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A NOTE ON WEAK w-PROJECTIVE MODULES

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Abstract. Let R be a ring. An R-module M is a weak w-projective module if $\mathrm{Ext}^1_R(M,N)=0$ for all N in the class of GV-torsion-free R-modules with the property that $\mathrm{Ext}^k_R(T,N)=0$ for all w-projective R-modules T and all integers $k\geq 1$. In this paper, we introduce and study some properties of weak w-projective modules. We use these modules to characterise some classical rings. For example, we will prove that a ring R is a DW-ring if and only if every weak w-projective is projective; R is a von Neumann regular ring if and only if every FP-projective module is weak w-projective if and only if every finitely presented R-module is weak w-projective; and R is w-semi-hereditary if and only if every finite type submodule of a free module is weak w-projective if and only if every finitely generated ideal of R is weak w-projective.

1. Introduction

In this paper, all rings are commutative with unity and all modules are unital. Let R be a ring and M be an R-module.

We review some definitions and notation. Let J be an ideal of R. Following [**24**], J is a *Glaz-Vasconcelos ideal* (a *GV-ideal* for short) if J is finitely generated and the natural homomorphism $\varphi : R \to J^* = \mathrm{Hom}_R(J,R)$ is an isomorphism. Note that the set $GV(R)$ of GV-ideals of R is a multiplicative system of ideals of R. If M is an R-module, then

$$\mathrm{tor}_{GV}(M) = \{x \in M \mid Jx = 0 \text{ for some } J \in GV(R)\}.$$

Clearly $\mathrm{tor}_{GV}(M)$ is a submodule of M. If $\mathrm{tor}_{GV}(M)=M$ (resp., $\mathrm{tor}_{GV}(M)=0$), then M is *GV-torsion* (resp., *GV-torsion-free*). A GV-torsion-free module M is a *w-module* if $\mathrm{Ext}^1_R(R/J,M)=0$ for every $J\in GV(R)$. Projective modules and reflexive modules are w-modules. In [**24**], it was shown that flat modules are w-modules. A GV-torsion-free R-module M is a w-module if and only if $\mathrm{Ext}^1_R(N,M)=0$ for every GV-torsion R-module N ([**14**], Theorem 6.2.7). The notion of w-modules was introduced firstly over a domain [**18**] in the study of strong Mori domains and was extended to commutative rings with zero divisors in [**24**]. If an ideal I of R is a w-module, then I is also a *w-ideal* of R. Let w-$Max(R)$ denote the set of w-ideals of R maximal among proper integral w-ideals of R (*maximal w-ideals*). Following [**24**, Proposition 3.8], every maximal w-ideal is prime. For every GV-torsion-free module M,

$$M_w := \{x \in E(M) \mid Jx \subseteq M \text{ for some } J \in GV(R)\}$$

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is a w-submodule of $E(M)$ containing M and is the *w-envelope* of M, where $E(M)$ denotes the injective hull of M. It is clear that a GV-torsion-free module M is a w-module if and only if $M_w = M$.

Let M and N be R-modules and let $f : M \to N$ be a homomorphism. Following [**13**], f is a *w-monomorphism* (resp., *w-epimorphism*, *w-isomorphism*) if $f_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is a *monomorphism* (resp., *an epimorphism*, *an isomorphism*) for all $\mathfrak{m} \in w\text{-}Max(R)$. A sequence $A \to B \to C$ of modules and homomorphisms is *w-exact* if the sequence $A_{\mathfrak{m}} \to B_{\mathfrak{m}} \to C_{\mathfrak{m}}$ is exact for all $\mathfrak{m} \in w\text{-}Max(R)$. An R-module M is of *finite type* if there exists a finitely generated free R-module F and a w-epimorphism $g : F \to M$. Similarly, an R-module M is of *finitely presented type* if there exists a w-exact sequence $F_1 \to F_0 \to M \to 0$, where F_1 and F_0 are finitely generated free.

In recent years, homological theoretic characterisation of w-modules has received much attention; see for example [**1, 19, 20, 22**]. The notion of w-projective modules and w-flat modules appeared first in [**12**] when R is an integral domain and was extended to an arbitrary commutative ring in [**2, 15, 19, 20**]. In [**15**], Wang and Kim generalised projective modules to w-projective modules by the w-operation. An R-module M is *w-projective* if $\mathrm{Ext}^1_R(L(M), N)$ is GV-torsion for every torsion-free w-module N, where $L(M) = (M/\operatorname{tor}_{GV}(M))_w$. Denote by $\mathcal{P}_w$ the class of all w-projective R-modules. Following [**21**], an R-module M is *w-split* if and only if $\mathrm{Ext}^1_R(M, N)$ is GV-torsion for all R-modules N. Denote by $\mathcal{S}_w$ the class of all w-split R-modules. Hence by [**21**, Corollary 2.5], every w-split module is w-projective. Following [**2**], an R-module M is *w-flat* if for every w-monomorphism $f : A \to B$, the induced sequence $1 \otimes f : M \otimes_R A \to M \otimes_R B$ is a w-monomorphism. Denote by $\mathcal{F}_w$ the class of all w-flat R-modules. Following [**19**], throughout this paper, $\mathcal{P}_w^{\dagger\infty}$ denotes the class of GV-torsion-free R-modules N with the property that $\mathrm{Ext}^k_R(M, N) = 0$ for all w-projective R-modules M and all integers $k \geq 1$. Clearly, every GV-torsion-free injective R-module belongs to $\mathcal{P}_w^{\dagger\infty}$. An R-module M is *weak w-projective* if $\mathrm{Ext}^1_R(M, N) = 0$ for all $N \in \mathcal{P}_w^{\dagger\infty}$. Wang and Qiao [**19**] introduce the notions of the weak w-projective dimension (w.w-pd) of a module and the global weak w-projective dimension (gl.w.w-dim) of a ring. Following [**19**], a GV-torsion-free module M is a *strong w-module* if $\mathrm{Ext}^i_R(N, M) = 0$ for every integer $i \geq 1$ and all GV-torsion modules N. Denote by $\mathcal{W}_\infty$ the class of all strong w-modules. Every GV-torsion-free injective module is a strong w-module. Clearly, $\mathcal{P}_w^{\dagger\infty} \subseteq \mathcal{W}_\infty$. In [**7**] it was shown that $\mathcal{W}_\infty$ properly contains $\mathcal{P}_w^{\dagger\infty}$.

Recall from [**4**] that an R-module M is *FP-projective* if $\mathrm{Ext}^1_R(M, N) = 0$ for every absolutely pure R-module N. Recall from [**3**] that an R-module A is *absolutely pure* if A is a pure submodule in every R-module which contains A as a submodule. Megibben [**5**] showed that an R-module A is absolutely pure if and only if $\mathrm{Ext}^1_R(F, A) = 0$ for every finitely presented module F. Hence an absolutely pure module is precisely an FP-injective module, as introduced in [**9**].

2. Results

In this section we introduce some characterisations of some classical rings. First we need the following result which is [**19**, Proposition 2.5].

Lemma 2.1. *An R-module M is weak w-projective if* $\mathrm{Ext}_R^k(M,N) = 0$ *for all* $N \in \mathcal{P}_w^{\dagger\infty}$ *and for all* $k > 0$.

Clearly the following containments hold:
$\{$ projective $\} \subseteq \{$ w-split $\} \subseteq \{$ w-projective $\} \subseteq \{$weak w-projective $\} \subseteq \{$ w-flat $\}$.
By [**1**, Proposition 2.5], if R is a perfect ring, then these five classes coincide.

We now give some characterisations of weak w-projective modules.

Proposition 2.2. *Let M be an R-module. The following are equivalent:*
(1) *M is weak w-projective.*
(2) *$M \otimes F$ is weak w-projective for every projective R-module F.*
(3) *$\mathrm{Hom}_R(F,M)$ is weak w-projective for every finitely generated projective R-module F.*
(4) *For every exact sequence of R-modules*

$$0 \to A \to B \to C \to 0$$

with $A \in \mathcal{P}_w^{\dagger\infty}$, the sequence

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to 0$$

is exact.
(5) *For every w-exact sequence of R-modules*

$$0 \to L \to E \to M \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(E,N) \to \mathrm{Hom}_R(L,N) \to 0$$

is exact for every R-module $N \in \mathcal{P}_w^{\dagger\infty}$.
(6) *For every exact sequence of R-modules*

$$0 \to L \to E \to M \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M,N) \to \mathrm{Hom}_R(E,N) \to \mathrm{Hom}_R(L,N) \to 0$$

is exact for every R-module $N \in \mathcal{P}_w^{\dagger\infty}$.

Proof. (1) $\Rightarrow$ (2). Let F be a projective R-module. If N is an R-module in $\mathcal{P}_w^{\dagger\infty}$, then $\mathrm{Ext}_R^1(F \otimes M, N) \cong \mathrm{Hom}_R(F, \mathrm{Ext}_R^1(M,N))$ by [**14**, Theorem 3.3.10]. Since M is weak w-projective, $\mathrm{Ext}_R^1(M,N) = 0$. Thus, $\mathrm{Ext}_R^1(F \otimes M, N) = 0$. Hence $F \otimes M$ is weak w-projective.

(2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1). These follow by letting $F = R$.

(1) $\Rightarrow$ (3). Let $N \in \mathcal{P}_w^{\dagger\infty}$. If F is a finitely generated projective R-module, then $F \otimes \mathrm{Ext}_R^1(M,N) \cong \mathrm{Ext}_R^1(\mathrm{Hom}_R(F,M),N)$ by [**14**, Theorem 3.3.12]. Since M is weak w-projective, $\mathrm{Ext}_R^1(M,N) = 0$. Hence $\mathrm{Ext}_R^1(\mathrm{Hom}_R(F,M),N) = 0$, which implies that $\mathrm{Hom}_R(F,M)$ is weak w-projective.

(1) $\Rightarrow$ (4). Let

$$0 \to A \to B \to C \to 0$$

be an exact sequence with $A \in \mathcal{P}_w^{\dagger\infty}$. Then we have the exact sequence

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C) \to \mathrm{Ext}_R^1(M,A).$$

Since M is weak w-projective and $A \in \mathcal{P}_w^{\dagger\infty}$, we deduce that $\mathrm{Ext}_R^1(M, A) = 0$. Thus,

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is exact.

(4) ⇒ (1). Let $N \in \mathcal{P}_w^{\dagger\infty}$. Let

$$0 \to N \to E \to L \to 0$$

be an exact sequence with E an injective module. Then we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M, L) \to \mathrm{Ext}_R^1(M, N) \to 0.$$

Keeping in mind that

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, E) \to \mathrm{Hom}_R(M, L) \to 0$$

is exact, we deduce that $\mathrm{Ext}_R^1(M, N) = 0$. Hence M is weak w-projective.

(1) ⇒ (5). Let

$$0 \to L \to E \to M \to 0$$

be a w-exact sequence. Let $N \in \mathcal{P}_w^{\dagger\infty}$. Then $N \in \mathcal{W}_\infty$. By [**19**, Lemma 2.1], we have the exact sequence

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(E, N) \to \mathrm{Hom}_R(L, N) \to \mathrm{Ext}_R^1(M, N).$$

Since M is weak w-projective, $\mathrm{Ext}_R^1(M, N) = 0$, and (5) holds.

(5) ⇒ (6). Trivial.

(6) ⇒ (1). Let

$$0 \to L \to E \to M \to 0$$

be an exact sequence with E projective. Hence for every R-module $N \in \mathcal{P}_w^{\dagger\infty}$,

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(E, N) \to \mathrm{Hom}_R(L, N) \to \mathrm{Ext}_R^1(M, N) \to 0$$

is an exact sequence. Keeping in mind that

$$0 \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(E, N) \to \mathrm{Hom}_R(L, N) \to 0$$

is exact, we deduce that $\mathrm{Ext}_R^1(M, N) = 0$, so M is weak w-projective. □

Recall from [**13**] that a ring is *w-coherent* if every finitely generated ideal of R is of finitely presented type. For more details of the functor Tor used below, see [**8, 14**].

Proposition 2.3. *Let R be a w-coherent ring, let E be an injective R-module, let M be a finitely presented type R-module and let N be an $R\{x\}$-module. If M is weak w-projective R-module, then* $\mathrm{Tor}_n^R(M, \mathrm{Hom}(N, E)) = 0$.

Proof. Let M be a weak w-projective R-module and let N be an $R\{x\}$-module and so in $\mathcal{P}_w^{\dagger\infty}$ by [**19**, Proposition 2.4]. Then $\mathrm{Ext}_R^n(M, N) = 0$ by [**19**, Proposition 2.5]. By [**17**, Proposition 2.13(6)],

$$\mathrm{Tor}_n^R(M, \mathrm{Hom}(N, E)) \cong \mathrm{Hom}(\mathrm{Ext}_R^n(M, N), E) = 0.$$

This implies that $\mathrm{Tor}_n^R(M, \mathrm{Hom}(N, E)) = 0$. □

Proposition 2.4. *Every weak w-projective R-module of finite type is of finitely presented type.*

Proof. Let M be a weak w-projective R-module of finite type. By [**19**, Corollary 2.9] M is w-projective of finite type. Thus, by [**14**, Theorem 6.7.22], M is of finitely presented type. □

Proposition 2.5. *Let M be a GV-torsion-free module.*
(1) *M_w/M is a weak w-projective module.*
(2) *M is weak w-projective if and only if M_w is.*

Proof. (1). Let M be a GV-torsion-free module. By [**14**, Proposition 6.2.5], M_w/M a GV-torsion module. Hence by [**19**, Proposition 2.3(2)], M_w/M is weak w-projective.

(2). Let N be an R-module in $\mathcal{P}_w^{\dagger\infty}$. Since M is GV-torsion-free, by (1) M_w/M is a weak w-projective module. Consider the following exact sequence

$$0 \to M \to M_w \to M_w/M \to 0,$$

which is w-exact. Hence by [**19**, Proposition 2.5], M is weak w-projective if and only if M_w is weak w-projective. □

Recall that a ring R is a *DW-ring* if every ideal of R is a w-ideal, or equivalently every maximal ideal of R is a w-ideal [**6**]. Examples of DW-rings are Prüfer domains, domains with Krull dimension one, and rings with Krull dimension zero. If R is a DW-ring, then every R-module is in $\mathcal{P}_w^{\dagger\infty}$.

In the following proposition, we will give new characterizations of DW-rings which are the only rings with these properties.

Proposition 2.6. *Let R be a ring. The following are equivalent:*
(1) *Every weak w-projective R-module is projective.*
(2) *Every w-projective R-module is projective.*
(3) *Every GV-torsion R-module is projective.*
(4) *Every GV-torsion-free R-module is a strong w-module.*
(5) *Every finitely presented type w-flat R-module is projective.*
(6) *Every weak w-projective R-module is a w-module.*
(7) *R is a DW-ring.*

Proof. (1) ⇒ (2) *and* (2) ⇒ (3). These are trivial.

(3) ⇒ (4). Let M be a GV-torsion-free R-module. For every GV-torsion R-module N and every $i > 0$, we deduce that $\mathrm{Ext}_R^i(N, M) = 0$ since N is projective. Hence M is a strong w-module.

(4) ⇒ (7). Since every strong w-module is a w-module, the implication follows by [**13**, Theorem 3.8].

(1) ⇒ (6). This is trivial, since every projective R-module is a w-module.

(2) ⇒ (5). Let M be a finitely presented type w-flat module. By [**19**, Corollary 2.9], M is finite type w-projective. Hence M is a projective R-module by (2).

(5) ⇒ (7). Let M be a finitely presented w-flat module. Now M is of finitely presented type w-flat, so M is projective by (5). Hence by [**10**, Proposition 2.1], R is a DW-ring.

(6) ⇒ (7). Let M be a GV-torsion-free R-module. Hence by Proposition 2.5, M_w/M is weak w-projective and so a w-module by (6). Thus, M_w/M is GV-torsion-free. Hence $M_w/M = 0$, and so $M_w = M$. Thus, M is a w-module. Then R is a DW-ring by [**13**, Theorem 3.8].

(7) ⇒ (1). Let M be a weak w-projective module. If N is an R-module, then $\mathrm{Ext}_R^1(M,N) = 0$ because $N \in \mathcal{P}_w^{\dagger_\infty}$ (since R is DW). Hence M is a projective module. □

Note that the equivalence (1) ⇔ (7) in Proposition 2.6 was given in [**7**, Proposition 4.4] for the domain case.

An R-module M has w-flat dimension at most n if $\mathrm{Tor}_{n+1}^R(M,N)$ is a GV-torsion R-module for all R-modules N (see [**20**]). Hence, the w-weak global dimension of R is defined to be

$$w\text{-w.gl.dim}(R) = \sup\{w\text{-fd}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Mao and Ding [**4**] proved that a ring R is *von Neumann regular* if and only if every FP-projective R-module is projective.

Next, we will give new characterisations of von Neumann regular rings by weak w-projective modules.

Proposition 2.7. *Let R be a ring. The following are equivalent:*
(1) *Every FP-projective R-module is weak w-projective.*
(2) *Every finitely presented R-module is weak w-projective.*
(3) *Every finitely presented R-module is w-flat.*
(4) *R is von Neumann regular.*

Proof. (1) ⇒ (2). This follows from the fact that every finitely presented R-module is FP-projective.

(2) ⇒ (3). This follows from [**19**, Corollary 2.11].

(3) ⇒ (4). Let I be a finitely generated ideal of R. Since R/I is finitely presented, by (3) it is w-flat. Hence R/I has w-flat dimension 0. Thus, w-w.gl.dim$(R) = 0$ by [**20**, Proposition 3.3]. Hence R is von Neumann regular by [**16**, Theorem 4.4].

(4) ⇒ (1). Let M be an FP-projective R-module. Then M is projective by [**4**, Remark 2.2]. Hence M is weak w-projective. □

Next, we will give an example of an FP-projective module which is not weak w-projective.

Example 2.8. Consider the local quasi-Frobenius ring $R := k[X]/(X^2)$, where k is a field, and denote by $\overline{X}$ the residue class in R of X. Then $(\overline{X})$ is an FP-projective R-module which is not weak w-projective. Indeed, since R is a quasi-Frobenius ring, every absolutely pure R-module is injective. Hence $\mathrm{Ext}_R^1((\overline{X}), N) = 0$ for every absolutely pure R-module N. So $(\overline{X})$ is FP-projective. But $(\overline{X})$ is not projective by [**11**, Example 2.2], and so not weak w-projective, since R is a DW-ring.

Recall from [**16**] that a ring R is *w-semi-hereditary* if every finite type ideal of R is w-projective.

Proposition 2.9. *The following are equivalent for a ring R:*
(1) *R is w-semi-hereditary.*
(2) *Every finite type submodule of a free R-module is weak w-projective.*
(3) *Every finite type ideal of R is weak w-projective.*
(4) *Every finitely generated submodule of a free R-module is weak w-projective.*
(5) *Every finitely generated ideal of R is weak w-projective.*

Proof. (1) $\Rightarrow$ (2). Let J be a finite type submodule of a free module. Hence J is w-projective by [**16**, Theorem 4.11]. Then J is weak w-projective by [**19**, Corollary 2.9].

(2) $\Rightarrow$ (3) $\Rightarrow$ (5) and (2) $\Rightarrow$ (4) $\Rightarrow$ (5). These are trivial.

(5) $\Rightarrow$ (1). Let J be a finite type ideal of R. Then J is w-isomorphic to a finitely generated subideal of J. Hence J is weak w-projective by hypothesis and [**19**, Corollary 2.7]. ☐

Proposition 2.10. *Every GV-torsion-free weak w-projective module is torsion-free.*

Proof. Let M be a GV-torsion-free weak w-projective module. Hence M is a GV-torsion-free w-flat module by [**19**, Corollary 2.11]. Thus by [**14**, Proposition 6.7.6], M is torsion-free. ☐

In the next example we show that a weak w-projective module need not be torsion-free.

Example 2.11. Let R be an integral domain and let J be a proper GV-ideal of R. Then $R \oplus R/J$ is a weak w-projective module but not torsion-free.

Proposition 2.12. *Let R be a ring and M be a finitely presented R-module. The following are equivalent:*

(1) *M is w-split.*
(2) *M is weak w-projective.*
(3) *For every w-exact sequence of R-modules*

$$0 \to A \to B \to C \to 0,$$

the sequence

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is w-exact.

Proof. (1) $\Rightarrow$ (2). This is trivial, since every w-split R-module is weak w-projective.

(2) $\Rightarrow$ (3). Let $0 \to A \to B \to C \to 0$ be a w-exact sequence of R-modules. For every maximal w-ideal $\mathfrak{m}$ of R, $0 \to A_\mathfrak{m} \to B_\mathfrak{m} \to C_\mathfrak{m} \to 0$ is an exact sequence of $R_\mathfrak{m}$-modules. Since $M_\mathfrak{m}$ is free by [**19**, Proposition 2.8], we have the exact sequence

$$0 \to \mathrm{Hom}_R(M_\mathfrak{m}, A_\mathfrak{m}) \to \mathrm{Hom}_R(M_\mathfrak{m}, B_\mathfrak{m}) \to \mathrm{Hom}_R(M_\mathfrak{m}, C_\mathfrak{m}) \to 0.$$

Since M is finitely presented, we have the commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, A_\mathfrak{m}) & \to & \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, B_\mathfrak{m}) & \to & \mathrm{Hom}_{R_\mathfrak{m}}(M_\mathfrak{m}, C_\mathfrak{m}) \\ \| \wr & & \| \wr & & \| \wr \\ \mathrm{Hom}_R(M, A)_\mathfrak{m} & \to & \mathrm{Hom}_R(M, B)_\mathfrak{m} & \to & \mathrm{Hom}_R(M, C)_\mathfrak{m} \end{array}$$

Thus,

$$0 \to \mathrm{Hom}_R(M, A)_\mathfrak{m} \to \mathrm{Hom}_R(M, B)_\mathfrak{m} \to \mathrm{Hom}_R(M, C)_\mathfrak{m} \to 0$$

is exact, and so

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to 0$$

is w-exact.

(3) $\Rightarrow$ (1). This follows from [**21**, Proposition 2.4]. ☐

Recall from [**23**] that a w-exact sequence of R-modules $0 \to A \to B \to C \to 0$ is *w-pure* exact if, for every R-module M, the induced sequence

$$0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$$

is w-exact.

Proposition 2.13. *Let C be a finitely presented type R-module. The following are equivalent:*
(1) *C is a weak w-projective R-module.*
(2) *Every w-exact sequence of R-modules $0 \to A \to B \to C \to 0$ is w-pure exact.*

Proof. (1) $\Rightarrow$ (2). The implication follows by [**19**, Corollary 2.11] and [**23**, Theorem 2.6].

(2) $\Rightarrow$ (1). Let $0 \to A \to B \to C \to 0$ be a w-exact sequence. Thus by hypothesis it is w-pure exact. Thus, C is w-flat by [**23**, Theorem 2.6]. Hence C is weak w-projective by [**19**, Corollary 2.9]. □

Proposition 2.14. *The following are equivalent for a finite type R-module M.*
(1) *M is a w-projective module.*
(2) *$\mathrm{Ext}_R^1(M, B) = 0$ for every $B \in \mathcal{P}_w^{\dagger\infty}$.*
(3) *$\mathrm{Ext}_R^1(M, N) = 0$ for every $R\{x\}$-module N.*
(4) *$M\{x\}$ is a projective $R\{x\}$-module.*

Proof. (1) $\Rightarrow$ (2). This is trivial.

(2) $\Rightarrow$ (3). This follows from [**19**, Proposition 2.4].

(3) $\Rightarrow$ (4). Let N be an $R\{x\}$-module. By [**17**, Proposition 2.5],

$$\mathrm{Ext}_{R\{x\}}^n(M\{x\}, N) \cong \mathrm{Ext}_R^n(M, N) = 0.$$

Thus, $M\{x\}$ is a projective $R\{x\}$-module.

(4) $\Rightarrow$ (1). This follows from [**14**, Theorem 6.7.18]. □

Recall from [**19**] that an R-module D is *$\mathcal{P}_w^{\dagger\infty}$-divisible* if it is isomorphic to E/N, where E is a GV-torsion-free injective R-module and $N \in \mathcal{P}_w^{\dagger\infty}$ is a submodule of E.

Proposition 2.15. *Let M be an R-module and let m be a positive integer. The following are equivalent.*
(1) $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leq m$.
(2) *$\mathrm{Ext}_R^m(M, D) = 0$ for every $\mathcal{P}_w^{\dagger\infty}$-divisible R-module D.*

Proof. (2) $\Rightarrow$ (1). Let $N \in \mathcal{P}_w^{\dagger\infty}$. There exists an exact sequence of R-modules $0 \to N \to E \to H \to 0$, where E is a GV-torsion-free injective R-module. Hence H is $\mathcal{P}_w^{\dagger\infty}$-divisible. Then we have the induced exact sequence

$$\mathrm{Ext}_R^m(M, H) \to \mathrm{Ext}_R^{m+1}(M, N) \to \mathrm{Ext}_R^{m+1}(M, E) = 0,$$

for every integer $m \geq 1$. The left term is zero by hypothesis. Hence $\mathrm{Ext}_R^{m+1}(M, N) = 0$, which implies that $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leq m$ by [**19**, Proposition 3.1].

(1) $\Rightarrow$ (2). Suppose $\mathrm{w}.w\text{-}\mathrm{pd}_R M \leq m$ and let D be a $\mathcal{P}_w^{\dagger\infty}$-divisible R-module. Then we have an exact sequence $0 \to N \to E \to H \to 0$, where E is a GV-torsion-free injective R-module and $N \in \mathcal{P}_w^{\dagger\infty}$. Hence we have the exact sequence

$$0 = \mathrm{Ext}_R^m(M, E) \to \mathrm{Ext}_R^m(M, H) \to \mathrm{Ext}_R^{m+1}(M, N).$$

The right term is zero by [**19**, Proposition 3.1]. Therefore, $\mathrm{Ext}_R^m(M,H)=0$. □

Proposition 2.16. *If M and N are two R-modules, then*

$$\mathrm{w.}w\text{-}\mathrm{pd}_R(M\oplus N)=\sup\{\mathrm{w.}w\text{-}\mathrm{pd}_R M,\mathrm{w.}w\text{-}\mathrm{pd}_R N\}.$$

Proof. The inequality

$$\mathrm{w.}w\text{-}\mathrm{pd}_R(M\oplus N)\leq\sup\{\mathrm{w.}w\text{-}\mathrm{pd}_R M,\mathrm{w.}w\text{-}\mathrm{pd}_R N\}$$

follows from the fact that the class of weak w-projective modules is closed under direct sums by [**19**, Proposition 2.5(1)]. For the reverse inequality, we may assume that $\mathrm{w.}w\text{-}\mathrm{pd}_R(M\oplus N)=n$ is finite. Thus, for every R-module $X\in\mathcal{P}_w^{\dagger_\infty}$,

$$\mathrm{Ext}_R^{n+1}(M\oplus N,X)\cong\mathrm{Ext}_R^{n+1}(M,X)\oplus\mathrm{Ext}_R^{n+1}(N,X).$$

By [**19**, Proposition 3.1], $\mathrm{Ext}_R^{n+1}(M\oplus N,X)=0$. Hence

$$\mathrm{Ext}_R^{n+1}(M,X)=\mathrm{Ext}_R^{n+1}(N,X)=0,$$

which implies that $\sup\{\mathrm{w.}w\text{-}\mathrm{pd}_R M,\mathrm{w.}w\text{-}\mathrm{pd}_R N\}\leq n$. □

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