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BENT HALF-SPACE MODEL PROBLEM FOR LAMÉ EQUATION WITH SURFACE TENSION

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Abstract. The study of fluid flow is a very fascinating area of fluid dynamics. Fluid motion has received more and more attention in recent years and numerous researchers have looked into this topic. However, they rarely used a mathematical analysis approach to analyse fluid motion; instead, they used numerical analysis. This serves as a significant justification for the researcher's decision to study fluid flow from the perspective of mathematical analysis. In this paper, we consider the $\mathcal{R}$-boundedness of the solution operator families of the Lamé equation with surface tension in bent half-space model problem by taking into account the surface tension in a bounded domain of N-dimensional Euclidean space ($N \geq 2$). The motion of the model problem can be described by linearizing an equation system of a model problem. This research is a continuation of [**13**]. They investigated the $\mathcal{R}$-boundedness of the solution operator families in the half-space case for the model problem of the Lamé equation with surface tension. First of all, by using Laplace transformation we consider the resolvent of the model problem, then treat the problem in bent half-space case. By using Weis's operator-valued Fourier multiplier theorem, we know that $\mathcal{R}$-boundedness implies the maximal L_p-L_q regularity for the initial boundary value. This regularity is an essential tool for the partial differential equation problem.

1. Introduction

An important field of research in recent years has been the analysis of fluid-structure interaction issues. At the end of the 1990s, research focusing on well-posedness of fluid-structure interaction problems began. For example, [**2**] in 2008 investigated the well-posed of the Navier-Stokes motion in the exterior of a rotating obstacle. The Lamé equation describes motion of the solid structure.

Let us consider the motion of Navier Lamé equation in N-dimensional space $\mathbb{R}^N$ $(N \geq 2)$. We define the velocity field $\mathbf{u}(x,t) = (u_1(x,t), \cdots, u_N(x,t))^T$ at position $x \in \Omega$ and time $t > 0$, with Ω a bounded domain in the N-dimensional space $\mathbb{R}^N$ $(N \geq 2)$. The formula of the Navier-Lamé equation with free surface in a bounded domain with surface tension is (see [**13**]):

$$\begin{cases} \partial_t \mathbf{u} - \operatorname{Div}(\alpha \mathbf{D}(\mathbf{u}) + (\beta - \alpha)\operatorname{div}\mathbf{u}\mathbf{I}) = \mathbf{f} & \text{in } \Omega, \\ \mathbf{S}(\mathbf{u})\mathbf{n}_t = \sigma \mathcal{H}(\Gamma_t)\mathbf{n}_t & \text{on } \Gamma_t, \\ V_N = \mathbf{n}_t \cdot \mathbf{u} & \text{on } \Gamma_t, \end{cases} \tag{1.1}$$

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for $0 < t < T$, where

$$\mathbf{S}(\mathbf{u}) = \alpha\mathbf{D}(\mathbf{v}) + (\beta - \alpha)\mathrm{div}\,\mathbf{u}\mathbf{I}, \tag{1.2}$$

$\mathbf{u} = (u_1, \ldots, u_N)$, $\mathbf{D}(\mathbf{u})$ is the doubled deformation tensor whose (i, j) components are

$$D_{ij}(\mathbf{u}) = \partial_i u_j + \partial_j u_i, \quad (\partial_i = \partial/\partial x_i), \tag{1.3}$$

$\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{n}_t$ is the unit outer normal to Γ_t. Furthermore, α and β are the first and second viscosity coefficients, respectively, $\mathcal{H}(\Gamma_t)$ is the $N-1$ times mean curvature of Γ_t, which is given by $\mathcal{H}(\Gamma_t)\mathbf{n}_t = \Delta_{\Gamma_t} x$ for $x \in \Gamma_t$, where Δ_{Γ_t} is the Laplace-Beltrami operator on Δ_{Γ_t}. For any matrix field $\mathbf{K}$ whose components are K_{ij}, the quantity $\mathrm{Div}\,\mathbf{K}$ is an N vector whose i-th component is $\sum_{j=1}^N \partial_j K_{ij}$. We define

$$\mathrm{div}\,\mathbf{u} = \sum_{j=1}^N \partial_j u_j, \quad (\mathbf{u} \cdot \nabla \mathbf{u}) = \sum_{j=1}^N u_j \partial_j u_i, \tag{1.4}$$

where the second equation of (1.4) is an N vector whose i-th component is written above. Furthermore, V_N denotes the evolution speed of the intersurface Γ_t in the $\mathbf{n}_t$ direction. By applying the Laplace transform to (1.1), we have following resolvent equation

$$\begin{cases} \lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{g} & \text{in } \Omega, \\ (\alpha\mathbf{D}(\mathbf{u}) + (\beta - \alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta'_\Gamma \eta)\mathbf{n} = \mathbf{k} & \text{on } \Gamma, \\ \lambda\eta + \mathbf{a}' \cdot \nabla'\eta - \mathbf{u} \cdot \mathbf{n} = d & \text{on } \Gamma, \end{cases} \tag{1.5}$$

where $\mathbf{a}' = (a_1, \ldots, a_{N-1}) \in \mathbb{R}^{N-1}$ and $\mathbf{a}' \cdot \nabla'\eta = \sum_{j=1}^{N-1} a_j \partial_j \eta$. The function α is uniformly continuous with respect to $x \in \Omega$ and satisfies the assumptions:

$$\rho_*/2 \le \alpha(x) \le 2\rho_*, \tag{1.6}$$

while β is a positive constant. There is much research in the existence of an $\mathcal{R}$-bounded solution operator to the resolvent problem related to the linearized problem of Navier-Lamé (1.5). It includes the $\mathcal{R}$-sectoriality for the model problem in the whole space by using the Green function [**4**], in half-space by using the estimate of Poisson kernels [**4**] based on a multiplier theorem due to Shibata and Shimizu [**15**].

In 2021, the existence of the $\mathcal{R}$-bounded solution operator of problem (1.5) in the case where $\sigma > 0$ and $\mathbf{a}' = 0$ has been proved in [**13**] by using the Weis operator-valued Fourier multiplier theorem [**16**]. The existence of the $\mathcal{R}$-boundedness solution operator of the problem (1.1) implies not only the maximal L_p-L_q regularity but also the generation of an analytic semigroup. In the present article, we shall prove the existence on an $\mathcal{R}$-bounded solution operator for the Problem (1.5) in the case where $\sigma > 0$ and $\mathbf{a}' \neq 0$ in a bent half-space.

We believe that our main result Theorem 2.9 is an important contribution to the analysis of regularity of the model problem, even though we only take into account the bent half-space case.

We would also like to emphasize that the challenging part of our proof is finding operator families of the model problem that are appropriate to deal with the Lamé equation system's nonlinearities and for which we are able to obtain $\mathcal{R}$-boundedness results for the linearized model of (1.5). Shibata [**14**] proved the existence of an

$\mathcal{R}$-bounded solution operator of the Stokes equations subject to free boundary conditions.

Many researchers have recently been interested in researching fluid motion in the compressible situation. Murata [**8**] looked into the $\mathcal{R}$-boundedness with slip boundary condition in 2014. However, we only know about the maximal L_p-L_q regularity on a bounded domain and some unbounded domains that satisfy some uniformity; and we know about global well-posedness in the bounded domain case for the compressible non-newtonian viscous barotropic fluid flow of the Oldroyd-B type without surface tension with free surface thanks to [**10, 11**].

By accounting for the surface tension at the boundary, this work extends Maryani's conclusion [**13**]. Investigating the $\mathcal{R}$-boundedness of the solution operator families for the Navier-Lamé equation with surface tension in bent half-space problem is the main goal of this research. We start with finding solving the case $\mathbf{a} \neq 0$ and $\sigma > 0$ in half-space. This kind of research has recently proven to be very helpful in the study of fluid mechanics. We then consider the equation system

$$\begin{cases} \lambda\mathbf{v} - \alpha_0(y_0)\Delta\mathbf{v} - \beta\nabla\mathrm{div}\,\mathbf{v} = \mathbf{f} & \text{in } \Omega_+, \\ (\alpha_0(y_0)\mathbf{D}(\mathbf{v}) - (\beta - \alpha_0(y_0))\mathrm{div}\,\mathbf{vI})\mathbf{n}_+ - \sigma_0(y_0)(\Delta'_\Gamma\eta)\mathbf{n}_+ = \mathbf{g}_d & \text{on } \Gamma_+, \\ \lambda\eta + \mathbf{a}'(y_0)\cdot\nabla'\eta - \mathbf{v}\cdot\mathbf{n}_+ = g_d & \text{on } \Gamma_+. \end{cases} \tag{1.7}$$

1.1. Notation Let $\mathbb{N}$ denote the set of natural numbers and we let $\mathbb{N}_0 = \mathbb{N}\cup\{0\}$. Let $\mathbb{C}$ and $\mathbb{R}$ denote the set of complex numbers, and real numbers, respectively. By $Sym(\mathbb{R}^N)$ and $ASym(\mathbb{R}^N)$ we denote the sets of all $N \times N$ symmetric and anti-symmetric matrices, respectively. Let $q' = q/(q-1)$ be the dual exponent of q with $1 < q < \infty$. For any multi-index $\kappa = (\kappa_1, \dots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial_x^k = \partial_1^{\kappa_1}\cdots\partial_N^{\kappa_N}$ with $x = (x_1, \dots, x_N)$. For a scalar function f and an N-vector function $\mathbf{g}$, we set

$$\nabla f = (\partial_1 f, \dots, \partial_N f),\ \nabla\mathbf{g} = (\partial_i g_j \mid i, j = 1, \dots, N),$$
$$\nabla^2 f = \{\partial_i\partial_j f \mid i, j = 1, \dots, N\},\ \nabla^2\mathbf{g} = \{\partial_i\partial_j g_k \mid i, j, k = 1, \dots, N\}.$$

For Banach spaces X and Y let $\mathcal{L}(X, Y)$ denote the set of all bounded linear operators from X into Y, and $\mathrm{Hol}\,(\mathrm{U}, \mathcal{L}(\mathrm{X}, \mathrm{Y}))$ the set of all $\mathcal{L}(X, Y)$-valued holomorphic functions defined on a domain U in $\mathbb{C}$. Let $L_q(D)$, $W_q^m(D)$, $B_{p,q}^s(D)$ and $H_q^s(D)$ denote the usual Lebesgue space, Sobolev space, Besov space and Bessel potential space, respectively, for any domain D in $\mathbb{R}^N$ and $1 \le p, q \le \infty$. We denote by $\|\cdot\|_{L_q(D)}$, $\|\cdot\|_{W_q^m(D)}$, $\|\cdot\|_{B_{q,p}^s(D)}$ and $\|\cdot\|_{H_q^s(D)}$ their respective norms. For $\theta \in (0, 1)$, $H_p^\theta(\mathbb{R}, X)$ denotes the standard X-valued Bessel potential space defined by

$$H_p^\theta(\mathbb{R}, X) = \{f \in L_p(\mathbb{R}, X) \mid \|f\|_{H_p^\theta(\mathbb{R}, X)} < \infty\},$$

where

$$\|f\|_{H_p^\theta(\mathbb{R}, X)} = \left(\int_{\mathbb{R}} \|\mathcal{F}^{-1}[(1+\tau^2)^{\theta/2}\mathcal{F}[f](\tau)](t)\|_X^p\,dt\right)^{1/p}.$$

We set $W_q^0(D) = L_q(D)$ and $W_q^s(D) = B_{q,q}^s(D)$. Let $C^\infty(D)$ denote the set all C^∞ functions defined on D. Further let $L_p((a, b), X)$ and $W_p^m((a, b), X)$ denote the usual Lebesgue space and Sobolev space of X-valued functions defined on an interval (a,b), and let $\|\cdot\|_{L_p((a,b),X)}$ and $\|\cdot\|_{W_p^m((a,b),X)}$ denote their respective norms. Moreover, we set

$$\|e^{\eta t}f\|_{L_p((a,b),X)} = \left(\int_a^b (e^{\eta t}\|f(t)\|_X)^p dt\right)^{1/p}$$

for $1 \le p < \infty$. The d-product space of X is defined by $X^d = \{f = (f, \ldots, f_d) \mid f_i \in X\ (i = 1, \ldots, d)\}$, where its norm is denoted by $\|\cdot\|_X$ instead of $\|\cdot\|_{X^d}$ for the sake of simplicity. We set

$$\begin{aligned}
W_q^{m,\ell}(D) &= \{(f, \mathbf{g}, \mathbf{H}) \mid f \in W_q^m(D), \mathbf{g} \in W_q^\ell(D)^N,\ \ \mathbf{H} \in W_q^m(D)^{N\times N}\},\\
\|(f,\mathbf{g},\mathbf{H})\|_{W_q^{m,\ell}(\Omega)} &= \|(f,\mathbf{H})\|_{W_q^m(\Omega)} + \|\mathbf{g}\|_{W_q^\ell(\Omega)},\\
L_{p,\gamma_1}(\mathbb{R}, X) &= \{f(t) \in L_{p,\mathrm{loc}}(\mathbb{R}, X) \mid e^{-\gamma_1 t}f(t) \in L_p(\mathbb{R}, X)\},\\
L_{p,\gamma_1,0}(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid f(t) = 0\ (t < 0)\},\\
W_{p,\gamma_1}^m(\mathbb{R}, X) &= \{f(t) \in L_{p,\gamma_1}(\mathbb{R}, X) \mid e^{-\gamma_1 t}\partial_t^j f(t) \in L_p(\mathbb{R}, X)\ (j = 1, \ldots, m)\},\\
W_{p,\gamma_1,0}^m(\mathbb{R}, X) &= W_{p,\gamma_1}^m \cap L_{p,\gamma_1,0}(\mathbb{R}, X).
\end{aligned}$$

Let $\mathcal{F}_x = \mathcal{F}$ and $\mathcal{F}_\xi^{-1} = \mathcal{F}^{-1}$ denote the Fourier transform and the Fourier inverse transform, respectively, which are defined by

$$\mathcal{F}_x[f](\xi) = \int_{\mathbb{R}^N} e^{-ix\cdot\xi}f(x)dx,\ \mathcal{F}_\xi^{-1}[g](x) = \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{ix\cdot\xi}g(\xi)d\xi.$$

We also write $\hat{f}(\xi) = \mathcal{F}_x[f](\xi)$. Let $\mathcal{L}$ and $\mathcal{L}^{-1}$ denote the Laplace transform and the Laplace inverse transform, respectively, which are defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t}f(t)dt,\ \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\lambda t}g(\tau)d\tau,$$

with $\lambda = \gamma + i\tau \in \mathbb{C}$. Given $s \in \mathbb{R}$ and X-valued function $f(t)$, we set

$$\Lambda_\gamma^s f(t) = \mathcal{L}_\lambda^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t).$$

We introduce the Bessel potential space of X-valued functions of order s as follows:

$$\begin{aligned}
H_{p,\gamma_1}^s(\mathbb{R}, X) &= \{f \in L_p(\mathbb{R}, X) \mid e^{-\gamma t}\Lambda_\gamma^s[f](t) \in L_p(\mathbb{R}, X) \text{ for any } \gamma \ge \gamma_1\},\\
H_{p,\gamma_1,0}^s(\mathbb{R}, X) &= \{f \in H_{p,\gamma_1}^s(\mathbb{R}, X) \mid f(t) = 0\ (t < 0)\}.
\end{aligned}$$

For $\mathbf{x} = (x_1, \ldots, x_N)$ and $\mathbf{y} = (y_1, \ldots, y_N)$, we set $\mathbf{x}\cdot\mathbf{y} = \langle\mathbf{x}, \mathbf{y}\rangle = \sum_{j=1}^N x_j y_j$. For scalar functions f, g and N-vectors of functions $\mathbf{k}$, $\mathbf{g}$ we set $(k, g)_D = \int_D kg\,dx$, $(\mathbf{k}, \mathbf{g})_D = \int_D \mathbf{k}\cdot\mathbf{g}\,dx$, $(k, g)_\Gamma = \int_\Gamma kg\,d\omega$, $(\mathbf{k}, \mathbf{g})_\Gamma = \int_\Gamma \mathbf{k}\cdot\mathbf{g}d\omega$, where ω is the surface element of Γ. For $N \times N$ matrices of functions $\mathbf{F} = (F_{ij})$ and $\mathbf{G} = (G_{ij})$, we set $(\mathbf{F}, \mathbf{G})_D = \int_D \mathbf{F} : \mathbf{G}\,dx$ and $(\mathbf{F}, \mathbf{G})_\Gamma = \int_\Gamma \mathbf{F} : \mathbf{G}\,d\omega$, where $\mathbf{F} : \mathbf{G} \equiv \sum_{i,j=1}^N F_{ij}G_{ij}$ and $|\mathbf{F}| \equiv \left(\sum_{i,j=1}^N F_{ij}F_{ij}\right)^{1/2}$. Moreover, $\mathbf{x}\cdot\mathbf{F}$ denotes the vector with components $\sum_{i=1}^N x_i F_{ij}$. Let $C_0^\infty(G)$ be the set of all C^∞ functions whose supports are compact and contained in G. The letter C denotes generic constants and the constant $C_{a,b,\ldots}$ depends on $a, b, \ldots$. The values of constants C and $C_{a,b,\ldots}$ denote a positive constant which may be different even in a single chain of inequalities. We use small boldface letters, e.g. $\mathbf{u}$ to denote vector-valued functions and capital boldface letters, e.g. $\mathbf{H}$ to denote matrix-valued functions, respectively. But, we also use the Greek letters, e.g. ρ, θ, τ, ω, to denote mass densities, and also elastic tensors, although they are $N \times N$ matrices.

2. Preliminaries and Statement of the Main Result

Before stating our main result, first we introduce the definition of $\mathcal{R}$-boundedness and the operator-valued Fourier multiplier theorem due to Weis [**16**].

Definition 2.1. A family of operators $\mathcal{T} \subset \mathcal{L}(X, Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X, Y)$, if there exist constants $C > 0$ and $p \in [1, \infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1, 1\}$-valued random variables on $[0, 1]$, we have the inequality:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u) T_j x_j\|_Y^p \, du\Big\}^{1/p} \le C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u) x_j\|_X^p \, du\Big\}^{1/p}.$$

The smallest such C is called the $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Let $\mathcal{D}(\mathbb{R}, X)$ and $\mathcal{S}(\mathbb{R}, X)$ be the set of all X-valued C^∞ functions having compact support and the Schwartz space of rapidly decreasing X-valued functions, respectively, while $\mathcal{S}'(\mathbb{R}, X) = \mathcal{L}(\mathcal{S}(\mathbb{R}, \mathbb{C}), X)$. Given $M \in L_{1,\mathrm{loc}}(\mathbb{R}\backslash\{0\}, X)$, we define the operator $T_M : \mathcal{F}^{-1}\mathcal{D}(\mathbb{R}, X) \to \mathcal{S}'(\mathbb{R}, Y)$ by

$$T_M \phi = \mathcal{F}^{-1}[M\mathcal{F}[\phi]], \quad (\mathcal{F}[\phi] \in \mathcal{D}(\mathbb{R}, X)). \tag{2.1}$$

The following theorem was obtained by Weis [**16**].

Theorem 2.2. *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let M be a function in $C^1(\mathbb{R}\backslash\{0\}, \mathcal{L}(X, Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{(\tau\frac{d}{d\tau})^\ell M(\tau) \mid \tau \in \mathbb{R}\backslash\{0\}\}) \le \kappa < \infty \quad (\ell = 0, 1)$$

with some constant κ. Then, the operator T_M defined in (2.1) extendeds to a bounded linear operator from $L_p(\mathbb{R}, X)$ into $L_p(\mathbb{R}, Y)$. Moreover, denoting this extension by T_M, we have

$$\|T_M\|_{\mathcal{L}(L_p(\mathbb{R},X), L_p(\mathbb{R},Y))} \le C\kappa$$

for some positive constant C depending on p, X and Y.

Remark 2.3. For the definition of UMD space, we refer to Amann [**1**]. For $1 < q < \infty$, the Lebesgue space $L_q(\Omega)$ and Sobolev space $W_q^m(\Omega)$ are both UMD spaces.

We quote [**4**, Proposition 3.4], which tells us that $\mathcal{R}$-bounds behave like norms.

Proposition 2.4. (1) *Let X and Y be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$. Then $\mathcal{T} + \mathcal{S} = \{T + S \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Y)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T} + \mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T}) + \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

(2) *Let X, Y and Z be Banach spaces and let $\mathcal{T}$ and $\mathcal{S}$ be $\mathcal{R}$-bounded families in $\mathcal{L}(X, Y)$ and $\mathcal{L}(Y, Z)$, respectively. Then $\mathcal{S}\mathcal{T} = \{ST \mid T \in \mathcal{T}, S \in \mathcal{S}\}$ is also an $\mathcal{R}$-bounded family in $\mathcal{L}(X, Z)$ and*

$$\mathcal{R}_{\mathcal{L}(X,Z)}(\mathcal{T}\mathcal{S}) \le \mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{S}).$$

Definition 2.5. Let V be a domain in $\mathbb{C}$, let $\Xi = V \times (\mathbb{R}^{N-1} \setminus \{0\})$, and let $m : \Xi \to \mathbb{C}; (\lambda, \xi') \mapsto m(\lambda, \xi')$ be C^1 with respect to τ, where $\lambda = \gamma + i\tau \in V$, and C^∞ with respect to $\xi' \in \mathbb{R}^{N-1} \setminus \{0\}$.

(1) $m(\lambda, \xi')$ is called a multiplier of order s with type 1 on Ξ, if the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|},$$
$$|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s-|\kappa'|}$$

hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and V.

(2) $m(\lambda, \xi')$ is called a multiplier of order s with type 2 on Ξ, if the estimates:

$$|\partial_{\xi'}^{\kappa'} m(\lambda, \xi')| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|},$$
$$|\partial_{\xi'}^{\kappa'}(\tau\partial_\tau m(\lambda, \xi'))| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|)^{s}|\xi'|^{-|\kappa'|}$$

hold for any multi-index $\kappa \in \mathbb{N}_0^N$ and $(\lambda, \xi') \in \Xi$ with some constant $C_{\kappa'}$ depending solely on κ' and V.

Let $\mathbf{M}_{s,i}(V)$ be the set of all multipliers of order s with type i on Ξ for $i = 1, 2$. For $m \in \mathbf{M}_{s,i}(V)$, we set $M(m, V) = \max_{|\kappa'| \leq N} C_{\kappa'}$.

Let $\mathcal{F}_{\xi'}^{-1}$ be the inverse partial Fourier transform defined by

$$\mathcal{F}_{\xi'}^{-1}[f(\xi', x_N)](x') = \frac{1}{(2\pi)^{N-1}} \int_{\mathbb{R}^{N-1}} e^{i\xi'\cdot\xi'} f(\xi', x_N)\, d\xi'.$$

Then, we have the following two lemmas which were proved essentially by Shibata and Shimizu [**15**, Lemma 5.4 and Lemma 5.6].

Lemma 2.6. *Let $\epsilon \in (0, \pi/2)$, $q \in (1, \infty)$ and $\lambda_0 > 0$. Given $m \in \mathbf{M}_{-2,1}(\Lambda_\kappa, \lambda_0)$, we define an operator $L(\lambda)$ by*

$$[L(\lambda)g](x) = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')\lambda^{1/2} e^{-B(x_n + y_N)} \hat{g}(\xi', y_N)](x')\, dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}_+^N), W_q^{2-j}(\mathbb{R}_+^N)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\partial_x^\alpha L(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq r_b(\lambda_0),$$

where $\ell = 0, 1, j = 0, 1, 2$ and τ denotes the imaginary part of λ, and $r_b(\lambda_0)$ is a constant depending on $M(m, \Sigma_{\kappa,\lambda_0})$, ϵ, λ_0, N, and q.

Lemma 2.7. *Let $1 < q < \infty$, $0 < \epsilon < \pi/2$ and $\lambda_0 > 0$. Let $m(\lambda, \xi')$ be a function defined on $\Sigma_{\epsilon,\lambda_0}$ such that for any multi-index $\kappa' \in \mathbb{N}_0^{N-1}$ there exists a constant $C_{\kappa'}$ such that*

$$|\partial_{\xi'}^{\kappa'}\{(\tau\frac{\partial}{\partial\tau})^\ell m(\lambda, \xi')\}| \leq C_{\kappa'}(|\lambda|^{1/2} + |\xi'|^{-2-|\kappa'|}) \quad (\ell = 0, 1) \tag{2.2}$$

for any $(\lambda, \xi') \in \Sigma_{\epsilon,\lambda_0,}$. Let $\Psi_j(\lambda)$ $(j = 1, 2)$ be operators defined by

$$\Psi_1(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')Be^{-B(x_N + y_N)}\mathcal{F}_{x'}[f](\xi', y_N)](x')\ dy_N,$$
$$\Psi_2(\lambda)f = \int_0^\infty \mathcal{F}_{\xi'}^{-1}[m(\lambda, \xi')B^2 M(x_N + y_N)\mathcal{F}_{x'}[f](\xi', y_N)](x')\ dy_N.$$

Then, we have

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N_+),L_q(\mathbb{R}^N_+)^{\tilde N})}(\{(\tau\frac{d}{d\tau})^\ell(G_\lambda\Psi_i(\lambda)) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\})\le C \quad (\ell=0,1,\ i=1,2)$$

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^n_+),W^{2-1/q}_q(\mathbb{R}^N_0))}(\{(\tau\frac{d}{d\tau})^\ell F_\lambda\Psi_i(\lambda) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\})\le C \quad (\ell=0,1,\ i=1,2)$$

with some constant C. Here and hereafter, $C_{\kappa'}$ denotes a generic constant depending on κ', ϵ, λ_0.

Proof. The proof of the lemma can be found in [**5**], [**3**] and [**8**]. □

Lemma 2.8. *Let $1<q<\infty$ and let Λ be a set in $\mathbb{C}$. Let $m=M(\lambda,\xi)$ be a function defined on $\Lambda\times(\mathbb{R}^N\setminus\{0\})$ which is infinitely differentiable with respect to $\xi\in\mathbb{R}^N\setminus\{0\}$ for each $\lambda\in\Lambda$. Assume that for any multi-index $\alpha\in\mathbb{N}^N_0$ there exists a constant C_α depending on α and Λ such that*

$$|\partial^\alpha_\xi m(\lambda,\xi)|\le C_\alpha|\xi|^{-|\alpha|} \tag{2.3}$$

for any $(\lambda,\xi)\in\Lambda\times(\mathbb{R}^N\setminus\{0\})$. Let K_λ be an operator defined by

$$K_\lambda f=\mathcal{F}^{-1}[m(\lambda,\xi)\mathcal{F}[f](\xi)]. \tag{2.4}$$

Then, the family of operators $\{K_\lambda\mid\lambda\in\Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{K_\lambda\mid\lambda\in\Lambda\})\le C_{q,N}\max_{|\alpha|\le N+1}C_\alpha \tag{2.5}$$

for some $C_{q,N}$ depending only on q and N.

The following theorem is the main result of this article.

Theorem 2.9. *Let Ω_+ be a bent half-space with boundary Γ_+. Let $1<q<\infty$ and $0<\epsilon<\pi/2$. Set*

$$Y_q(\Omega_+)=\{(\mathbf{f}_+,\mathbf{g}_{d+},g_{d+})\mid\mathbf{f}_+\in L_q(\Omega_+)^N,\mathbf{g}_{d+}\in W^1_q(\Omega_+),g_{d+}\in W^{2-1/q}_q(\Gamma_+)\}$$

$$\mathcal{Y}_q(\Omega_+)=\{(F_1,F_2,F_3,F_4)\mid F_1,F_2\in L_q(\Omega_+)^N,F_3\in W^1_q(\Omega_+),F_4\in W^{2-1/q}_q(\Gamma_+)\}.$$

Then, there exist $M_1\in(0,1)$, $\tilde\lambda_0\ge1$ and operator families $\mathcal{A}_d(\lambda)$ and $\mathcal{H}_d(\lambda)$ with

$$\mathcal{A}_d(\lambda)\in Hol(\Sigma_{\epsilon,\tilde\lambda_0},\mathcal{L}(\mathcal{Y}_q(\Omega_+),W^2_q(\Omega_+)^N)),$$
$$\mathcal{H}_d(\lambda)\in Hol(\Sigma_{\epsilon,\tilde\lambda_0},\mathcal{L}(\mathcal{Y}_q(\Omega_+),W^3_q(\Omega_+)))$$

such that for any $\lambda=\gamma+i\tau\in\Sigma_{\epsilon,\tilde\lambda_0}$ and $(\mathbf{f}_+,\mathbf{g}_{d+},g_{d+})\in Y_q(\Omega_+)$,

$$\mathbf{u}=\mathcal{A}_d(\lambda)(\mathbf{f}_+,\lambda^{1/2}\mathbf{g}_{d+},\mathbf{g}_{d+},g_{d+}),$$
$$\eta=\mathcal{H}_d(\lambda)(\mathbf{f}_+,\lambda^{1/2}\mathbf{g}_{d+},\mathbf{g}_{d+},g_{d+})$$

are unique solutions of equation (1.7) *and*

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\Omega_+),W^{2-j}_q(\Omega_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\mathcal{A}_d(\lambda))\mid\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\})\le r_b\ (\ell=0,1,\ j=0,1,2)$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\Omega_+),W^{3-j}_q(\Omega_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^k\mathcal{H}_d(\lambda))\mid\lambda\in\Sigma_{\epsilon,\tilde\lambda_0}\})\le r_b\ (\ell=0,1,\ k=0,1), \tag{2.6}$$

where, r_b is a constant depending on m_0, m_1, m_2, N, q, and ϵ but independent of M_1 and M_2 and moreover, $\tilde\lambda_0$ is a constant depending on M_2.

Remark 2.10. The F_1, F_2, F_3, and F_4 are variables corresponding to $\mathbf{f}_+$, $\lambda^{1/2}\mathbf{g}_{d+}$, $\mathbf{g}_{d+}$ and g_{d+}, respectively.

3. Proof of the Main Theorem

The proof of Theorem 2.9 consists of the following steps. Let $\Phi : \mathbb{R}^N \to \mathbb{R}^N$ be such that $x \mapsto y = \Phi(x)$ be a bijection of C^1 class and let Φ^{-1} be its inverse map. Define

$$\Omega_+ = \Phi(\mathbb{R}^N_+)\ , \Gamma_+ = \Phi(\mathbb{R}^N_0),$$

where $\mathbb{R}^N_0 = \{(x_1, \dots, x_N) \in \mathbb{R}^N \mid x_N = 0\}$. We assume that $\nabla\Phi$ and $\nabla\Phi^{-1}$ have forms

$$\nabla\Phi = \mathcal{A} + B(x), \quad \nabla\Phi^{-1} = \mathcal{A}_{-1} + B_{-1}(y),$$

where $\mathcal{A}$ and $\mathcal{A}_{-1}$ are $N \times N$ orthogonal matrices with constant coefficients and $B(x)$ and $B_{-1}(y)$ are matrices of functions in $C^2(\mathbb{R}^N)$ such that

$$\begin{aligned}
&\|(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_1, \quad \|\nabla(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le C_a,\\
&\|\nabla^2(B, B_{-1})\|_{L_\infty(\mathbb{R}^N)} \le M_2.
\end{aligned}$$

We consider the following equation system:

$$\begin{cases}
\lambda\mathbf{v} - \alpha_0(y_0)\Delta\mathbf{v} - \beta\nabla\mathrm{div}\,\mathbf{v} = \mathbf{f} & \text{in } \Omega_+,\\
(\alpha_0(y_0)\mathbf{D}(\mathbf{v}) + \beta\mathrm{div}\,\mathbf{vI})\mathbf{n}_+ - \sigma_0(y_0)(\Delta'_\Gamma\eta)\mathbf{n}_+ = \mathbf{g}_d & \text{on } \Gamma_+,\\
\lambda\eta + \mathbf{a}'(y_0)\cdot\nabla'\eta - \mathbf{v}\cdot\mathbf{n}_+ = g_d & \text{on } \Gamma_+.
\end{cases} \tag{3.1}$$

By the change of variables $y = \Phi(x)$, we transform (1.7) to a problem in the half-space which can be seen in [**13**]. Let

$$\begin{aligned}
&y_0 = \Phi(x_0), \quad \tilde{\alpha}(x) = \phi(\Phi(x))\mu(\Phi(x)),\\
&\tilde{\sigma}(x) = \phi(\Phi(x))\sigma(\Phi(x)), \quad \tilde{\mathbf{a}} = \phi(\Phi(x))\sigma(\Phi(x)).
\end{aligned}$$

Notice that

$$\begin{aligned}
\alpha_{y_0}(\Phi(x)) &= \alpha(y_0) + \tilde{\alpha}(x) - \tilde{\alpha}(x_0),\\
\sigma_{y_0}(\Phi(x)) &= \sigma(y_0) + \tilde{\sigma}(x) - \tilde{\sigma}(x_0),\\
\mathbf{a}(\Phi(x', 0)) &= \mathbf{a}(y_0) + \tilde{\mathbf{a}}(x) - \tilde{\mathbf{a}}(x_0).
\end{aligned}$$

In addition, α, σ, and $\mathbf{a}$ satisfy the following conditions:

$$\begin{aligned}
&m_0 \le \alpha(y), \sigma(y) \le m_1, \quad |\nabla\alpha(y)|, |\nabla\sigma(y)| \le m_1 \quad \text{for any } y \in \overline{\Omega_+},\\
&|\mathbf{a}(y)| \le m_2 \quad \text{for any } y \in \Gamma_+,\\
&\|\mathbf{a}\|_{W_r^{2-1/q}(\Omega_+)} \le m_3\kappa^{-b}
\end{aligned} \tag{3.2}$$

for some $m_1, m_2, m_3 > 0$. For a positive number of d_0, we have

$$\begin{aligned}
&|\alpha(y) - \alpha(y_0)| \le m_1M_1, \quad |\sigma(y) - \sigma(y_0)| \le m_1M_1 \quad \text{for any } y \in \overline{\Omega_+} \cap B_{d_0}(y_0),\\
&|\mathbf{a}(y) - \mathbf{a}(y_0)| \le m_2M_1 \quad \text{for any } y \in \Gamma_+ \cap B_{d_0}(y_0)
\end{aligned} \tag{3.3}$$

for some constants $M_1, M_2 > 0$. We may assume that m_1, m_2, $m_3 \le M_2$. Recall that $\|\nabla\phi\|_{W^1_\infty(\mathbb{R}^N)} \le M_2$. According to (3.2) and (3.3) we have

$$
\begin{aligned}
&|\tilde\alpha(x) - \tilde\alpha(x_0)| \le m_1 M_1, \quad |\tilde\sigma(x) - \tilde\sigma(x_0)| \le m_1 M_1,\\
&\|(\tilde\alpha, \tilde\sigma)\|_{L_\infty(\mathbb{R}^N)} \le m_1, \quad \|\nabla(\tilde\alpha, \tilde\sigma)\|_{L_\infty(\mathbb{R}^N)} \le C_{M_2},\\
&|\tilde{\mathbf{a}}(x) - \tilde{\mathbf{a}}(x_0)| \le m_2 M_1, \quad \|\tilde{\mathbf{a}}\|_{L_\infty(\mathbb{R}^N)} \le m_2,\\
&\|\nabla\tilde{\mathbf{a}}\|_{W^{1-1/q}_q(\mathbb{R}^N_0)} \le C_{M_2}\kappa^{-b} \quad \text{for } \kappa \in (0,1).
\end{aligned} \tag{3.4}
$$

Using the change of the unknown functions: $\mathbf{u} = \mathcal{R}_{-1}\mathbf{v} \circ \Phi$ as well as the change variable: $y = \Phi(x)$, we have

$$
\frac{\partial}{\partial y_j} = \sum_{k=1}^N (\mathcal{R}_{kj} + R_{kj}(x))\frac{\partial}{\partial x_k} \text{ and } \mathbf{n}_+ = -(a_{N1}, \dots, a_{NN})^T + \mathbf{b}_+(x), \tag{3.5}
$$

where $(\nabla\Phi^{-1})(\Phi(x)) = (\mathcal{R}_{ij} + R_{ij}(x))$. Then we will derive the problem in $\mathbb{R}^N_+$ from (1.7). Nothing that $\mathcal{R} = \mathcal{R}^T_{-1}$, by (3.5), we can rewrite the equation (1.7) in $\mathbb{R}^N_+$ to be

$$
\begin{cases}
\lambda\mathbf{u} - \alpha(y_0)\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} + \mathcal{H}^1(\mathbf{u}) = \mathbf{f}_+ \text{in } \mathbb{R}^N_+,\\
(\alpha(y_0)\mathbf{D}(\mathbf{u}) + \beta\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n}_+ - \sigma(y_0)(\Delta'_\Gamma\eta)\mathbf{n}_+ + \mathcal{H}^2(\mathbf{u}, \eta) = \mathbf{g}_{d+} \text{ on } \mathbb{R}^N_0,\\
\lambda\eta + \mathbf{a}'(y_0)\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n}_+ + \mathcal{H}^3_\kappa(\mathbf{u}, \eta) = g_{d+} \text{ on } \mathbb{R}^N_0,
\end{cases} \tag{3.6}
$$

where

$$
\begin{aligned}
\mathbf{f}_+ = \mathcal{R}_{-1}\mathbf{f}\circ\Phi,\ \mathbf{g}_{d+} = \mathcal{R}_{-1}\mathbf{g}_d\circ\Phi,\ g_{d+} = g_d\circ\Phi,\\
\mathcal{H}^1(\mathbf{u}) = \mathcal{P}_1\Delta\mathbf{u} + \mathcal{P}_2\nabla^2\mathbf{u} + \mathcal{P}_3\nabla\mathbf{u},\\
\mathcal{H}^2(\mathbf{u}, \eta) = \mathcal{P}_4\nabla^2\mathbf{u} + \mathcal{P}_5\eta + \mathcal{P}_6\nabla\mathbf{u},\\
\mathcal{H}^3_0(\mathbf{u}, \eta) = -\mathbf{u}\cdot(\mathcal{R}_{-1}\mathbf{b}_0) \quad \text{for } \kappa = 0,\\
\mathcal{H}^3_{\kappa'}(\mathbf{u}, \eta) = (\tilde{\mathbf{a}}(x) - \tilde{\mathbf{a}}(x_0))\nabla'\eta - \mathbf{u}\cdot(\mathcal{R}_{-1}\mathbf{b}_0) \text{ for } \kappa' = (0,1),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{H}^1(\mathbf{u}) =& \sum_{k=1}^N \frac{\partial}{\partial x_k}\{(\tilde\mu(x) - \tilde\mu(x_0))D_{sk}(\mathbf{u})\} + \sum_{i,j,k}^N a_{si}a_{kj}\frac{\partial}{\partial x_k}(\tilde\mu(x)b^d_{ij} : \nabla\mathbf{u}),\\
\mathcal{H}^2(\mathbf{u}, \eta) =& -(\tilde\mu(x) - \tilde\mu(x_0))D_{sN}(\mathbf{u}) + (\mu(y_0)\\
&+ \tilde\mu(x) - \tilde\mu(x_0))\sum_{i,j=1}^N (a_{ij}b_{+j}D_{si}(\mathbf{u}) + a_{si}b^d_{ij} : \nabla\mathbf{u}(-a_{Nj} + b_{+j}))\\
&- (\tilde\sigma(x) - \tilde\sigma(x_0))(\Delta'\eta)\mathbf{n}_0\\
&- \tilde\sigma(x)\{(\Delta'\eta)(\mathcal{R}_{-1}\mathbf{b}_+) + (\mathcal{D}_+\eta)(\mathbf{n}_0 + \mathcal{R}_{-1}\mathbf{b}_+)\}.
\end{aligned}
$$

In addition, $\mathcal{H}^1$, $\mathcal{H}^2$ and $\mathcal{H}^3$ are nonlinear in $\mathbf{u}$ and η, and satisfy the estimates:

$$\begin{aligned}
\|\mathcal{H}^1(\mathbf{u},\eta)\|_{L_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla^2\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\eta\|_{L_q(\mathbb{R}^N_+)})\\
&\quad + C_{M_2}(\|\mathbf{u}\|_{W^1_q(\mathbb{R}^N_+)} + \|\eta\|_{W^2_q(\mathbb{R}^N_+)}),\\
\|\mathcal{H}^2(\mathbf{u},\eta)\|_{L_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^2\eta\|_{L_q(\mathbb{R}^N_+)})\\
&\quad + C_{M_2}(\|\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\eta\|_{W^1_q(\mathbb{R}^N_+)}),\\
\|\mathcal{H}^2(\mathbf{u},\eta)\|_{W^1_q(\mathbb{R}^N_+)} &\le CM_1(\|\nabla^2\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\eta\|_{L_q(\mathbb{R}^N_+)})\\
&\quad + C_{M_2}(\|\mathbf{u}\|_{W^1_q(\mathbb{R}^N_+)} + \|\eta\|_{W^2_q(\mathbb{R}^N_+)}),\\
\|\mathcal{H}^3_0(\mathbf{u},\eta)\|_{W^{2-1/q}_q(\mathbb{R}^N_0)} &\le CM_1(\|\nabla^2\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + C_{M_2}(\|\mathbf{u}\|_{W^1_q(\mathbb{R}^N_+)},\\
\|\mathcal{H}^3_{\kappa'}(\mathbf{u},\eta)\|_{W^{2-1/q}_q(\mathbb{R}^N_0)} &\le CM_1(\|\nabla^2\mathbf{u}\|_{L_q(\mathbb{R}^N_+)} + \|\nabla^3\eta\|_{L_q(\mathbb{R}^N_+)})\\
&\quad + C_{M_2}(\|\mathbf{u}\|_{W^1_q(\mathbb{R}^N_+)} + \kappa^{-b}\|\eta\|_{W^2_q(\mathbb{R}^N_+)}).
\end{aligned}$$

Here and in the following, C denotes a generic constant depending on N, q, m_1, m_2 and C_{M_2} denotes a generic constant depending on N, q, m_1, m_2, m_3 and M_2. By the $\mathcal{R}$-boundedness in half-space [**13**], there exists a large number λ_0 and operator families $\mathcal{A}_0(\lambda)$ and $\mathcal{H}_0(\lambda)$ with

$$\begin{aligned}
\mathcal{A}_0(\lambda) &\in \mathrm{Hol}\ (\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^2_q(\mathbb{R}^N_+)^N)),\\
\mathcal{H}_0(\lambda) &\in \mathrm{Hol}\ (\Sigma_{\epsilon,\lambda_0}, \mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^3_q(\mathbb{R}^N_+)^N)),
\end{aligned}$$

such that for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and $(\mathbf{f},\mathbf{g},d) \in Y_q(\mathbb{R}^N_+)$, $\mathbf{u}$ and η with

$$\mathbf{u} = \mathcal{A}_0(\lambda)F_\lambda(\mathbf{f},\mathbf{g},d), \qquad \eta = \mathcal{H}_0(\lambda)F_\lambda(\mathbf{f},\mathbf{g},d),$$

where $F_\lambda(\mathbf{f},\mathbf{g},d) = (\mathbf{f}, \lambda^{1/2}\mathbf{g}, \nabla\mathbf{g}, d)$ are unique solutions of the equations:

$$\begin{cases}
\lambda\mathbf{u} - \alpha\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} = \mathbf{f} & \text{in } \mathbb{R}^N_+,\\
(\alpha\mathbf{D}(\mathbf{u}) - (\beta-\alpha)\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n} - \sigma(\Delta'_\Gamma\eta)\mathbf{n} = \mathbf{g} & \text{on } \mathbb{R}^N_0,\\
\lambda\eta + \mathbf{a}'\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n} = d & \text{on } \mathbb{R}^N_0,
\end{cases} \tag{3.7}$$

and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\mathcal{A}_0(\lambda)) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \le r_b \quad (\ell = 0,1, j = 0,1,2)$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^{3-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{k}\mathcal{H}_0(\lambda)) \mid \lambda\in\Sigma_{\epsilon,\lambda_0}\}) \le r_b \quad (\ell = 0,1, k = 0,1).$$

Let

$$\begin{aligned}
\mathbf{u} &= \mathcal{A}_0(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}),\\
\eta &= \mathcal{H}_0(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})
\end{aligned}$$

in (1.7). Then, (1.7) is written as

$$\begin{cases}
\lambda\mathbf{u} - \alpha(y_0)\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} + \mathcal{H}^1(\mathbf{u})\\
\quad = \mathbf{f}_+ + \mathcal{H}^4(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) \text{ in } \mathbb{R}^N_+,\\
(\alpha(y_0)\mathbf{D}(\mathbf{u}) + \beta\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n}_+ - \sigma(y_0)(\Delta'_\Gamma\eta)\mathbf{n}_+ + \mathcal{H}^2(\mathbf{u},\eta)\\
\quad = \mathbf{g}_{d+} + \mathcal{H}^5(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) \text{ on } \mathbb{R}^N_0,\\
\lambda\eta + \mathbf{a}'(y_0)\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n}_+ + \mathcal{H}^3_\kappa(\mathbf{u},\eta)\\
\quad = g_{d+} + \mathcal{H}^6(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) \text{ on } \mathbb{R}^N_0,
\end{cases} \tag{3.8}$$

where we have set

$$\begin{aligned}
&\mathcal{H}^4(\lambda)(F_1,F_2,F_3,F_4) = \mathcal{H}^4(\lambda)(\mathcal{A}_0(\lambda)(F_1,F_2,F_3,F_4), \mathcal{H}_0(\lambda)(F_1,F_2,F_3,F_4)),\\
&\mathcal{H}^5(\lambda)(F_1,F_2,F_3,F_4) = \mathcal{H}^5(\lambda)(\mathcal{A}_0(\lambda)(F_1,F_2,F_3,F_4), \mathcal{H}_0(\lambda)(F_1,F_2,F_3,F_4)),\\
&\mathcal{H}^6(\lambda)(F_1,F_2,F_3,F_4) = \mathcal{H}^6(\lambda)(\mathcal{A}_0(\lambda)(F_1,F_2,F_3,F_4), \mathcal{H}_0(\lambda)(F_1,F_2,F_3,F_4)).
\end{aligned}$$

Let

$$\mathcal{H}^7(\lambda)F = (\mathcal{H}^4(\lambda)F, \mathcal{H}^5(\lambda)F, \mathcal{H}^6(\lambda)F)$$

for $F = (F_1,F_2,F_3,F_4) \in \mathcal{Y}_q(\mathbb{R}^N_+)$. Notice that

$$\mathcal{H}^7(\lambda)F = (\mathcal{H}^4(\lambda)F, \lambda^{1/2}\mathcal{H}^5(\lambda)F, \nabla\mathcal{H}^5(\lambda)F, \mathcal{H}^6(\lambda)F).$$

Let

$$\mathcal{A}_\lambda(\mathbf{u},\eta) = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3),$$

where

$$\begin{aligned}
&\mathcal{A}_1 = \lambda\mathbf{u} - \alpha(y_0)\Delta\mathbf{u} - \beta\nabla\mathrm{div}\,\mathbf{u} + \mathcal{H}^1(\mathbf{u}),\\
&\mathcal{A}_2 = (\alpha(y_0)\mathbf{D}(\mathbf{u}) + \beta\mathrm{div}\,\mathbf{u}\mathbf{I})\mathbf{n}_+ - \sigma(y_0)(\Delta'_\Gamma\eta)\mathbf{n}_+ + \mathcal{H}^2(\mathbf{u},\eta),\\
&\mathcal{A}_3 = \lambda\eta + \mathbf{a}'(y_0)\cdot\nabla'\eta - \mathbf{u}\cdot\mathbf{n}_+ + \mathcal{H}^3_\kappa(\mathbf{u},\eta)
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{G}_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\\
&\quad = (\mathcal{H}^4(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}), \mathcal{H}^5(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}), \mathcal{H}^6(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})).
\end{aligned}$$

Then we may write

$$\mathcal{A}_\lambda(\mathbf{u},\eta)(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) = (\mathbf{I} + \mathcal{G}_\lambda F_\lambda)(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}).$$

By (3.2), (3.3), the $\mathcal{R}$-boundedness theorem in half-space, and Proposition 2.4, we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial\tau)^\ell F_\lambda\mathcal{H}^7(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le CM_1 + C_{M_2}(\lambda_1^{1/2} + \lambda_1^{-1}\gamma_\kappa) \tag{3.9}$$

for any $\lambda_1 \ge \lambda_0$. Choosing M_1 so small that $CM_1 \le 1/4$ and choosing $\lambda_1 > 0$ so large that $C_{M_2}\lambda_1^{-1/2} \le 1/8$ and $\lambda_1^{-1}\gamma_\kappa \le 1/8$, by (3.9) we have

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial\tau)^\ell F_\lambda\mathcal{H}^7(\lambda) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \le 1/2, \ (\ell = 0,1). \tag{3.10}$$

Let

$$F = (F_1,F_2,F_3,F_4) \in \mathcal{Y}_q(\mathbb{R}^N_+), \ (\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) \in Y_q(\mathbb{R}^N_+).$$

Then

$$\begin{aligned}
\|(F_1,F_2,F_3,F_4)\|_{\mathcal{Y}_q(\mathbb{R}^N_+)} &= \|(F_1,F_2)\|_{L_q(\mathbb{R}^N_+)^N} + \|F_3\|_{W^1_q(\mathbb{R}^N_+)} + \|F_4\|_{W^{2-1/q}_q(\mathbb{R}^N_0)},\\
\|(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\|_{Y_q(\mathbb{R}^N_+)} &= \|\mathbf{f}_+\|_{L_q(\mathbb{R}^N_+)^N} + \|\mathbf{g}_{d+}\|_{W^1_q(\mathbb{R}^N_+)} + \|g_{d+}\|_{W^2_q(\mathbb{R}^N_+)}.
\end{aligned} \tag{3.11}$$

By (3.10), we have

$$\|F_\lambda(\mathcal{H}^7(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}))\|_{\mathcal{Y}_q(\mathbb{R}^N_+)} \le 1/2\|F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\|_{\mathcal{Y}_q(\mathbb{R}^N_+)}. \tag{3.12}$$

In view of (3.11), $\|F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}))\|_{\mathcal{Y}_q(\mathbb{R}^N_+)}$ is equivalent to $\|(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\|_{Y_q(\mathbb{R}^N_+)}$ provided that $\lambda \neq 0$. Thus, $\mathcal{H}^7(\lambda)F_\lambda$ is a contraction map from $Y_q(\mathbb{R}^N_+)$ into itself, that is $\mathcal{H}^7(\lambda)F_\lambda : Y_q(\mathbb{R}^N_+) \to Y_q(\mathbb{R}^N_+)$. Let

$$\begin{aligned}\mathbf{u} &= \mathcal{A}_0(\lambda)F_\lambda(\mathbf{I} + \mathcal{H}^7(\lambda)F_\lambda)^{-1}(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}),\\ \eta &= \mathcal{H}_0(\lambda)F_\lambda(\mathbf{I} + \mathcal{H}^7(\lambda)F_\lambda)^{-1}(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}).\end{aligned} \tag{3.13}$$

By (3.8), we see that $\mathbf{u}$ and η are solutions of equation (3.7). In view of equation (3.8), $(\mathbf{I} + F_\lambda\mathcal{H}^7(\lambda))^{-1} = \sum_{j=0}^\infty(-F_\lambda\mathcal{H}^7(\lambda))^j$ exists in $\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))$ and

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}_q(\mathbb{R}^N_+))}(\{(\tau\partial\tau)^\ell(\mathbf{I} + F_\lambda\mathcal{H}^7(\lambda))^{-1} \mid \lambda \in \Sigma_{\epsilon,\lambda_1}\}) \leq 4, \quad (\ell = 0, 1). \tag{3.14}$$

Now

$$\begin{aligned}F_\lambda(\mathbf{I} + \mathcal{H}^7(\lambda)F_\lambda)^{-1} &= F_\lambda\sum_{j=0}^\infty(-\mathcal{H}^7(\lambda)F_\lambda)^j\\ &= (\sum_{j=0}^\infty(-F_\lambda\mathcal{H}^7(\lambda))^j)F_\lambda\\ &= (\mathbf{I} + F_\lambda\mathcal{H}^7(\lambda))^{-1}F_\lambda.\end{aligned}$$

We define $\mathcal{A}_1(\lambda)$ and $\mathcal{H}_1(\lambda)$ as operators which respect to $F = (F_1, F_2, F_3, F_4) \in \mathcal{Y}_q(\mathbb{R}^N_+)$, such that

$$\begin{aligned}\mathcal{A}_1(\lambda)F &= \mathcal{A}_0(\lambda)(\mathbf{I} + F_\lambda\mathcal{H}^7(\lambda))^{-1}F,\\ \mathcal{H}_1(\lambda)F &= \mathcal{H}_0(\lambda)(\mathbf{I} + F_\lambda\mathcal{H}^7(\lambda))^{-1}F.\end{aligned}$$

By using equation (3.13),

$$\begin{aligned}\mathbf{u} &= \mathcal{A}_1(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\\ \eta &= \mathcal{H}_1(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})\end{aligned}$$

are solutions of equation (3.7). Moreover, by (3.14) and the result in half-space [**13**],

$$\begin{aligned}&\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^{2-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\mathcal{A}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq 4r_b \quad (\ell = 0, 1,\ j = 0, 1, 2),\\ &\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\mathbb{R}^N_+), W^{3-j}_q(\mathbb{R}^N_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^k\mathcal{H}_1(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq 4r_b \quad (\ell = 0, 1,\ k = 0, 1).\end{aligned} \tag{3.15}$$

Recalling that

$$\begin{aligned}\mathbf{v} = (\mathcal{R}_{-1}\mathbf{u}) \circ \Phi^{-1},\ \mathbf{f}_+ = \mathcal{R}_{-1}\mathbf{f} \circ \Phi,\\ \mathbf{g}_{d+} = \mathcal{R}_{-1}\mathbf{g}_d \circ \Phi,\ g_{d+} = g_d \circ \Phi,\end{aligned}$$

we define operators $\mathcal{A}_d(\lambda)$ and $\mathcal{H}_d(\lambda)$ acting on $F = (F_1, F_2, F_3, F_4) \in \mathcal{Y}_q(\Omega_+)$ by

$$\begin{aligned}\mathcal{A}_d(F_1, F_2, F_3, F_4) &= \mathcal{R}^T_{-1}\big[\mathcal{A}_1(\lambda)(\mathcal{R}_{-1}F_1 \circ \Phi, F_2 \circ \Phi, \mathcal{R}_{-1}F_3 \circ \Phi, F_4 \circ \Phi)\big] \circ \Phi^{-1}\\ \mathcal{H}_d(F_1, F_2, F_3, F_4) &= \mathcal{R}^T_{-1}\big[\mathcal{H}_1(\lambda)(\mathcal{R}_{-1}F_1 \circ \Phi, F_2 \circ \Phi, \mathcal{R}_{-1}F_3 \circ \Phi, F_4 \circ \Phi)\big] \circ \Phi^{-1}.\end{aligned}$$

Obviously, given any $(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+}) \in Y_q(\Omega_+)$, $\mathbf{u} = \mathcal{A}_d(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})$ and $\eta = \mathcal{H}_d(\lambda)F_\lambda(\mathbf{f}_+, \mathbf{g}_{d+}, g_{d+})$ are solutions of equation (1.7). By using (3.15), we can choose $\tilde{\lambda}_2 \geq \lambda_1$ suitably large such that $\mathcal{A}_d(\lambda)$ and $\mathcal{H}_d(\lambda)$ satisfy the estimates:

$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\Omega_+), W_q^{2-j}(\Omega_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{j/2}\mathcal{A}_d(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq Cr_b \, (\ell = 0, 1, \ j = 0, 1, 2)$$
$$\mathcal{R}_{\mathcal{L}(\mathcal{Y}(\Omega_+), W_q^{3-j}(\Omega_+)^N)}(\{(\tau\partial\tau)^\ell(\lambda^{k}\mathcal{H}_d(\lambda)) \mid \lambda \in \Sigma_{\epsilon,\lambda_0}\}) \leq Cr_b \quad (\ell = 0, 1, \ k = 0, 1). \tag{3.16}$$

This completes the proof of Theorem 2.9.

4. Conclusions

In this work, we developed a bent half-space model problem for Lamé equation. Furthermore, we can consider a general domain and the maximal L_p-L_q regularity class. For further research, we can analyse not only the local well-posedness of the model problem but also the global well-posedness.

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References

[1] H. Amann, *Linear and Quasilinear Parabolic Problems. Vol. 1: Abstract Linear Theory*, Monographs in Mathematics 89, Birkhäuser, Basel, 1995.

[2] P. Cumsille and M. Tucsnak, *Well-posedness for the Navier-Stokes flow in the exterior of a rotating obstacle*, Math. Methods Appl. Sci. **29** (2006), 595–623. Doi: 10.1002/mma.702

[3] D. Götz, and Y. Shibata, *On the $\mathcal{R}$-boundedness of the solution operators in the study of the compressible viscous fluid flow with free boundary conditions*, Asymptotic. Anal. **90** (2014), 207–236. Doi: 10.3233/ASY-141238

[4] R. Denk, M. Hieber and J. Prüss, *$\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Mem. Am. Math. Soc. **166** (2003), Paper no. 788. Doi: 10.1090/memo/0788

[5] Y. Enomoto, and Y. Shibata, *On the $\mathcal{R}$-sectoriality and the initial boundary value problem for the viscous compressible fluid flow*, Funkc. Ekvacioj, Ser. Int. **56** (2013), 441–505. Doi: 10.1619/fesi.56.441

[6] Y. Enomoto, L. von Below and Y. Shibata, *On some free boundary problem for a compressible barotropic viscous fluid flow*, Ann. Univ. Ferrara, Sez. VII, Sci. Mat. **60** (2014), 55–89. Doi: 10.1007/s11565-013-0194-8

[7] J. S. Martín, J. F. Scheid, T. Takahashi, and M. Tucsnak, *An initial and boundary value problem modeling of fish-like swimming*, Arch. Ration. Mech. Anal. **188** (2008), 429–455. Doi: 10.1007/s00205-007-0092-2

[8] M. Murata, *On a maximal $L_p - L_q$ approach to the compressible viscous fluid flow with slip boundary condition*, Nonlinear Anal. **106** (2014), 86–109. Doi: 10.1016/j.na.2014.04.012

[9] J.-P. Raymond, and M. Vanninathan, *A fluid-structure model coupling the Navier-Stokes equations and the Lamé system*, J. Math. Pures Appl. (9) **102** (2014), 546–596. Doi: 10.1016/j.matpur.2013.12.004

[10] S. Maryani, *On the free boundary problem for the Oldroyd-B model in the maximal $L_p - L_q$ regularity class*, Nonlinear Anal., **141** (2016), 109–129. Doi: 10.1016/j.na.2016.03.024

[11] S. Maryani, *Global well-posedness for free boundary problem of the Oldroyd-B model fluid flow*, Mathematical Math. Methods Appl. Sci. **39** (2016), 2202–2219. Doi: 10.1002/mma.3634

[12] S. Maryani, and H. Saito, *On the $\mathcal{R}$-boundedness of solution operator families for two-phase Stokes resolvent equations*, Differ. Integral Equ. **30** (2017), 1–52.

[13] S. Maryani, B. H. Guswanto, and H. Gunawan, *Half-space model problem for Navier-Lamé equations with surface tension*, Math. Stat. **10** (2022) 498–514. Doi: 10.13189/ms.2022.100305

[14] Y. Shibata, *On the $\mathcal{R}$-boundedness of solution operators for the Stokes equations with free boundary condition*, Differ. Integral Equ. **27** (2014), 313–368.

[15] Y. Shibata, and S. Shimizu, *On the L_p-L_q maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, J. Reine Angew. Math. **615** (2008), 157–209. Doi: 10.1515/CRELLE.2008.013

[16] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann. **319** (2001), 735–758. Doi: 10.1007/PL00004457

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