NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 48 (2018), 115-119

SPLITTING OF CLOSED SUBGROUPS OF LOCALLY COMPACT ABELIAN GROUPS

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(Received August 24, 2018)

Abstract. Let $\pounds$ be the category of all locally compact abelian (LCA) groups. In this paper, we investigate the splitting of the identity component, the subgroup of all compact elements and the maximal torsion subgroup of a group $G \in \pounds$. A group $G \in \pounds$ will be called split full if every closed subgroup of G splits. In this paper, we give a necessary condition for an LCA group to be split full.

1. Introduction

All groups considered in this paper are Hausdorff topological abelian groups and they will be written additively. For a group G, we denote by tG, the maximal torsion subgroup of G defined by $tG = \bigcup_{n\in\mathbb{N}} G[n]$ where $G[n]$ is the subgroup of G defined by $G[n] = \{x \in G; nx = 0\}$ for all positive integers n. An element $g \in G$ is called compact if the smallest closed subgroup which its contains is compact [**4**, Definition 9.9]. We denote by bG, the subgroup of all compact elements of G. For any group G, G_0 is the identity component of G. Let H be a closed subgroup of G. We say that H splits in G, if G contains a closed subgroup K such that $H \bigcap K = 0$ and the map $H \times K \to G$, $(h, k) \longmapsto h + k$ is a homeomorphism. Let $\pounds$ denote the category of all locally compact abelian groups with continuous homomorphisms as morphisms. A morphism is called proper if it is open onto its image, and a short exact sequence $0 \to A \xrightarrow{\phi} B \xrightarrow{\psi} C \to 0$ in $\pounds$ is said to be an extension of A by C if ϕ and ψ are proper morphism. We let $Ext(C, A)$ denote the group of extensions of A by C [**3**]. For a group G in $\pounds$, tG need not be split in G [**8**]. Let G^* be the minimal divisible extension of G [**4**, A.15] . We proved that if G is torsion-closed, then, tG^* the maximal torsion subgroup of G^* splits in G^* ([**7**, Theorem 4.3]). In [**5**], Loth studied the splitting of G_0 and bG of an LCA group G. In Section 1, we study the splitting of G_0, bG and tG under certain conditions on $G \in \pounds$. We show that if G is torsion-free, then G_0 splits in G (see Theorem (2.6)). If bG is compact, then bG splits in G (Theorem (2.8)). We also show that if G is densely divisible, then bG splits in G (Lemma 2.13). A group G is called torsion-closed if tG is closed in G. In [**7**], we determined the LCA torsion-closed groups. We prove that the subgroup tG of a torsion-closed group G splits if and only if G is a direct sum of a torsion group and a torsion-free group (see Theorem 2.17).

We say that an LCA group G is split full if every closed subgroup of G splits. We show that if an LCA group G is split full, then $G \cong \prod_{p\in I\subseteq P}(\prod_{n_p} \mathbb{Z}(p)) \oplus$

2010 *Mathematics Subject Classification* 20K35, 22B05.
Key words and phrases: Split, Split Full, Locally Compact Abelian Groups.
Part of this work is supported by Mollasadra Technical and Vocational College.

$(\bigoplus_{q\in I\subseteq P}(\bigoplus_{n_q}\mathbb{Z}(q)))$ where P is the set of all prime numbers, n_p and n_q are some cardinals and $I, J \subseteq P$ (see Theorem 3.9).

The additive topological group of real numbers is denoted by $\mathbb{R}$, $\mathbb{Z}$ is the group of integers with the discrete topology and $Z(n)$ is the cyclic group of order n. For a closed subgroup H of G, $H \hookrightarrow G$ is the inclusion and $\pi : G \to G/H$ is the natural mapping. For any group G and H, $Hom(G, H)$ is the group of all continuous homomorphisms from G to H, endowed with the compact-open topology. The dual group of G is $\hat{G} = Hom(G, \mathbb{R}/\mathbb{Z})$ and $(\hat{G}, S)$ denotes the annihilator of $S \subseteq G$ in $\hat{G}$. For more on locally compact abelian groups, see [**4**].

2. Splitting of G_0, bG and tG in an LCA Group G

Let G be a group in $\pounds$. In this section, we prove the following statements:

(1) If G is a torsion-free group, then G_0 splits in G.
(2) If bG is compact, then bG splits in G.
(3) Let G be torsion-closed. tG splits in G if and only if G is a direct sum of a torsion group and a torsion-free group.

Lemma 2.1. *Let $G \in \pounds$ and H be a closed subgroup of G. If $Ext(G/H, H) = 0$, then H splits in G.*

Proof. It is clear by the definition of Ext. □

Definition 2.2. A subgroup H of G is called pure if $nH = H \bigcap nG$ for every positive integer n [**1**].

Remark 2.3. (a) Every divisible subgroup of G is pure.
(b) Let $G \in \pounds$. Then, G_0 is connected. By [**4**, Theorem 24.25], a connected LCA group is divisible. So G_0 is a pure subgroup of G.

Lemma 2.4. *Let G be a torsion-free group in $\pounds$ and H a pure subgroup of G. Then G/H is a torsion-free group.*

Proof. Let $n(x + H) = H$ for some positive integer n and $x \in G$. Then $nx \in H$. Since H is a pure subgroup of G, so $nx = nh$ for some $h \in H$. Hence $n(x - h) = 0$. Since G is torsion-free, so $x = h$. So $x+H = H$ and G/H is a torsion-free group. □

Definition 2.5. A group $G \in \pounds$ is called an $\pounds-$cotorsion group if $Ext(X, G) = 0$ for all torsion-free groups $X \in \pounds$ [**2**].

Theorem 2.6. *Let G be a torsion-free group in $\pounds$. Then G_0 splits in G.*

Proof. Let G be a torsion-free group. By (b) of Remark 2.3, G_0 is a pure subgroup of G. Since G is torsion-free, G/G_0 is a torsion-free group (see Lemma 2.4). By [**2**, Corollary 6], G_0 is an $\pounds-$cotorsion group. So, $Ext(G/G_0, G_0) = 0$. Hence, by Lemma 2.1, G_0 splits in G. □

Lemma 2.7. *Let $G \in \pounds$. If G contains a compact open subgroup, then G/bG is a discrete torsion-free group.*

Proof. Let K be a compact open subgroup of G. Then $(\hat{G}, K)$ is a compact open subgroup of $\hat{G}$. By [**4**, Theorem 24.17], $(\hat{G})_0 = (\hat{G}, bG)$. Since $(\hat{G})_0$ is the intersection of all open subgroups of $\hat{G}$, so $(\hat{G}, bG) \subseteq (\hat{G}, K)$. Hence, $(\hat{G}, bG)$ is a compact group. By [**4**, Theorem 23.25], $\widehat{(G/bG)} \cong (\hat{G}, bG) = (\hat{G})_0$. So, $\widehat{(G/bG)}$ is a compact connected group. Hence, by [**4**, Theorem 24.25], G/bG is a discrete torsion free group. □

Theorem 2.8. *Let G be a group in $\pounds$ such that bG is compact. Then bG splits in G.*

Proof. Let G be a group in $\pounds$ such that bG is compact. By [**4**, Theorem 24.30], $G = \mathbb{R}^n \oplus G^{'}$ where $G^{'}$ contains a compact open subgroup K. So, $bG' = bG$ is compact. On the other hand, G/bG is a discrete torsion-free group (see Lemma 2.7). So, by [**1**, Proposition 53.4], $Ext(G/bG, bG) = Pext(G/bG, bG) = 0$. Hence, bG splits in G. □

Lemma 2.9. *Let $G \in \pounds$ and H be a closed subgroup of G. Then, $0 \to H \hookrightarrow G \to G/H \to 0$ splits if and only if $0 \to \widehat{(G/H)} \to \hat{G} \to \hat{H} \to 0$ splits.*

Proof. It is clear by [**3**, Theorem 2.12]. □

Corollary 2.10. *Let $G \in \pounds$ and H be a closed subgroup of G. Then, H splits in G if and only if $(\hat{G}, H)$ splits in G.*

Definition 2.11. A group $G \in \pounds$ is called densely divisible if G contains a dense divisible subgroup [**6**].

Lemma 2.12. *A group G is densely divisible if and only if $\hat{G}$ is a torsion-free group* [**6**].

Lemma 2.13. *Let G be a densely divisible group in $\pounds$. Then, bG splits in G.*

Proof. Let G be a densely divisible group in $\pounds$. By Lemma 2.12, $\hat{G}$ is a torsion-free group. So,$(\hat{G})_0 = (\hat{G}, bG)$ splits in G (see Theorem 2.6). Hence, by Corollary 2.10, bG splits in G. □

Definition 2.14. A group G is called torsion-closed if tG is closed in G [**7**].

Let $G \in \pounds$. In general, tG need not be closed in G [**7**].

Theorem 2.15. *A group G is torsion-closed if and only if $G = \mathbb{R}^n \oplus M \oplus G^{'}$ where n is a nonnegative integer, M a compact connected torsion-free group and G' containing a compact open subgroup $K \cong \prod_{i \in I} \mathbb{Z}/p_i^{r_i}\mathbb{Z} \times \prod_p {\triangle_p}^{n_p}$* [**7**, Theorem 3.5].

Let G be a torsion-closed group in $\pounds$. In general, tG need not be split in G.

Example 2.16. Let p be a prime number. Consider the group $G = \prod_{n \in \mathbb{N}} \mathbb{Z}(p^n)$ with discrete topology. If tG splits in G, then G should be contain a torsion-free subgroup which is a contradiction. Hence, tG do not split in G.

Theorem 2.17. *Let G be a torsion-closed group in $\pounds$. Then, tG splits in G if and only if G is a direct sum of a torsion group and a torsion-free group.*

Proof. First, suppose that tG splits in G. Then the extension $0 \to tG \hookrightarrow G \xrightarrow{\pi} G/tG \to 0$ splits. So $G \cong tG \oplus G/tG$. Conversely is clear. □

3. Split Full LCA Groups

In this section, we define the concept of a split full LCA group and give a necessary condition for an LCA group to be split full.

Definition 3.1. A discrete abelian group is said to be elementary if it can be expressed as a sum of simple subgroups [**1**].

Remark 3.2. Let G be a discrete abelian group. If G is split full, then every subgroup of G is a direct summand. So, a discrete abelian group G is split full if and only if G is an elementary abelian group. But, elementary abelian groups has the form $\bigoplus_{p\in I\subseteq P}(\bigoplus_{n_p}\mathbb{Z}(p))$ where P is the set of all prime numbers, n_p are some cardinals and $I \subseteq P$ [**1**].

Definition 3.3. A group $G \in \pounds$ is called split full if every closed subgroup of G splits.

Lemma 3.4. *Let G be a group in $\pounds$. Then, G is split full if and only if $\hat{G}$ is split full.*

Proof. Let G be a split full group in $\pounds$ and H a closed subgroup of $\hat{G}$. Then, (G, H) splits in G (see Corollary 2.10). By [**4**, Theorem 24.10], $H = (\hat{G}, (G, H))$. So, H splits in $\hat{G}$. Hence, $\hat{G}$ is split full. Conversely, suppose that $\hat{G}$ is split full and H a closed subgroup of G. Then, $(\hat{G}, H)$ splits in $\hat{G}$. By Corollary 2.10, H splits in G. □

Theorem 3.5. *A compact group G is split full if and only if $G \cong \prod_{p\in I\subseteq P}(\prod_{n_p}\mathbb{Z}(p))$ where P is the set of all prime numbers, n_p are some cardinals and $I \subseteq P$.*

Proof. It is clear by Remark 3.2 and Lemma 3.4. □

Lemma 3.6. *A closed subgroup H of $G \in \pounds$ splits if and only if there exists a morphism $f : G \to H$ such that $f(x) = x$ for every $x \in H$.*

Proof. It is clear. □

Lemma 3.7. *Let G_1 and G_2 be two groups in $\pounds$. If $G_1 \oplus G_2$ is split full, then G_1 and G_2 are split full.*

Proof. Let H be a closed subgroup of G_1. Then, $H \oplus G_2$ is a closed subgroup of $G_1 \oplus G_2$. By Lemma 3.6, there exists a morphism $f : G_1 \oplus G_2 \to H \oplus G_2$ such that $f\,|_{H\oplus G_2} = 1\,|_{H\oplus G_2}$. It is clear that $g = \pi_1 f l_1 : G_1 \to H$ is a morphism such that $g\,|_H = 1\,|_H$. Similarly, it can be show that G_2 is split full. □

Remark 3.8. It is clear that $\mathbb{Z}$ does not split in $\mathbb{R}$. So, $\mathbb{R}$ is not split full.

Theorem 3.9. *Let G be a split full group in $\pounds$. Then,*

$$G \cong \prod_{p\in I\subseteq P}(\prod_{n_p}\mathbb{Z}(p)) \oplus (\bigoplus_{q\in I\subseteq P}(\bigoplus_{n_q}\mathbb{Z}(q)))$$

Where P is the set of all prime numbers, n_p and n_q are some cardinals and $I, J \subseteq P$.

Proof. Let G be a split full group in $\pounds$. By [**4**, Theorem 24.30], $G = \mathbb{R}^n \oplus G'$ where G' contains a compact open subgroup K. By Lemma 3.7, $\mathbb{R}^n$ and G' are split full. So, $n = 0$ and $G \cong K \oplus (G/K)$. Hence, by Remark 3.2 and Theorem 3.5, $G \cong \prod_{p\in I\subseteq P}\mathbb{Z}(p) \oplus (\bigoplus_{q\in J\subseteq P}\mathbb{Z}(q))$. □

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