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DISTANCE-DEPENDENT CHASE-ESCAPE ON TREES

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Abstract. We give a necessary and sufficient condition for species coexistence in a parasite-host growth process on infinite d-ary trees. The novelty of this work is that the spreading and death rates for hosts depend on the distance to the nearest parasite.

1. Introduction

Chase-escape is an interacting particle system inspired by certain parasite-host dynamics [**15, 12, 14**]. The same dynamics have been reinterpreted in a variety of applications: predator-prey systems, rumour scotching, infection spread, and malware repair in a device network [**5, 4, 6, 11, 3**]. Red (host) particles occupy and spread to adjacent vertices of a graph according to exponential clocks while facing the threat of blue (parasite) particles. When a host is infected by a parasite, the host perishes. The vertex where this occurs is occupied by the parasite for all time thereafter.

The authors of [**2**] introduced a variant called *chase-escape with death* in which red particles die at a given rate, independently of the spread of parasites. After the death of a red particle, its vertex is blocked and cannot be occupied by others. They studied this process on infinite d-ary trees and characterised the phase behaviour with the occurrence of death (the phase transitions of chase-escape were previously understood in [**13, 5**]). Much of the analysis in [**2**] relied on a novel connection to weighted Catalan numbers. We deepen this connection by generalising to the setting in which the host spreading and death rates depend on the distance to the nearest parasite.

1.1. Model definition.

We begin by defining *generalised chase-escape* on a d-ary tree. Fix $d \geq 1$ and let $\mathbb{T}_d$ denote the infinite, rooted d-ary tree in which every vertex has d children. Denote the root by $\mathbf{0}$ and let $\mathcal{T}_d$ be $\mathbb{T}_d$ augmented with an additional vertex $\mathfrak{b}$ attached to $\mathbf{0}$. Vertices of $\mathcal{T}_d$ are in one of four states $\{w, b, r, \dagger\}$. The vertices of $\mathcal{T}_d$ are the possible positions for the particles in the dynamic; to indicate this role, a *site* is a vertex of $\mathcal{T}_d$ with an associated state. State w is a "white" unoccupied site, state b is a "blue" site occupied by a parasite, state r is a "red" site occupied by a host, and state $\dagger$ is a site containing a "dead" host. Given vertices $u, v \in \mathcal{T}_d$, define $|u - v|$ to be the number of edges on the unique shortest

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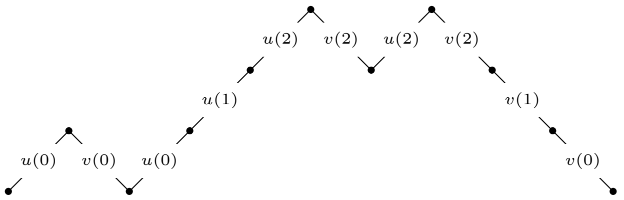

FIGURE 1. A Dyck path of length 10 with weight $u(0)^2v(0)^2u(1)v(1)u(2)^2v(2)^2$.

length path connecting u and v. We write $\pi(u)$ to be the set of vertices on the shortest path connecting $\mathfrak{b}$ to u.

The dynamics of *distance-dependent chase-escape* are as follows. Site $\mathfrak{b}$ is initially in state b. The root is initially in state r. All other sites begin in state w. Adjacent sites in states (b, r) transition to (b, b) according to a rate 1 Poisson process i.e., the time for each event is an exponential random variable with mean 1. To specify below the red spreading and death rates we take vectors

$$\vec{\lambda} = (\lambda_1, \lambda_2, \dots) \text{ and } \vec{\rho} = (\rho_1, \rho_2, \dots) \tag{1.1}$$

of nonnegative real numbers.

Given a site $u \in \mathcal{T}$, define the distance to the nearest site in state b as

$$\ell(u) = \min\{|u - v| : v \in \pi(u) \text{ and } v \text{ is in state } b\}. \tag{1.2}$$

Note that ℓ depends on the current configuration of the tree. Adjacent sites (u, v) with u in state r and v in state w have v transition to state r according to a Poisson process with intensity $\lambda_{\ell(u)}$ i.e., after exponentially distributed times with mean $1/\lambda_{\ell(u)}$. Meanwhile, a site u in state r transitions to state † according to a Poisson process with intensity $\rho_{\ell(u)}$. Unless the dependence is important to highlight, we will typically write $\mathbf{P}(A)$ rather than $\mathbf{P}_{\vec{\lambda},\vec{\rho}}(A)$ for events A pertaining to distance-dependent chase-escape.

In words, blue may *only* spread to red sites and does so at rate 1. Red may *only* spread to white sites and does so at a rate that depends on the distance to the nearest blue site. Similarly, the rate that a red particle dies depends on its distance to the nearest blue particle. Once a site is dead, it stays in the state †; in particular, it blocks the advancement of blue and red through such site. The chase-escape with death process from [**2**] is the special case of our model with $\lambda_i \equiv \lambda$ and $\rho_i \equiv \rho$ for all $i \geq 1$ with $\lambda, \rho > 0$.

We are interested in the persistence of both species. Let $\mathcal{B}$ equal the sites that are at some time colored blue. Since blue may only occupy sites that were at some time red, we say that *coexistence* occurs if $\mathbf{P}(|\mathcal{B}| = \infty) > 0$. We say that *expected coexistence* occurs if the weaker condition $\mathbf{E}[|\mathcal{B}|] = \infty$ occurs. The work [**2**] provided a necessary and sufficient condition for coexistence to occur and described the behaviour at the phase transition in detail. For the sake of a simpler, less technical generalisation of a main idea from [**2**], we focus on expected coexistence (rather than coexistence).

As observed in [**2**], the phase structure of chase-escape with death is connected to weighted Catalan numbers. We generalise this connection. Let

$$D_j = \sum_{i=1}^{j} \rho_i$$

and define the weights

$$u(j) = \frac{\lambda_{j+1}}{1+\lambda_{j+1}+D_{j+1}} \text{ and } v(j) = \frac{1}{1+\lambda_{j+2}+D_{j+2}}. \tag{1.3}$$

Given a lattice path γ consisting of unit rise and fall steps, each rise step from (x, j) to $(x+1, j+1)$ has weight $u(j)$, while a fall step from $(x, j+1)$ to $(x+1, j)$ has weight $v(j)$. If γ is a Dyck path of length $2k$ (nonnegative lattice path starting at $(0,0)$ consisting of k rise and k fall steps for some $k \geq 0$) then its *weight* $\omega(\gamma)$ is the product of the rise and fall step weights along γ. See Figure 1. The corresponding *weighted Catalan number* is

$$C_k^{\vec{\lambda},\vec{\rho}} = \sum_{\gamma \in \mathcal{D}_k} \omega(\gamma) \tag{1.4}$$

where $\mathcal{D}_k$ is the set of all Dyck paths of length $2k$. Denote the generating function by

$$g(z) = \sum_{k \geq 0} C_k^{\vec{\lambda},\vec{\rho}} z^k. \tag{1.5}$$

Let M be the largest value such that $|g(z)| < \infty$ for all complex numbers $|z| < M$.

The weights at (1.3) make the $C_k^{\vec{\lambda},\vec{\rho}}$ into probabilities related to *renewal events*, in a version of distance-dependent chase-escape on the nonnegative integers (see (2.2)). These renewal events correspond to a repetition of the initial conditions. On the nonnegative integers, a renewal occurs whenever blue is once again adjacent to the rightmost red site.

1.2. Result. In the setting from [**2**], it is proven that the radius of convergence of g relative to the degree of the tree determines the phase. This continues to hold in distance-dependent chase-escape.

Theorem 1.1. *Suppose that* $\lambda_1 > 0$, *that there exist constants* $c > 0$ *and* $m \geq 1$ *such that*

$$\lambda_{\ell+1} \prod_{i=1}^{\ell} \left(1 + \frac{\lambda_i}{1+D_i}\right) \leq c\ell^m \quad \textit{for all } \ell \geq 1, \tag{1.6}$$

and that for u *and* v *as at* (1.3)

$$\lim_{j \to \infty} u(j)v(j) = 0. \tag{1.7}$$

Then for $d \geq 2$, *expected coexistence on* $\mathcal{T}_d$ *occurs if and only if* $M \leq d$ *with* M *the radius of convergence of the generating function defined at* (1.5).

The hypotheses (1.6) and (1.7) are consequences of generalising the main result of [**2**]. It is unclear how much (1.6) can be relaxed. It is fairly robust. For example, (1.6) and (1.7) hold so long as $\lambda_i/D_i = O(i^{-\epsilon})$ and $\lambda_i = O(i^{\eta})$ for some $\epsilon, \eta > 0$. The hypothesis at (1.7) is essential to our argument as it allows us to apply the

Worpitzky Circle Theorem ([**16, 1**] and presented as Theorem 3.1 below). Note that (1.6) and (1.7) are easily verified in the case that $\vec{\lambda}$ and $\vec{\rho}$ are constant. Since we focus on the weaker requirement of expected coexistence, we are able to sidestep many technical difficulties. As a result, we give a streamlined presentation that clarifies and builds on some of the main ideas from [**2**].

We give a few remarks on generalisations. One, our results remain valid even if the λ_i are such that red reaches infinitely many sites in finite time. The hypothesis at (1.6) requires that $\lambda_i = O(D_i)$. Thus, at least one red will die along any explosive path before the explosion occurs. At this point, from the perspective of coexistence, it does not matter how quickly red reaches infinity. Two, one might be interested in the further generalisation in which the blue spreading rate varies. Defining this is trickier than the red spreading rate since it is not clear what distance to use. Indeed, there are often multiple red sites all adjacent to white sites whose paths to the nearest blue site overlap.

1.3. Proof overview. In Section 2, we analyse distance-dependent chase-escape on the non-negative integers. This lets us connect renewal events in the one-dimensional process to weighted Catalan numbers. In Lemma 2.2, we prove that the probability blue reaches beyond a given distance is comparable to the probability a renewal occurs at that distance. Section 3 contains the proof of Theorem 1.1. Lemma 2.2 gives us an upper bound $\mathbf{E}[|\mathcal{B}|]$ in terms of $g(d)$. This lets us deduce that $d < M$ implies $\mathbf{E}[|\mathcal{B}|] < \infty$. When $\mathbf{E}[|\mathcal{B}|] < \infty$, it is easy to see that $d \leq M$. To handle the boundary case $M = d$, we employ two classical results. One is the aforementioned Worpitsky's Circle Theorem, which we apply via a continued fraction characterisation of $g(z)$. We then apply Pringsheim's Theorem (see [**9**, Theorem IV.6] or Theorem 3.3 below) to deduce that $\mathbf{E}[|\mathcal{B}|] = \infty$ when $M = d$.

2. Distance-Dependent Chase-Escape on the Integers

We begin by defining distance-dependent chase-escape on the non-negative integers, which is equivalent to the case $\mathcal{T}_1$ with $\mathfrak{b} = 0$ and $\mathbf{0} = 1$. We indicate the state of the site n at time t by $s_t(n) \in \{w, b, r, \dagger\}$, which indicates if the site is white, blue, red or dead. Initially, $s_0(0) = b$, $s_0(1) = r$ and $s_0(n) = w$ for all $n > 1$, and the process follows the dynamics of distance dependent-chase escape with rates $\vec{\lambda}$ and $\vec{\rho}$. For each time $t \geq 0$, we write $B_t = \sup\{n \colon s_t(n) = b\}$ and define

$$Y = \sup\{B_t \colon t \geq 0\}. \tag{2.1}$$

If Y is finite, its value corresponds to the maximum integer reached by the blue particles.

Remark 2.1. Fixing $\rho_i = 0$ and $\lambda_i = \lambda$ for all $i \geq 1$, then distance-dependent chase-escape on the integers recovers the original chase-escape model on the integers. In this case $Y = \infty$ with positive probability if and only if $\lambda > 1$, see [**13, 5**] or [**7**]. However, chase-escape fails condition (1.6).

A natural case that we do not consider here is $\rho_i \equiv 0$ for all $i \geq 1$ i.e., distance-dependent chase-escape with no death. Our theorem does not apply since (1.6) is not satisfied. Nonetheless, we suspect that exact necessary and sufficient criteria for coexistence should be possible to obtain for the process on d-ary trees.

For $t \geq 0$, define the event

$$A_t = \{s_t(k) \neq \dagger \text{ for all } k\}$$

that there are no dead red sites at time t. Let $R_t = \sup\{n \colon s_j(n) = r \text{ for } j \leq t\}$ be the largest index of a site that has ever been red up to time t. Although $s_j(R_t)$ may be a dead site, on the event A_t, R_t represents the largest red site at time t. Since there are no dead sites blocking the way on the event A_t, the site R_t may be reachable by blue at a later time. Therefore, we will work on the event A_t in our subsequent analysis.

As in [**2**], we are mainly interested in the sites at which the process renews. For each site $k \geq 0$, call

$$\mathfrak{R}_k = \{B_t = k,\ R_t = k+1 \text{ and } s_t(n) \neq \dagger \text{ for all } n > k+1\}$$

a *renewal event at the site* k. At these points, the process exhibits its initial conditions with a translation by k.

We follow the evolution of distance-dependent chase-escape with a discrete-time Markov chain. Since we are tracking renewals, we will follow the process only when all the red particles are alive i.e. on the event A_t. Let

$$S_t = \begin{cases} R_t - B_t & \text{if } A_t \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

indicate the distance between the rightmost blue and red particles at time $t \geq 0$, as long as A_t holds. We define a discrete version of $(S_t)_{t\geq 0}$ by considering the collection of times when a particle changes its state. Let $\tau(0) = 0$ and

$$\tau(i) = \inf\{t \geq \tau(i-1) \,:\, S_t \neq S_{\tau(i-1)} \text{ or } \mathbb{1}(A_t) = 0\}.$$

The *jump chain* $J = (J_i)_{i\in\mathbb{Z}_+}$ of $(S_t)_{t\geq 0}$ is defined by

$$J_i = \begin{cases} S_{\tau(i)}, & \text{if } A_{\tau(i)} \text{ occurs} \\ 0 & \text{otherwise.} \end{cases}$$

Writing $D_j = \sum_{i=1}^{j} \rho_i$, the transition probabilities of the jump Markov chain are, for each $j > 1$,

$$p_{j,j+1} = \frac{\lambda_j}{1+\lambda_j+D_j}, \quad p_{j,j-1} = \frac{1}{1+\lambda_j+D_j}, \quad p_{j,0} = \frac{D_j}{1+\lambda_j+D_j};$$

for $j = 1$, we have

$$p_{1,2} = \frac{\lambda_1}{1+\lambda_1+D_1}, \quad p_{1,0} = \frac{1+D_1}{1+\lambda_1+D_1}.$$

We say that a jump chain is *living at step* n if $J_i > 0$ for all $0 \leq i \leq n$.

If $\gamma = (\gamma_0, \dots, \gamma_m)$ is a path on $\mathbb{Z}$ that may be realised by the jump chain J until time m, we thus refer to γ as a *jump chain path (of length m)*. Additionally if γ satisfies $\gamma_i > 0$ for all $0 \leq i \leq m$ then γ is a *living jump path*

On the event $\mathfrak{R}_k$, the path of the jump chain J (up to the renewal time) can be identified with a Dyck path of length $2k$ translated by one vertical unit. Moreover, the weights that we considered in (1.3) correspond to the transition probabilities of the jump chain: $p_{j,j+1} = u(j-1)$ and $p_{j,j-1} = v(j-2)$. It follows that

$$\mathbf{P}_{\vec{\lambda},\vec{\rho}}(\mathfrak{R}_k) = C_k^{\vec{\lambda},\vec{\rho}}, \tag{2.2}$$

the weighted Catalan number defined at (1.4) that uses the weights at (1.3).

We use ideas from [**2**, Lemma 2.2] to prove that $\mathbf{P}(Y \geq k)$ is comparable to $\mathbf{P}(\mathfrak{R}_k)$. The difficulty is that the event $\{Y \geq k\}$ includes *all* realisations for which blue reaches k, while $\mathfrak{R}_k$ only includes realisations which have a renewal at k.

Lemma 2.2. *Let c and m be as in* (1.6). *There exists c_0 that does not depend on k such that*

$$\mathbf{P}(Y \geq k) \leq c_0 k^{1+m} \mathbf{P}(\mathfrak{R}_k)$$

for all $k \geq 1$.

Proof. Define the *height profile* of a jump chain path of length m, $\gamma = (\gamma_0, \dots, \gamma_m)$, to be the vector $h(\gamma) = (h_1(\gamma), \dots, h_{m+1}(\gamma))$ which indicates the frequency of each height reached by γ. Formally, $h_i(\gamma) := \sum_{n=0}^{m} \mathbb{1}(\gamma_n = i)$.

Let γ be a living jump chain path of length m with $k-1$ upward steps. The probability that the distance-dependent chase-escape follows the path γ in its associated jump chain until the m-th step (and without specifying the behaviour afterwards) is

$$p_k(\gamma) = \prod_{i=1}^{k-1} \lambda_i^{h_i(\gamma)} \prod_{j=1}^{m+1} \left(\frac{1}{1 + \lambda_j + D_j} \right)^{h_j(\gamma)}.$$

It is necessary and sufficient for the occurrence of $\{Y \geq k\}$ to observe the following two events: *first* we need for the rightmost red particle to reach k; when red reaches this site, the rightmost blue particle is at $k - \ell$, for some $\ell > 0$. *Secondly*, we need for the rightmost blue particle to reach k after ℓ (space) steps. An advantage of this perspective is that the occurrence of the second stage only depends on the configuration in the interval $[k - \ell, \dots, k]$. In particular, it does not depend on the behaviour of red, and its potential deaths, beyond k. We now formally decompose jump chain paths corresponding to $\{Y \geq k\}$ into these two stages.

For a jump chain path γ corresponding to a configuration from $\{Y \geq k\}$, $\ell \in \{1, \dots, k\}$ is the value of the jump chain when red arrives to k. After this, we enter the second stage of $\{Y \geq k\}$: from the point of view of the particle system, the rightmost blue particle must advance ℓ space steps to reach k. Formally, for a given $\ell \in \{1, \dots, k\}$, the first and the second stages correspond to the events

$$\mathfrak{F}(\ell, k) = \{\text{for some } t > 0,\ R_t = k, B_t = k - \ell \text{ and } s_t(n) \neq \dagger \text{ for all } n\} \text{ and}$$
$$\mathfrak{S}(\ell, k) = \{\text{for some } t_1 > t_0 > 0,\ R_{t_0} = k, B_{t_0} = k - \ell \text{ and } B_{t_1} = k\},$$

respectively. Hence $\{Y \geq k\} = \bigcup_{\ell=1}^{k} (\mathfrak{F}(\ell, k) \cap \mathfrak{S}(\ell, k))$. The conditional probability of the event $\mathfrak{S}(\ell, k)$ given $\mathfrak{F}(\ell, k)$ is

$$\sigma(\ell, k) = \prod_{n=1}^{\ell} \mathbf{P} \left(\begin{array}{c} \text{blue advances 1 step before} \\ \text{any site } k - \ell + n, \dots, k \text{ dies} \end{array} \right) = \prod_{n=1}^{\ell} \frac{1}{1 + D_n}. \tag{2.3}$$

We now partition the event $\mathfrak{F}(\ell, k)$ according to the path observed in the jump chain. Let $\Gamma_{\ell,k}$ be the set of all living jump chain paths of length $2k - \ell - 1$ that start at $(0, 1)$ and end with an upward step to $(2k - \ell - 1, \ell)$, see Figure 2. The event

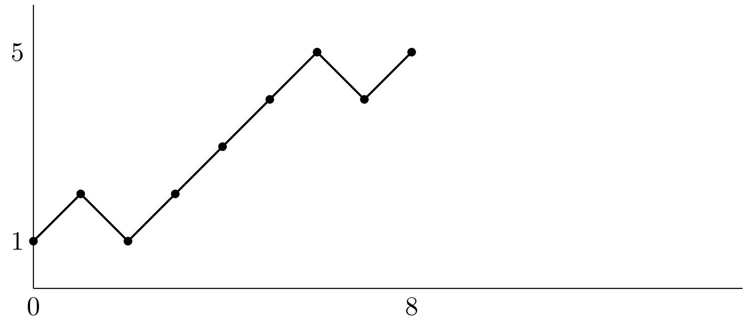

FIGURE 2. A path $\gamma \in \Gamma_{5,7}$ ($\ell = 5$ and $k = 7$). If this path is realised by the jump chain it indicates the evolution of the particle system up to the time when red first reaches location $k = 7$, while the rightmost blue particle is at 2 and the sites $\{3, \dots, 7\}$ are red.

$\mathfrak{F}(\ell, k)$ happens if, and only if, the jump chain realises a path in $\Gamma_{\ell,k}$. Furthermore, since we stop the jump chain at the first hitting time of site k (i.e. at an upward step to $(2k - \ell - 1, \ell)$), it suffices to consider $\ell \in \{2, \dots, k\}$.

For $\gamma \in \Gamma_{\ell,k}$, the probability that $Y \geq k$ and the first $2k - \ell - 1$ steps of the jump chain follow γ is $q_{\ell,k}(\gamma) := p_k(\gamma)\sigma(\ell, k)$. Setting $Q_{\ell,k} := \sum_{\gamma \in \Gamma_{\ell,k}} q_{\ell,k}(\gamma)$, we arrive at the following decomposition of $\{Y \geq k\}$:

$$\mathbf{P}(Y \geq k) = \sum_{\ell=2}^{k} \sum_{\gamma \in \Gamma_{\ell,k}} p_k(\gamma)\sigma(\ell, k) = \sum_{\ell=2}^{k} Q_{\ell,k}. \tag{2.4}$$

A subset of $\mathfrak{R}_k$ is the collection of processes which follow jump chain paths in $\Gamma_{2,k}$, and subsequently have blue advance by one, then red advance by one, followed by blue advancing one. This gives the bound

$$Q_{2,k} \frac{\lambda_2}{(1 + \lambda_1 + D_1)(1 + \lambda_2 + D_2)^2} \leq \mathbf{P}(\mathfrak{R}_k).$$

We will now prove that, given a fixed k, $Q_{2,k}$ is comparable to $Q_{\ell,k}$ for all $\ell \geq 5$. The choice of $\ell \geq 5$ is for convenience in the proof, the terms for $\ell = 2, 3, 4$ are finite and can thus be absorbed into the constant c_0.

Given $\gamma \in \Gamma_{\ell,k}$, we obtain $\tilde{\gamma} \in \Gamma_{2,k}$ by inserting $\ell - 2$ downward steps before the last upward step. See Figure 3. The paths γ and $\tilde{\gamma}$ agree on the first $2k - \ell - 2$ steps, so

$$q_{\ell,k}(\gamma) = \frac{\lambda_{\ell-1}}{\lambda_1} \frac{\sigma(\ell, k)}{\sigma(2, k) \prod_{i=1}^{\ell-2} (1 + \lambda_i + D_i)^{-1}} q_{2,k}(\tilde{\gamma}) \tag{2.5}$$

$$= \frac{\lambda_{\ell-1}}{\lambda_1} \frac{\prod_{n=1}^{\ell} \frac{1}{1+D_n}}{\prod_{n=1}^{2} \frac{1}{1+D_n} \prod_{i=1}^{\ell-2} (1 + \lambda_i + D_i)^{-1}} q_{2,k}(\tilde{\gamma}) \tag{2.6}$$

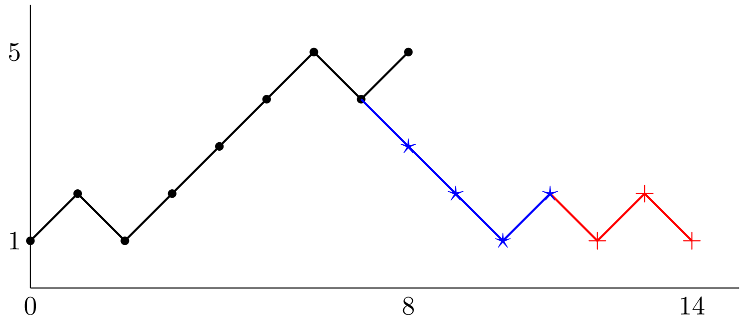

FIGURE 3. Let $k = 7$. The black line with dots is a path $\gamma \in \Gamma_{5,7}$. The black line with dots (up to time 7) followed by the blue line with stars is the modified path $\tilde{\gamma} \in \Gamma_{2,7}$. The concatenation of the black line with dots (up to time 7), the blue line with stars and the red line with pluses is a jump chain path in $\mathfrak{R}_7$.

Simplifying gives

$$q_{\ell,k}(\gamma) = q_{2,k}(\tilde{\gamma}) \frac{(1 + D_1 + \lambda_1)(1 + D_2 + \lambda_2)}{\lambda_1(1 + D_\ell)(1 + D_{\ell-1})} \lambda_{\ell-1} \prod_{i=3}^{\ell-2} \left(1 + \frac{\lambda_i}{1 + D_i}\right). \tag{2.7}$$

Taking c and m as at (1.6) set

$$c_0' = c\frac{(1 + D_1 + \lambda_1)(1 + D_2 + \lambda_2)}{\lambda_1} \geq c\frac{(1 + D_1 + \lambda_1)(1 + D_2 + \lambda_2)}{\lambda_1(1 + D_\ell)(1 + D_{\ell-1})},$$

and the inequality holds since $D_{\ell-1}, D_\ell \geq 0$. It follows from our hypothesis (1.6), and the inequality above, that for all $\gamma \in \Gamma_{\ell,k}$ with $\ell \geq 5$

$$q_{\ell,k}(\gamma) \leq c_0' \ell^m q_{2,k}(\tilde{\gamma}).$$

Choose c_0 possibly larger than c_0' so that the above inequality is true for all $\ell \geq 2$. As the map $\gamma \in \Gamma_{\ell,k} \mapsto \tilde{\gamma} \in \Gamma_{2,k}$ is injective, we have for $\ell \geq 2$

$$Q_{\ell,k} \leq \sum_{\gamma \in \Gamma_{\ell,k}} c_0 \ell^m q_{2,k}(\tilde{\gamma}) \leq c_0 \ell^m \sum_{\gamma \in \Gamma_{2,k}} q_{2,k}(\gamma) = c_0 \ell^m Q_{2,k}. \tag{2.8}$$

Applying (2.8) at (2.4) and summing completes the lemma.

□

3. Proof of Theorem 1.1

3.1. Preliminaries. This treatment is similar to that in [**2**, Section 3]. For completeness, we reproduce and generalize the necessary elements. Given a sequence $(c_n)_{n \geq 0}$, define the formal continued fraction

$$K[c_0, c_1, \ldots] := \cfrac{c_0}{1 - \cfrac{c_1}{1 - \ddots}}. \tag{3.1}$$

Let

$$a_j := u(j)v(j) = \frac{\lambda_{j+1}}{(1+\lambda_{j+1}+D_{j+1})(1+\lambda_{j+2}+D_{j+2})}. \tag{3.2}$$

It follows from Flajolet's relation between continued fractions and generating functions related to path enumeration [**8**] that the generating function g from (1.5) admits a continued fraction representation, so that

$$g(z) = f(z) := K[1, a_0 z, a_1 z, \ldots] \tag{3.3}$$

for all $|z| < M$. (See also [**10**, Chapter 5] for a comprehensive account of this theory.)

A classical theorem of Worpitzky [**16, 1**] lets us prove that f is *meromorphic* i.e. holomorphic outside of a set of isolated poles.

Theorem 3.1 (Worpitzky Circle Theorem)**.** *Let* $c_j \colon D \to \{|w| < 1/4\}$ *be a family of analytic functions over a domain* $D \subseteq \mathbb{C}$*. Then* $K[1, c_0(z), c_1(z), \ldots]$ *converges uniformly for* z *in any compact subset of* D*.*

Corollary 3.2. *Assuming* (1.7)*, it holds that* f *is meromorphic on* $\mathbb{C}$*.*

Proof. We will prove that f is meromorphic for all $z \in \Delta = \{|z| < r_0\}$ with $r_0 > 0$ arbitrary. Let $T_j(z) := K[a_j z, a_{j+1} z, \ldots]$ be the tail of the continued fraction so that $f(z) = K[1, a_0 z, \ldots, a_{j-1} z, T_j(z)]$. By the hypothesis (1.7), we have $|a_j| \downarrow 0$ as $j \to \infty$. It follows that for some $j = j(r_0)$ large enough, $|a_k z| \leq 1/4$ for all $k \geq j$ and $z \in \Delta$. Theorem 3.1 ensures that $|T_j(z)| < \infty$ and the partial continued fractions $K[a_j z, \ldots, a_n z]$ are analytic (again by Theorem 3.1) and converge uniformly to T_j for $z \in \Delta$. Thus, T_j is a uniform limit of analytic functions and is therefore analytic on Δ. We can then write $f(z) = K[1, a_0 z, \ldots, a_{j-1} z, T_j(z)]$. Since each $a_i z$ is a linear function in z, f is a quotient of two analytic functions. □

Our next lemma requires a classical theorem from complex variable theory (see [**9**, Theorem IV.6] for example).

Theorem 3.3 (Pringsheim's Theorem)**.** *If* $\varphi(z)$ *is representable at the origin by a power series* $\varphi(z) = \sum_{n=0}^{\infty} a_n z^n$ *that has real coefficients* $a_n \geq 0$ *and radius of convergence* M*, then the point* $z = M$ *is a singularity of* $\varphi(z)$*.*

Lemma 3.4. *Let* $\rho > 0$*. Then* $M \leq d$ *if and only if* $g(d) = \infty$*.*

Proof. We first note that the implication "$M < d$ implies $g(d) = \infty$" as well as the reverse direction "$g(d) = \infty$ implies $M \leq d$" both follow immediately from the definition of the radius of convergence. It remains to show that $M = d$ implies $g(d) = \infty$. Corollary 3.2 proves that the only singularities of f are poles. Moreover, Theorem 3.3 gives $z = d$ is a singularity of a function whose power series with positive coefficients, thus $f(d) = \infty$. Since $g = f$ for $|z| < d$, we must also have $g(d) = \infty$. □

3.2. Proof of Theorem 1.1.

Proof. Index the d^k sites at distance k from the root of $\mathcal{T}_d$ by $(v_{k,i})_{i=1}^{d^k}$. Let Y be as at (2.1). Since the tree contains no cycles, the probability a vertex $v_{k,i}$ at distance k from the root is eventually blue is the same as its analogue in chase-escape on the line. This observation and self-similarity of the d^k vertices at distance k from the root of $\mathcal{T}_d$ allow us to write

$$\mathbf{E}[|\mathcal{B}|] = 1 + \mathbf{E}\left[\sum_{k=0}^{\infty}\sum_{i=1}^{d^k} \mathbf{1}\{v_{k,i} \text{ is eventually blue}\}\right] \tag{3.4}$$

$$= 1 + \sum_{k=0}^{\infty} \mathbf{P}(Y \geq k) d^k. \tag{3.5}$$

Now, suppose that $M > d$. Using (3.5) and the comparison in Lemma 2.2 gives

$$E[|\mathcal{B}|] \leq 1 + \sum_{k=1}^{\infty} c_0 k^{1+m} \mathbf{P}(\mathfrak{R}_k) d^k. \tag{3.6}$$

Since $M > d$, the sum on the right converges even with the polynomial prefactor. Thus, $\mathbf{E}[|\mathcal{B}|] < \infty$.

Next, suppose that $\mathbf{E}[|\mathcal{B}|] < \infty$. Since $\{Y \geq k\}$ contains the event that a renewal occurs at k followed by blue advancing one step, we have

$$\mathbf{P}(\mathcal{R}_{k-1}) \frac{1}{1 + \lambda_1 + \rho_1} \leq \mathbf{P}(Y \geq k).$$

Applying this bound to (3.5), and reindexing the sum while applying the identity at (2.2) gives

$$\mathbf{E}[|\mathcal{B}|] \geq \frac{d}{1 + \lambda_1 + \rho_1} \sum_{k=0}^{\infty} \mathbf{P}(\mathfrak{R}_k) d^k = \frac{1}{1 + \lambda_1 + \rho_1} g(d). \tag{3.7}$$

Hence, $g(d) < \infty$, which gives $M > d$ by Lemma 3.4. ☐

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