's inequality and inquality (3.12) , we infer that

$$\|\nabla u\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}t\le\|\nabla u_0\|_{L^2}^2+\int_0^t\|\nabla_h u\|_{L^2}\|\nabla u\|_{L^2}^{\frac12}\|\nabla\nabla_h u\|_{L^2}\|\Delta u\|_{L^2}^{\frac12}\mathrm{d}t+C$$

$$\le C+\|\nabla u_0\|_{L^2}^2+C(\sup_{0\le s\le t}\|\nabla_h u\|_{L^2})\left(\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}t\right)^{\frac14}\times\left(\int_0^t\|\nabla\nabla_h u\|_{L^2}^2\mathrm{d}t\right)^{\frac12}$$

$$\times\left(\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}t\right)^{\frac14}$$

$$\le C+\|\nabla u_0\|_{L^2}^2+C\left(1+\|\nabla u_0\|_{L^2}^2+\int_0^t\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2\mathrm{d}t\right)$$

$$\times\left(\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}t\right)^{\frac14}. \tag{3.15}$$

By utilising Hölder's inequality and Young's inequality, one has

$$\begin{aligned}\|\nabla u\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}t&\le C(1+\|\nabla u_0\|_{L^2}^2)+C\|\nabla u_0\|_{L^2}^{\frac83}\\&\quad+C\left(\int_0^t\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^2\|\nabla u\|_{L^2}^2\mathrm{d}t\right)^{\frac43}\\&\le C(1+\|\nabla u_0\|_{L^2}^2)+C\|\nabla u_0\|_{L^2}^{\frac83}\\&\quad+C\left(\int_0^t\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac83}\|\nabla u\|_{L^2}^2\mathrm{d}t\right)\left(\int_0^t\|\nabla u\|_{L^2}^2\mathrm{d}t\right)^{\frac13}\\&\le C(1+\|\nabla u_0\|_{L^2}^2)+C\|\nabla u_0\|_{L^2}^{\frac83}\\&\quad+C\left(\int_0^t\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac83}\|\nabla u\|_{L^2}^2\mathrm{d}t\right).\end{aligned} \tag{3.16}$$

Finally, we deduce from the Gronwall's inequality that

$$\|\nabla u\|_{L^2}^2+\int_0^t\|\Delta u\|_{L^2}^2\mathrm{d}t\le C(1+\|\nabla u_0\|_{L^2}^2+\|\nabla u_0\|_{L^2}^{\frac83})\times\exp\left(\int_0^t\|\nabla_h u\|_{\dot{B}^{-1}_{\infty,\infty}}^{\frac83}\mathrm{d}t\right). \tag{3.17}$$

Then applying Lemma 2.5, we can get the H^1 norm of ω, which completes the proof of (ii).

Now we prove (iii) and we estimate I_1 again in another way. Using Lemma 2.4, it follows that

$$\begin{aligned}I_1\le&C\int_{\mathbb{R}^3}|\nabla_h u_h||\nabla u||\nabla u|\mathrm{d}x\\&\le C\|\nabla_h u_h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}\|\nabla u\|_{L^2}^2+\frac18\|\Delta u\|_{L^2}^2.\end{aligned} \tag{3.18}$$

Combining (3.1), (3.6) and (3.18), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|_{L^2}^2+\int_0^T\|\Delta u\|_{L^2}^2\mathrm{d}t\le C\|\nabla_h u_h\|_{\dot{B}^{-\alpha}_{\infty,\infty}}^{\frac{2}{2-\alpha}}\|\nabla u\|_{L^2}^2+C\|\nabla\omega\|_{L^2}^2. \tag{3.19}$$

Then employing the Gronwall's inequality for (3.19), the H^1 norm of u is available. Finally, applying Lemma 2.5, the H^1 norm of ω can be achieved. This completes the proof of Theorem 1.1.

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