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THE PROJECTIVE SYMMETRY GROUP OF A FINITE FRAME

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Abstract. We define the *projective symmetry group* of a finite sequence of vectors (a frame) in a natural way as a group of permutations on the vectors (or their indices). This definition ensures that the projective symmetry group is the same for a frame and its complement. We give an algorithm for computing the projective symmetry group from a small set of projective invariants when the underlying field is a subfield of $\mathbb{C}$ which is closed under conjugation. This algorithm is applied in a number of examples including equiangular lines (in particular SICs), MUBs, and harmonic frames.

1. Introduction

1.1. Motivation. Finite frames provide redundant and stable expansions, which have numerous applications [**6**]. The 1–dimensional projections in the frame expansion are unchanged if the vectors are multiplied by scalars of unit modulus. In many applications of finite frames, e.g., robustness to erasures, the cross correlation between the vectors (the modulus of their inner products) is is vitally important. This is unchanged if the vectors are multiplied by a unitary matrix, or by scalars of unit modulus, i.e., the frame is considered up to projective unitary equivalence. Many frames which are optimal for applications are projectively unitary equivalent to one from a special class of frames, e.g., harmonic frames, SICs (equiangular tight frames with a maximal number of vectors) and MUBs (used in quantum information theory). Understanding such a frame, which usually comes as a group orbit, is often helped by knowing its projective symmetry group. For example, the standard method for finding an analytic form of a SIC is to find a generating vector which is an eigenvector of a projective symmetry. This allows one to simplify the system of equations which determine such a vector. It may not be clear that a given frame is a group frame, e.g., the original constructions of harmonic frames and MUBs were not as group orbits. There is a growing body of evidence that complex (projective) spherical t–designs with the minimal number of vectors are group frames, or the orbit a small number of vectors. This can be determined (for any frame) by calculating the projective symmetry group. Moreover, knowing the full symmetry group simplifies the counting of frames up to projective unitary equivalence. In this paper, we give an explicit parallelisable algorithm for computing the projective symmetry

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group of any finite frame. We then apply this to SICs, MUBs, and harmonic frames (including calcuating their number of erasures).

1.2. Key definitions. Let $\Phi = (v_j)_{j\in J}$ be a finite sequence of vectors. A "projective symmetry" of (v_j) is given by an invertible linear map L for which (Lv_j) equals (v_j) up to reordering and multiplication by unit scalars, i.e.,

$$Lv_j = c_j v_{\sigma j}, \qquad |c_j| = 1, \quad \sigma \in S_J. \tag{1.1}$$

The example $\Phi = (e_1, -e_1, e_2, -e_2)$ shows that the choice for L and (c_j) is far from unique, and the example $\Phi = (e_1, e_1, e_2, e_2)$ that L and (c_j) are not sufficient to determine the permutation σ. For these reasons, we define the projective symmetry group as a group of permutations on the index set J (or the vectors themselves). Throughout, we assume all vector spaces X are over a subfield of $\mathbb{C}$. If the span of a finite sequence (v_j) is X, then we will refer to (v_j) as a **frame** for X, as is commonly done when X is an inner product space.

Definition 1.1. *Let $\Phi = (v_j)_{j\in J}$ be a finite sequence of vectors with span X. Then*
(1) The **projective symmetry group** *of Φ is*

$$\mathrm{Sym}_P(\Phi) := \{\sigma \in S_J : \exists L \in \mathcal{G}L(X), |c_j| = 1 \text{ with } Lv_j = c_j v_{\sigma j}, \forall j \in J\}.$$

(2) The **symmetry group** *of Φ is*

$$\mathrm{Sym}(\Phi) := \{\sigma \in S_J : \exists L \in \mathcal{G}L(X) \text{ with } Lv_j = v_{\sigma j}, \forall j \in J\}.$$

These are clearly subgroups of S_J (the symmetric group on J), and the symmetry group is a subgroup of the projective symmetry group, i.e.,

$$\mathrm{Sym}(\Phi) \subset \mathrm{Sym}_P(\Phi) \subset S_J.$$

One can also consider projective symmetries induced by antilinear maps (see Section 5). We say that $(v_j)_{j\in J}$ and $(w_j)_{j\in J}$ which span X and Y are **projectively similar** if

$$w_j = c_j Q v_j, \qquad \forall j, \tag{1.2}$$

and **similar** if

$$w_j = Q v_j, \qquad \forall j, \tag{1.3}$$

where $Q : X \to Y$ is invertible, and $|c_j| = 1$. We observe,

- Projectively similar sequences of vectors have the same projective symmetry group.
- Similar sequences of vectors have the same symmetry group.

The symmetry group and its calculation from the Gram matrix of the canonical tight frame was studied in [**20**], [**21**]. At that time (cf. §5 of [**21**]), it was believed that the projective symmetry group could not be calculated in general (except over $\mathbb{R}$) because of the nonuniqueness of the L and (c_j) in (1.1). A recent characterisation of projective similarity in terms of a small number of projective invariants (m–products) [**4**], [**9**] now makes this possible.

1.3. Outline. The paper is set out as follows. In Section 2, we recall some basic frame theory [**6**], [**24**], and then extend it to vector spaces without inner products. In particular, a sequence of vectors is similar to a (canonical) tight frame. This allows the complement of a sequence of vectors (v_j) to be defined (up to projective similarity). We show that a frame and is complement have the same projective symmetry group. In Section 3, we give a set of projective invariants which determine a sequence of vectors up to projective similarity. This requires that the underlying field be closed under complex conjugation. In Section 4, we give an algorithm for calculating the projective symmetry group from a small set of the projective invariants. The only known algorithm [**16**] is for the special case of d^2 equiangular vectors in $\mathbb{C}^d$ which are given as a group orbit. In Section 5, we consider "antilinear symmetries", and the corresponding extended projective symmetry group and its calculation. In Section 6, we consider simplifications in our algorithm that occur for group frames, and apply the algorithm to find the extended projective symmetry group of certain SICs and MUBs. In Section 7, we give the results of our extensive calculations of the projective symmetry group, and extended projective symmetry group of harmonic frames.

2. Tight Frames and the Complement of a Frame

We now give the basic theory of tight frames we require (see [**25**] for further detail). A sequence of vectors $\Phi = (v_j)_{j\in J}$ in a real or complex Hilbert space $\mathcal{H}$ is said to be a **frame** for $\mathcal{H}$ with **frame bounds** $A, B > 0$ if

$$A\|f\|^2 \leq \sum_{j\in J} |\langle f, v_j\rangle|^2 \leq B\|f\|^2, \qquad \forall f \in \mathcal{H}. \tag{2.4}$$

For J finite this is equivalent to the (v_j) having span $\mathcal{H}$. When $A = B$, Φ is said to be a **tight frame**, and when $A = B = 1$ it is a **normalised tight frame**. For tight frames the polarisation identity implies (2.4) is equivalent to

$$f = \frac{1}{A}\sum_{j\in J}\langle f, v_j\rangle v_j, \qquad \forall f \in \mathcal{H}. \tag{2.5}$$

A tight frame is said to be **equiangular** if its vectors have equal norms, and there is some $C > 0$ with

$$|\langle v_j, v_k\rangle| = C, \qquad \forall j \neq k.$$

The **synthesis operator** for a finite sequence $(v_j)_{j\in J}$ in $\mathcal{H}$ is the linear map

$$V := [v_j]_{j\in J} : \ell_2(J) \to \mathcal{H} : a \mapsto \sum_{j\in J} a_j v_j,$$

and its **frame operator** is the linear map $S = S_V = VV^* : \mathcal{H} \to \mathcal{H}$ given by

$$Sf := \sum_{j\in J}\langle f, v_j\rangle v_j, \qquad \forall f \in \mathcal{H}.$$

If $\Phi = (v_j)_{j\in J}$ is a frame, then S is invertible, and the **canonical dual frame** $\tilde{\Phi} = (\tilde{v}_j)$ is defined by

$$\tilde{v}_j := S^{-1}v_j, \qquad \forall j \in J, \tag{2.6}$$

and the **canonical tight frame** $\Phi^{\text{can}} = (v_j^{\text{can}})$ by

$$v_j^{\text{can}} = S^{-\frac{1}{2}}v_j, \qquad \forall j \in J. \tag{2.7}$$

A frame and its canonical dual satisfy the expansion

$$f = \sum_{j\in J} \langle f, v_j \rangle \tilde{v}_j = \sum_{j\in J} \langle f, \tilde{v}_j \rangle v_j, \qquad \forall f \in \mathcal{H},$$

and the canonical tight frame is a normalised tight frame, i.e.,

$$f = \sum_{j\in J} \langle f, v_j^{\rm can} \rangle v_j^{\rm can}, \qquad \forall f \in \mathcal{H}.$$

In view of definitions (2.6) and (2.7), a frame, its canonical dual and canonical tight frame are all similar. A simple calculation shows that normalised tight frames are similar if and only if they are unitarily equivalent, i.e., the Q in (1.3) can be taken to be unitary. The **Gramian** of a sequence of n vectors $\Phi = (v_j)_{j\in J}$ in $\mathcal{H}$ is the $n \times n$ matrix

$$\mathrm{Gram}(\Phi) := V^* V = [\langle v_k, v_j \rangle]_{j,k\in J}.$$

The injective linear map

$$R_\Phi = V^* S_V^{-1} = V^* (VV^*)^{-1}$$

takes the vectors of a frame Φ to the columns of

$$\mathrm{Gram}(\Phi^{\rm can}) = (S_V^{-\frac{1}{2}} V)^* S_V^{-\frac{1}{2}} V = V^*(VV^*)^{-1} V = V^* S_V^{-1} V, \tag{2.8}$$

i.e.,

$$R_\Phi v_j = V^*(VV^*)^{-1} v_j = \mathrm{Gram}(\Phi^{\rm can}) e_j,$$

where e_j is the j–th standard basis vector. A sequence of vectors is a normalised tight frame (for its span) if and only if its Gramian matrix P is an orthogonal projection matrix, i.e., $P^2 = P$ and $P = P^*$. Hence for the purpose of determining similarity, Φ can be replaced by the normalised tight frame given by the columns of the orthogonal projection matrix $\mathrm{Gram}(\Phi^{\rm can})$. In view of (2.8), $\mathrm{Gram}(\Phi^{\rm can})$ can be calculated without taking the square root of the frame operator S_V, a preprocessing step which is usually not numerically stable. Using the theory of frames for vector spaces [**22**], this association can be extended to vector spaces (without an inner product), where $\mathrm{Gram}(\Phi^{\rm can})$ is replaced by the orthogonal projection matrix P_Φ (cf. [**9**]). This requires that the underlying field $\mathbb{F}$ of the vector space X be a subfield of $\mathbb{C}$ which is closed under conjugation, e.g., $\mathbb{F} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$. We assume this from now on.

Definition 2.1. *Let $\Phi = (v_j)_{j\in J}$ be a finite sequence of vectors in X, with synthesis operator $V = [v_j]_{j\in J}$. The subspace of all linear dependencies between the vectors of Φ is*

$$\mathrm{dep}(\Phi) := \ker(V) = \{a \in \mathbb{F}^J : \textstyle\sum_j a_j v_j = 0\},$$

and we denote the orthogonal projection onto $\mathrm{dep}(\Phi)^\perp$ *(orthogonal complement) by* P_Φ.

If $X = \mathrm{span}(\Phi)$ and $\Lambda : X \to \mathbb{F}^m$ is an injective linear map, then

$$P_\Phi = (\Lambda V)^\dagger \Lambda V,$$

where $A^\dagger$ is the pseudoinverse of A. For Φ a frame, taking $\Lambda = V^*$, gives

$$P_\Phi = \mathrm{Gram}(\Phi^{\rm can}).$$

The sequence Φ is similar to the normalised tight frame (for a subspace of $\mathbb{F}^J$) given by (Pe_j), the columns of $P = P_\Phi$, via the well defined injective linear map $R_\Phi : X \to \mathbb{F}^J$ with

$$R_\Phi v_j = P_\Phi e_j, \qquad \forall j. \tag{2.9}$$

The Euclidean inner product between these vectors is

$$\langle Pe_j, Pe_k\rangle = P_{kj},$$

i.e., P is the Gram matrix of (Pe_j).

Proposition 2.2. *Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite frames for X and Y. Then*

(1) Φ and Ψ are similar if and only if

$$P_\Psi = P_\Phi.$$

(2) Φ and Ψ are projectively similar, i.e., $w_j = c_j Q v_j$, if and only if

$$P_\Psi = C^* P_\Phi C,$$

where C is the diagonal matrix with diagonal entries c_j of unit modulus.

Proof. It suffices to show that Φ and Ψ are projectively similar, i.e., $w_j = c_j Q v_j$, equivalently $W = QVC$, if and only if $P_\Psi = C^* P_\Phi C$. ($\Longrightarrow$) Suppose that Φ and Ψ are projectively similar. Let $\Lambda : X \to \mathbb{F}^m$ be an injective linear map. Then $\Lambda Q^{-1} : Y \to \mathbb{F}^m$ is an injective linear map, and so

$$\begin{aligned} P_\Psi &= (\Lambda Q^{-1} W)^\dagger \Lambda Q^{-1} W = (\Lambda Q^{-1} QVC)^\dagger \Lambda Q^{-1} QVC \\ &= (\Lambda VC)^\dagger \Lambda VC = C^* (\Lambda V)^\dagger \Lambda VC = C^* P_\Phi C. \end{aligned}$$

($\Longleftarrow$) Suppose that $P_\Psi = C^* P_\Phi C$. By the implication just proved,

$$P_\Phi = C P_\Psi C^* = P_{(\overline{c_j} w_j)}.$$

Thus $Q := R^{-1}_{(c_j w_j)} R_\Phi$ maps v_j to $\overline{c_j} w_j$, i.e., $w_j = c_j Q v_j$. □

We say that two frames Φ and Ψ are **complementary** (or **complements** of each other) if P_Φ and P_Ψ are complementary projection matrices, i.e.,

$$P_\Phi + P_\Psi = I. \tag{2.10}$$

The complement of a frame is well defined up to similarity. In view of Proposition 2.2, the complement of a tight frame in the class of normalised tight frames is well defined up to unitary equivalence, and complement of a frame in the class of projectively similar frames is defined up to projective similarity. There is a bijection between permutations $\sigma \in S_J$ and the $J \times J$ *permutation matrices*, given by $\sigma \mapsto P_\sigma$, where

$$P_\sigma e_j := e_{\sigma j}.$$

We can express a symmetry σ in terms of P_σ as follows.

Lemma 2.3. *Let $\Phi = (v_j)_{j\in J}$ be a finite frame for X. Then*

(1) $\sigma \in \mathrm{Sym}(\Phi) \iff P_\sigma^* P_\Phi P_\sigma = P_\Phi$.

(2) $\sigma \in \mathrm{Sym}_P(\Phi) \iff P_\sigma^* P_\Phi P_\sigma = C^* P_\Phi C$, for some unitary diagonal matrix C.

Proof. We observe that $\sigma \in \mathrm{Sym}(\Phi)$ if and only if $\Phi = (v_j)$ is similar to $\Psi = (v_{\sigma j})$, and $\sigma \in \mathrm{Sym}_P(\Phi)$ if and only if Φ is projectively similar to $\Psi = (v_{\sigma j})$. Let $V = [v_j]$, and $\Lambda : X \to \mathbb{F}^m$ be an injective linear map. Then

$$[v_{\sigma j}] = [Ve_{\sigma j}] = V[e_{\sigma j}] = VP_\sigma,$$

so that

$$P_\Psi = (\Lambda VP_\sigma)^\dagger \Lambda VP_\sigma = P_\sigma^*(\Lambda V)^\dagger \Lambda VP_\sigma = P_\sigma^* P_\Phi P_\sigma,$$

and so we obtain the result by applying Proposition 2.2. □

In [**21**] (Theorem 3.7) it was shown that if Ψ is a complement of Φ up to similarity, i.e., $P_\Phi + P_\Psi = I$, then

$$\mathrm{Sym}(\Psi) = \mathrm{Sym}(\Phi).$$

We now prove the corresponding result for the projective symmetry group.

Theorem 2.4. *Suppose that $\Phi = (v_j)_{j\in J}$ is a finite frame for X. If Ψ is a complement of Φ up to projective similarity, i.e., $P_\Phi + CP_\Psi C^* = I$, where C is a unitary diagonal matrix, then*

$$\mathrm{Sym}_P(\Psi) = \mathrm{Sym}_P(\Phi).$$

Proof. Since the definition of frames being complementary is symmetric, it suffices to show that $\mathrm{Sym}_P(\Phi) \subset \mathrm{Sym}_P(\Psi)$. Suppose $\sigma \in \mathrm{Sym}_P(\Phi)$. Then, by Lemma 2.3,

$$P_\sigma^* P_\Phi P_\sigma = C_\sigma^* P_\Phi C_\sigma,$$

for some unitary diagonal matrix C_σ. Now

$$P_\sigma^* P_\Psi P_\sigma = P_\sigma^*(I - C^* P_\Phi C)P_\sigma = I - P_\sigma^* C^* P_\Phi C P_\sigma.$$

Let c_j be the diagonal entries of the matrix C. Since $(c_j v_j)$ is projectively similar to $\Phi = (v_j)$, $P_{(c_j v_j)} = C^* P_\Phi C$, and $\sigma \in \mathrm{Sym}_P((c_j v_j)) = \mathrm{Sym}_P(\Phi)$, we have

$$P_\sigma^* C^* P_\Phi C P_\sigma = \tilde{C}_\sigma^* C^* P_\Phi C \tilde{C}_\sigma,$$

for some unitary diagonal matrix $\tilde{C}_\sigma$. Thus

$$P_\sigma^* P_\Psi P_\sigma = I - \tilde{C}_\sigma^* C^* P_\Phi C \tilde{C}_\sigma = I - \tilde{C}_\sigma^*(I - P_\Psi)\tilde{C}_\sigma = \tilde{C}_\sigma^* P_\Psi \tilde{C}_\sigma,$$

and so, by Lemma 2.3, we have $\sigma \in \mathrm{Sym}_P(\Psi)$. □

Example 2.5. *Let $\Phi = (v_j)_{j\in J}$ be an equal–norm tight frame of $d+1$ vectors for $\mathbb{C}^d$, e.g., the vertices of the regular simplex. Since P_Φ has a constant diagonal, the complement Ψ of Φ consists of $d+1$ equal–norm vectors (w_j) in $\mathbb{C}^1$. Let $w_j = [a_j]$. For any $\sigma \in S_J$,*

$$w_j = c_j w_{\sigma j}, \qquad c_j := a_j a_{\sigma j}^{-1},$$

so that $\sigma \in \mathrm{Sym}_P(\Phi)$, and we obtain

$$\mathrm{Sym}_P(\Phi) = \mathrm{Sym}_P(\Psi) = S_J.$$

Thus all equal–norm tight frames $\Phi = (v_j)_{j\in J}$ of $d+1$ vectors in $\mathbb{C}^d$ are projectively similar, with projective symmetry group $\mathrm{Sym}_P(\Phi) = S_J$.

3. Projective Invariants

For the purpose of determining projective similarity between $\Phi = (v_j)$ and $\Psi = (w_j)$, and hence calculating $\mathrm{Sym}_P(\Phi)$, it suffices to assume that Φ and Ψ are the *normalised tight frames* given by the columns of P_Φ and P_Ψ. Under this assumption, if Φ and Ψ are projectively similar, i.e., $w_j = c_j Q v_j$, equivalently $W = QVC$, then Q is unitary, since

$$I = WW^* = QVCC^*V^*Q^* = QVV^*Q^* = QQ^*.$$

Further, the conditions of Proposition 2.2 become

$$\langle w_j, w_k\rangle = (P_\Psi)_{kj} = (P_\Phi)_{jk} = \langle v_j, v_k\rangle \qquad \text{(similarity)}.$$

$$\langle w_j, w_k\rangle = (P_\Psi)_{kj} = (C^* P_\Phi C)_{jk} = c_j\bar{c}_k\langle v_j, v_k\rangle \qquad \text{(projective similarity)}.$$

The second of these shows that the inner product between vectors is not a projective invariant (preserved by a projective similarity), indeed

$$\langle w_j, w_k\rangle = \langle c_j Q v_j, c_k Q v_k\rangle = c_j\overline{c_k}\langle Qv_j, Qv_k\rangle = c_j\overline{c_k}\langle v_j, v_k\rangle.$$

There do exist projective invariants, e.g.,

$$\begin{aligned}\langle w_{j_1}, w_{j_2}\rangle\langle w_{j_2}, w_{j_3}\rangle\langle w_{j_3}, w_{j_1}\rangle &= \langle c_{j_1}Qv_{j_1}, c_{j_2}Qv_{j_2}\rangle\langle c_{j_2}Qv_{j_2}, c_{j_3}Qv_{j_3}\rangle\langle c_{j_3}Qv_{j_3}, c_{j_1}Qv_{j_1}\rangle \\ &= c_{j_1}\overline{c_{j_2}}\langle v_{j_1}, v_{j_2}\rangle c_{j_2}\overline{c_{j_3}}\langle v_{j_2}, v_{j_3}\rangle c_{j_3}\overline{c_{j_1}}\langle v_{j_3}, v_{j_1}\rangle \\ &= \langle v_{j_1}, v_{j_2}\rangle\langle v_{j_2}, v_{j_3}\rangle\langle v_{j_3}, v_{j_1}\rangle.\end{aligned}$$

Generalising this example gives the following projective invariants.

Definition 3.1. *Let $\Phi = (v_j)_{j\in J}$ be a sequence of n vectors (in a vector space over a subfield of $\mathbb{C}$ which is closed under complex conjugation), and $P = P_\Phi = [p_{jk}]$. Then the (***canonical***) m–***products*** of Φ are*

$$\begin{aligned}\Delta_C(v_{j_1},\ldots,v_{j_m}) &:= \langle Pe_{j_1}, Pe_{j_2}\rangle\langle Pe_{j_2}, Pe_{j_3}\rangle\cdots\langle Pe_{j_{m-1}}, Pe_{j_m}\rangle \\ &= p_{j_2 j_1}p_{j_3 j_2}\cdots p_{j_1 j_m}, \qquad j_1,\ldots,j_m\in J, \quad 1\le m\le n.\end{aligned} \tag{3.11}$$

Clearly, $\Delta_C(v_{j_1},\ldots,v_{j_m})$ is invariant under cyclic shifts of $j_1,\ldots,j_m$, and so it is often convenient to think of it being defined on the m–cycle $(j_1,\ldots,j_m)$. A subset of the m–products is called a **determining set** for Φ if it characterise Φ up to projective similarity. In [**9**] certain determining sets were studied.

Theorem 3.2. *([**9**]) Let $\Phi = (v_j)$ and $\Psi = (w_j)$ be finite sequences of vectors in vector spaces over a subfield $\mathbb{F}$ of $\mathbb{C}$ which is closed under complex conjugation. Then*

(1) Φ and Ψ are similar if and only if $P_\Phi = P_\Psi$.

(2) Φ and Ψ are projectively similar if and only if their canonical m–products (for a determining set) are equal.

Here the first equivalence is included for the purpose of comparison (see Proposition 2.2). The subsets of the m–products which are determining sets depend on the zeros of P_Φ. The **frame graph** of Φ is the graph with vertices the vectors of $\Phi = (v_j)$ and

$$\text{an edge between } v_j \text{ and } v_k,\ j\ne k \quad\Longleftrightarrow\quad \langle v_j, v_k\rangle \ne 0.$$

One choice for a determining set is the m–products which correspond to a basis for the cycle space of the frame graph. In particular, when P_Φ has no zero entries, i.e., the frame graph is complete, a determining set is given by the 3–products. From Theorem 3.2, we obtain the following computable condition for a permutation $\sigma \in S_J$ to be in the projective symmetry group of $\Phi = (v_j)_{j\in J}$.

Proposition 3.3. *Let $\Phi = (v_j)_{j\in J}$ be a finite frame for X. Then $\sigma \in \mathrm{Sym}_P(\Phi)$ if and only if*

$$\Delta_C(v_{j_1}, \dots, v_{j_m}) = \Delta_C(v_{\sigma j_1}, \dots, v_{\sigma j_m}),$$

for all cycles $(j_1, \dots, j_m)$ from a determining set for Φ.

Proof. Since $\sigma \in \mathrm{Sym}_P(\Phi)$ if and only if Φ is projectively similar to $\Psi = (v_{\sigma j})$, this follows from Theorem 3.2. $\square$

4. The Algorithm

For frames $\Phi = (v_j)_{j\in J}$ and $\Psi = (w_j)_{j\in J}$ of n vectors, we now give an algorithm which determines the set of σ for which Φ and $(w_{\sigma j})$ are projectively similar, i.e.,

$$\Delta_C(v_{j_1}, \dots, v_{j_m}) = \Delta_C(w_{\sigma j_1}, \dots, w_{\sigma j_m}), \tag{4.12}$$

for all cycles $(j_1, \dots, j_m)$ from a determining set for Φ. In particular, for $\Psi = \Phi$ it calculates $\mathrm{Sym}_P(\Phi)$, and if there exists some σ then Φ and Ψ are projectively equivalent. A priori, the calculation of (4.12) requires one to consider all $n!$ permutations $\sigma \in S_J$. To make this feasible (for large n), we seek an algorithm which checks the m–product condition efficiently, i.e, for many permutations at a time. There are two cases:
(1) $\mathrm{Sym}_P(\Phi)$ is large, i.e., the m–products take few different values.
(2) $\mathrm{Sym}_P(\Phi)$ is small, i.e., the m–products take many different values.
An extreme example of the first is the vertices of a regular d–simplex (cf. Example 2.5), where

$$(P_\Phi)_{jk} = \begin{cases} \frac{d}{d+1}, & j = k; \\ -\frac{1}{d+1}, & j \neq k, \end{cases} \qquad \mathrm{Sym}_P(\Phi) = S_J.$$

Here the m–products are all equal (for fixed m), and so it is easy to check each $\sigma \in S_J$ is a projective symmetry. In such cases, where $\mathrm{Sym}_P(\Phi)$ is large, one could try to build it using generators: starting with the identity subgroup, check whether a random permutation not in the subgroup of $\mathrm{Sym}_P(\Phi)$ known so far is in $\mathrm{Sym}_P(\Phi)$, and if so use it to generate a larger subgroup. If the index of $\mathrm{Sym}_P(\Phi)$ in S_J is small, then this process has a high probability of quickly finding generators for $\mathrm{Sym}_P(\Phi)$. Henceforth, we will concern ourselves only with the second case: when $\mathrm{Sym}_P(\Phi)$ is small, and the m–products take many different values. This is the generic situation. Indeed, if the diagonal entries of P_Φ are distinct, then so are the 1–products, and hence $|\mathrm{Sym}_P(\Phi)| = 1$. For an index set J of size n, we define a k–**flag** f to be an ordering of k distinct elements of J

$$f = (j_1, j_2, \dots, j_k).$$

For a given fixed n–flag

$$f_b = (j_1, \dots, j_n),$$

we can represent the permutation $\sigma : j_\ell \mapsto \sigma j_\ell$ (giving a projective similarity or symmetry) by the n–flag

$$f_\sigma = (\sigma j_1, \ldots, \sigma j_n).$$

Thus determining whether $\Phi = (v_j)$ and $\Psi = (w_{\sigma j})$ are projectively similar is equivalent to determining which of the $n!$ permutations σ, i.e., n–flags f_σ, satisfy (4.12). We think of each possible n–flag $f_\sigma = (\sigma j_1, \ldots, \sigma j_n)$ as being built up from the 0–flag () by successively adding entries

$$f_\sigma^0 = (), \quad f_\sigma^1 = (\sigma j_1), \quad f_\sigma^2 = (\sigma j_1, \sigma j_2), \quad \ldots \quad f_\sigma^n = (\sigma j_1, \sigma j_2, \ldots, \sigma j_n).$$

We will call the operation of going from a set $\mathcal{F}_{k-1}$ of $(k-1)$–flags to a set $\mathcal{F}_k$ of k–flags as **growing**. At the k–th stage there are $n - k + 1$ choices for the next entry, so that

$$|\mathcal{F}_k| \le (n - k + 1)|\mathcal{F}_{k-1}|.$$

If $|\operatorname{Sym}(\Phi)| < n!$, then, at some stage, not all $f_\sigma^k \in \mathcal{F}_k$ will be extendable to an n–flag satisfying (4.12). A necessary condition for such an extension to exist is that (4.12) hold for all cycles (of length $\le k$) on the first k indices of the fixed flag $f_b = (j_1, \ldots, j_n)$ from a determining set for $(v_{j_1}, \ldots, v_{j_k})$. Removing elements from $\mathcal{F}_k$ because they fail this condition (either in full or in part) will be called **pruning**. When the condition is imposed for all eligible cycles then we have a **full pruning**, otherwise a **partial pruning**. In these terms, our algorithm for finding the set $\mathcal{F}_n$ of n–flags f_σ giving the permutations σ that Φ and $(w_{\sigma j})$ are projectively similar is: **Algorithm**[1] (to determine the n–flags $\mathcal{F}_n$ giving a projective similarity).

```
Let 𝓕₀ := {()} consist of the empty flag
for k from 1 to n do
  Grow 𝓕ₖ₋₁ to 𝓕ₖ
  Prune 𝓕ₖ
end for
Fully prune 𝓕ₙ, if necessary.
```

The art is in balancing the cost of pruning, with that of growing the set of possible k–flags overly large. One can do this on a case by case basis, or program an adaptive algorithm. For our calculations, detailed in the next sections, we used full pruning, which is easily programmed. We stored each k–flag $(j_1, \ldots, j_k)$ as an n–vector

$$(j_1, \ldots, j_k | J \setminus \{j_1, \ldots, j_k\}),$$

$$(j_1, \ldots, j_k | j_{k+1}, \ldots, j_n), \qquad \{j_{k+1}, \ldots, j_n\} = J \setminus \{j_1, \ldots, j_k\},$$

so that the $(k+1)$–flags could easily be constructed. The algorithm can easily be parallelised: simply partition $\mathcal{F}_k$ in any way, at any stage k, and apply the algorithm to each subset. We now illustrate our algorithm with a couple of examples, where $\Psi = \Phi$. As a pruning rule we ask that a k–flag $(j_1, \ldots, j_k)$ match

$$\Delta_C(v_{j_1}, \ldots, v_{j_k}) = \Delta_C(w_{\sigma j_1}, \ldots, w_{\sigma j_k}).$$

Thus at each stage we check only one new m–product, which is easily calculated.

[1]The code (in magma and maple) used in our implementation of this algorithm can be found on the second author's homepage under a link to this paper.

Example 4.1. *The simplest example of a SIC (see Section 6) is the equiangular tight frame* $\Phi := (v, Sv, \Omega v, S\Omega v)$ *of four vectors for* $\mathbb{C}^2$*, where*

$$v := \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{3+\sqrt{3}} \\ e^{\frac{\pi}{4}i}\sqrt{3-\sqrt{3}} \end{pmatrix}, \qquad S := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \Omega := \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}.$$

We have

$$P_\Phi = \frac{1}{2} \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & 1 & \frac{-i}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{i}{\sqrt{3}} & 1 & -\frac{1}{\sqrt{3}} \\ \frac{i}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 1 \end{pmatrix}.$$

Take the base flag to be $(1, 2, 3, 4)$*. The empty flag (0–flag)*

$$\mathcal{F}_0 = \{()\}, \qquad () = (|1, 2, 3, 4)$$

grows to the set of 1*-flags*

$$\mathcal{F}_1 = \{(1), (2), (3), (4)\}.$$

The pruning rule is that $\langle v_1, v_1\rangle = \langle w_{\sigma 1}, w_{\sigma_1}\rangle$*, i.e., the norm is preserved, and so there is no pruning. Growing gives*

$$\mathcal{F}_2 = \{(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3)\},$$

and pruning gives no reduction since Φ *is equiangular. We now consider growing the* 2*–flag* $(3, 2)$*, the others being similar. This grows to the* 3*–flags* $(3, 2, 1)$*,* $(3, 2, 4)$*. Since*

$$\Delta_C(v_1, v_2, v_3) = \frac{i}{24\sqrt{3}}, \qquad \Delta_C(w_3, w_2, w_1) = -\frac{i}{24\sqrt{3}}, \quad \Delta_C(w_3, w_2, w_4) = \frac{i}{24\sqrt{3}},$$

the 3*–flag* $(3, 2, 1)$ *is pruned. Continuing in this way gives*

$$\begin{aligned} \mathcal{F}_3 = \{&(1,2,3), (1,3,4), (1,4,2), (2,1,4), (2,3,1), (2,4,3), \\ &(3,1,2), (3,2,4), (3,4,1), (4,1,3), (4,2,1), (4,3,2)\}. \end{aligned}$$

The final stage $k = n$*, growing does not increase the size of* $\mathcal{F}_{n-1}$*, and in this case nothing gets pruned, by the rule used, or a full prune. Thus we have*

$$\begin{aligned} \mathrm{Sym}_P(\Phi) = \mathcal{F}_4 = \{&(1,2,3,4), (1,3,4,2), (1,4,2,3), (2,1,4,3), (2,3,1,4), (2,4,3,1), \\ &(3,1,2,4), (3,2,4,1), (3,4,1,2), (4,1,3,2), (4,2,1,3), (4,3,2,1)\}. \end{aligned}$$

This is the alternating group A_4*. Generators for* $\mathrm{Sym}_P(\Phi)$ *(written in cycle notation), and matrices* L *which induce them are*

$$(12)(34) \quad S, \qquad (13)(24) \quad \Omega, \qquad (132) \quad B := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}.$$

Example 4.2. *Let* $\Phi = (v_j)$ *be the following two MUBs (see Section 6) in* $\mathbb{C}^2$

$$v_1 = e_1, \quad v_2 = e_2, \quad v_3 = \frac{1}{\sqrt{2}}(e_1 + e_2), \quad v_4 = \frac{1}{\sqrt{2}}(e_1 - e_2).$$

Here

$$P_\Phi = \frac{1}{2}\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \end{pmatrix}.$$

We arrive at the same $\mathcal{F}_2$ as in Example 4.1, without pruning. The pruning rule says that the modulus of the inner product between v_1 and v_2 must be preserved. Since this is zero, the index pairs in $\mathcal{F}_2$ must correspond to pairs of orthogonal vectors, which leads to the pruning

$$\mathcal{F}_2 = \{(1,2),(2,1),(3,4),(4,3)\}.$$

Growing this gives

$$\mathcal{F}_3 = \{(1,2,3),(1,2,4),(2,1,3),(2,1,4),(3,4,1),(3,4,2),(4,3,1),(4,3,2)\}.$$

All 3–products for distinct vectors are zero, and so there is no pruning at this stage. Growing, then full pruning leads to

$$\begin{aligned}\mathrm{Sym}_P(\Phi) = \mathcal{F}_4 = \{&(1,2,3,4),(1,2,4,3),(2,1,3,4),(2,1,4,3),\\ &(3,4,1,2),(3,4,2,1),(4,3,1,2),(4,3,2,1)\}.\end{aligned}$$

This group is the dihedral group of order 8 (the only subgroup of S_4 of order 8), which is generated by the following permutations

$$(1324)\quad \textit{(rotation through 90 degrees)},\qquad (34)\quad \textit{(reflection in the } x\textit{–axis)}.$$

5. The Extended Projective Symmetry Group

Let $K:\mathbb{C}^d\to\mathbb{C}^d$ be the **complex conjugate operator**

$$Kz := \overline{z} = (\overline{z_j}).$$

A product of a linear/unitary map with complex conjugation is called an **anti-linear/antiunitary** map (these are not linear maps). Antiunitary maps preserve the modulus of the inner product, and so take SICs to SICs, and MUBs to MUBs. For this reason it is useful to extend to projective symmetry group (see [**2**] for SICs). Since the product of two antilinear/antiunitary maps is linear/unitary, we can extend the groups of linear maps and unitary maps

$$\mathcal{E}GL(\mathbb{C}^d) := \{LK^s : L\in\mathcal{G}L(\mathbb{C}^d), s=0,1\},\qquad \mathcal{E}U(\mathbb{C}^d) := \{UK^s : L\in\mathcal{U}(\mathbb{C}^d), s=0,1\},$$

and the projective symmetry group as follows.

Definition 5.1. *Let $\Phi=(v_j)_{j\in J}$ be a finite sequence of vectors with span $\mathbb{C}^d$. Then the* **extended projective symmetry group** *of Φ is*

$$\mathrm{Sym}_{EP}(\Phi) := \{\sigma\in S_J : \exists A\in\mathcal{E}GL(\mathbb{C}^d),\ |c_j|=1 \textit{ with } Av_j = c_j v_{\sigma j},\ \forall j\in J\},$$

and we will refer to σ as an **anti projective symmetry** *if the A above is antilinear.*

A permutation σ is an anti projective symmetry of $\Phi=(v_j)$ if and only if for some linear L

$$L\overline{v_j} = c_j v_{\sigma j},\qquad \forall j\in J,$$

i.e., $\overline{\Phi}=(\overline{v_j})$ and $(v_{\sigma j})$ are projectively similar. Since $P_{\overline{\Phi}}=\overline{P_\Phi}$,

$$\Delta_C(\overline{v_{j_1}},\ldots,\overline{v_{j_m}}) = \overline{\Delta_C(v_{j_1},\ldots,v_{j_m})} = \Delta_C(v_{j_m},\ldots,v_{j_1}).$$

Thus (by Theorem 3.2), σ is an anti projective symmetry of $\Phi = (v_j)$ if and only if

$$\Delta_C(v_{j_m}, \ldots, v_{j_1}) = \Delta_C(v_{\sigma j_1}, \ldots, v_{\sigma j_m}), \tag{5.13}$$

for all cycles $(j_1, \ldots, j_m)$ from a determining set for Φ. Since condition (5.13) is just (4.12) with the ordering of $(v_{j_1}, \ldots, v_{j_m})$ reversed, our algorithm can be modified to calculate the anti projective symmetries by simply replacing P_Φ by its transpose P_Φ^T.

Example 5.2. *Applying the full pruning algorithm to Example 4.1, with the m–products $\Delta_C(v_{j_1}, \ldots, v_{j_m})$ replaced by their conjugates, and base flag $(1, 2, 3, 4)$ gives the following anti projective symmetries*

$$\begin{aligned}\mathcal{F}_4 = \{&(1,2,4,3), (1,3,2,4), (1,4,3,2), (2,1,3,4), (2,3,4,1), (2,4,1,3), \\ &(3,1,4,2), (3,2,1,4), (3,4,2,1), (4,1,2,3), (4,2,3,1), (4,3,1,2)\}.\end{aligned}$$

Hence, we have

$$\mathrm{Sym}_P(\Phi) = A_4 \subset \mathrm{Sym}_{EP}(\Phi) = S_4.$$

Since the product of two anti projective symmetries is a projective symmetry, if there exist antiprojective symmetries of Φ, then

$$|\,\mathrm{Sym}_{EP}(\Phi)| = 2|\,\mathrm{Sym}_P(\Phi)|,$$

and $\mathrm{Sym}_{EP}(\Phi)$ is generated by the anti projective symmetries, or by $\mathrm{Sym}_P(\Phi)$ together with any anti projective symmetry. In all cases

$$\mathrm{Sym}_P(\Phi) \triangleleft \mathrm{Sym}_{EP}(\Phi), \qquad [\mathrm{Sym}_{EP}(\Phi) : \mathrm{Sym}_P(\Phi)] = 1 \text{ or } 2.$$

6. Group Frames, Nice Error Bases, SICs and MUBs

Many tight frames of interest come as the orbit of a unitary group action on $\mathcal{H} = \mathbb{R}^d, \mathbb{C}^d$ (cf. [**23**], [**24**]). For these, the calculation of $\mathrm{Sym}_P(\Phi)$ can be simplified. Let G be a finite group. A **group frame** or G–**frame** for $\mathcal{H}$ is a frame $\Phi = (v_g)_{g\in G}$ for which there exists a unitary representation $\rho : G \to \mathcal{U}(\mathcal{H})$ with

$$gv_h := \rho(g)v_h = v_{gh}, \qquad \forall g, h \in G.$$

When $\rho(G)$ contains scalar matrices other than the identity, then the vectors in $\Phi = (gv)_{g\in G}$ are repeated (up to unit scalar multiples). In this case it is often convenient to consider the frame $(gv)_{g\in\rho(G)/Z}$ where Z is the group of scalar matrices in $\rho(G)$. This setup can also be described in the language of *projective representations*, and in some special cases *nice error bases* (cf. [**14**], [**17**], [**10**]). Without loss of generality, we now assume Z is a subgroup of $\rho(G)$ consisting of scalar matrices, so that if $(gv)_{g\in G}$ is a group frame, then $\Phi = (gv)_{gZ\in\rho(G)/Z}$ is "group frame", where the vectors gv are defined up to multiplication by a unit scalar (and so the m–products are well defined). For simplicity, we will write $g \in G$ for $gZ \in \rho(G)/Z$. Recent work of [**7**], [**25**] extends the theory of group frames to this situation, with the corresponding frames being called *twisted group frames* or *projective group frames* for the (abstract) group $\rho(G)/Z$. Let $\Phi = (gv)_{g\in G}$ be a group frame. Since the index set is G, the projective symmetries are permutations of G. We observe that each $h \in G$ induces a projective symmetry

$$\tau_h : G \to G : g \mapsto hg,$$

and the subgroup $\tau_G = \{\tau_h\}_{h\in G}$ of $\mathrm{Sym}_P(\Phi)$ acts transitively on G. Moreover, if $\sigma \in \mathrm{Sym}_P(\Phi)$ and $h \in G$, then

$$\sigma = \tau_{\sigma(h)h^{-1}}\sigma_h, \qquad \sigma_h := \tau_{h\sigma(h)^{-1}}\sigma,$$

i.e., σ is a product of an element of τ_G and a $\sigma_h \in \mathrm{Sym}_P(\Phi)$ which fixes h (the vector hv). Thus we obtain the following.

Proposition 6.1. *Let $\Phi = (gv)_{g\in G}$ be a group frame, then $\mathrm{Sym}_P(\Phi)$ is generated by the "translations" τ_G, and the stabiliser of any vector hv, i.e.,*

$$\mathrm{Stab}(h) := \{\sigma \in \mathrm{Sym}_P(\Phi) : \sigma(h) = h\}.$$

In particular, to calculate $\mathrm{Sym}_P(\Phi)$ it suffices to find the stabiliser of some vector.

For $|G| = n$, the calculation of $\mathrm{Stab}(h)$ as above, requires the examination of the $(n-1)!$ permutations which fix h, and checking (4.12) for only those cycles from a determining set for $\Phi = (gv)_{g\in G}$ which do not involve h. Thus our algorithm can be applied (with a reduced set of permutations to examine, and a simpler pruning rule).

Example 6.2. *Consider Example 4.1. This is a group frame with $\rho(G) = \langle S, \Omega\rangle$ having order 8, and centre $Z = \{I, -I\}$. The group $\rho(G)/Z$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ (since $\Omega S = -S\Omega$), and we will write it elements in the order $(Z, SZ, \Omega Z, S\Omega Z)$. The corresponding translations are*

$$\tau_Z = I, \quad \tau_{SZ} = (12)(34), \quad \tau_{\Omega Z} = (13)(24), \quad \tau_{S\Omega Z} = (14)(23).$$

If we fix the first vector, i.e., consider only possible projective symmetries with $\sigma 1 = 1$, then applying our algorithm as before gives

$$\mathcal{F}_2 = \{(1, 2), (1, 3), (1, 4)\}.$$

Growing and pruning gives

$$\mathcal{F}_3 = \{(1, 2, 3), (1, 3, 4), (1, 4, 2)\}, \qquad \mathcal{F}_4 = \{(1, 2, 3, 4), (1, 3, 4, 2), (1, 4, 2, 3)\}.$$

Thus the stabliser of the first vector is generated by the cycle (234), and $\mathrm{Sym}_P(\Phi)$ is generated by generators for $\tau_{\rho(G)/Z}$, say (12)(34) and (13)(24) and the generator (234) for Stab(1).

We now present the results of our calculation of the symmetry group for SICs and MUBs. These were done in the computer algebra packages `Maple` and `Magma`. We use the notation `<n,k>` to denote the k–th group of order n, as used in `Magma`. For values of n for which the Magma command `IdentifyGroup(G)` is unable to identify the group, which simply give its order n. Example 6.2 can be generalised as follows.

Definition 6.3. *A* **symmetric informationally complete positive operator valued measure***, or* **SIC** *for short, is an equiangular tight frame of d^2 vectors for $\mathbb{C}^d$.*

SICs are usually considered up to projective equivalence, i.e., as sequences of equiangular lines. There is a growing body of work (cf. [**18**], [**2**], [**19**], [**4**]) indicating that these exist in every dimension, as group frames, where the $\rho(G)$ is the Weyl–Heisenberg group which is generated by the cyclic shift and modulation operators

$$S := \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad \Omega := \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{d-1} \end{pmatrix}, \quad \omega := e^{2\pi i/d}. \tag{6.14}$$

and has centre $Z = \{\omega^j I\}_{j=0}^{d-1}$. Here the index group is $\rho(G)/Z \approx \mathbb{Z}_d \times \mathbb{Z}_d$, which we denote by G. Such a "Weyl–Heisenberg" SIC Φ is determined (up to projective unitary equivalence) by its 3–products (cf. [**4**], [**9**])

$$\Lambda^j_{g,h} := d^3 \|v\|^6 \Delta_C(jv, gv, hv) = \langle jv, gv\rangle\langle gv, hv\rangle\langle hv, jv\rangle, \qquad j, g, h \in G.$$

In particular, a permutation $\sigma : G \to G$ is a projective symmetry if and only if

$$\Lambda^{\sigma j}_{\sigma g, \sigma h} = \Lambda^j_{g,h}, \qquad \forall j, g, h.$$

By Proposition 6.1, it suffices to find the stabiliser of some fixed $j \in G$, i.e., the subgroup of σ satisfying

$$\Lambda^j_{\sigma g, \sigma h} = \Lambda^j_{g,h}, \qquad \forall g, h.$$

This observation is made in [**16**] (Chapter 10), where the Hermitian $G \times G$ matrix $[\Lambda^j_{g,h}]$ is replaced by the real antisymmetric matrix $\Lambda^{(j)}$, with entries

$$\Lambda^{(j)}_{g,h} := \arg\Big(\frac{\Lambda^j_{g,h}}{|\Lambda^j_{g,h}|}\Big).$$

For Example 6.2 (with the same ordering), we have

$$\Lambda^I = \frac{1}{8}\begin{pmatrix} 1 & \frac13 & \frac13 & \frac13 \\ \frac13 & \frac13 & \frac{i}{3\sqrt3} & -\frac{i}{3\sqrt3} \\ \frac13 & -\frac{i}{3\sqrt3} & \frac13 & \frac{i}{3\sqrt3} \\ \frac13 & \frac{i}{3\sqrt3} & -\frac{i}{3\sqrt3} & \frac13 \end{pmatrix}, \qquad \Lambda^{(I)} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\pi}{2} & -\frac{\pi}{2} \\ 0 & -\frac{\pi}{2} & 0 & \frac{\pi}{2} \\ 0 & \frac{\pi}{2} & -\frac{\pi}{2} & 0 \end{pmatrix}.$$

The stabiliser of some j, i.e., the set of permutations satisfying

$$\Lambda^{(j)}_{\sigma g, \sigma h} = \Lambda^{(j)}_{g,h}, \qquad \forall g, h,$$

is called an automorphism of the angle matrix $\Lambda^{(j)}$. An algorithm, akin to ours, for computing the automorphism group of a symmetric or antisymmetric matrix is given in [**16**]. It is observed that such a matrix can be thought of as a weighted (directed) graph with vertices G, and the calculation of the automophism group is equivalent to the (well studied) weighted graph isomorphism problem. Using our algorithm, we obtained the same extended projective symmetry groups for the numerically known SICs in dimensions $d \neq 3$ as in [**16**], namely the subgroup of it that lies in the extended Clifford group (the normaliser of the Weyl–Heisenberg group in the unitary and antiunitary matrices), as given in [**19**]. As a consequence of Proposition 6.1, we obtain a necessary condition for a frame to be a group frame.

Corollary 6.4. *Let Φ be a finite frame. Then a necessary condition for Φ to be a group frame for G/Z (up to scalar multiples of its vectors) is that* $\mathrm{Sym}_P(\Phi)$ *has a transitive subgroup which is isomorphic to G/Z.*

This is a sufficient condition in the following situation.

Lemma 6.5. *Let $\Phi = (v_j)$ be a finite frame. If the frame graph of $(P_\Phi e_j)$ is connected, and $\sigma \in \mathrm{Sym}_P(\Phi)$, i.e.,*

$$Lv_j = c_j v_{\sigma j}, \qquad \forall j,$$

then the linear map $L = L_\sigma$ is unique up to multiplication by a unit scalar, i.e.,the map $\sigma \mapsto L_\sigma$ gives a projective representation of $\mathrm{Sym}_P(\Phi)$.

Proof. First, we assume that (v_j) is a normalised tight frame, so that $L = U$ is unitary, and $\langle v_k, v_j\rangle = \langle P_\Phi e_k, P_\Phi e_j\rangle = (P_\Phi)_{kj}$. Suppose that

$$Uv_j = c_j v_{\sigma j}, \quad \tilde{U} v_j = \tilde{c}_j v_{\sigma j}, \qquad \forall j, \tag{6.15}$$

with U and $\tilde{U}$ unitary, and c_j and $\tilde{c}_j$ unit scalars. Then

$$\langle \overline{c_j} U v_j, \overline{c_k} U v_k\rangle = \langle v_{\sigma j}, v_{\sigma k}\rangle = \langle \overline{\tilde{c}_j}\tilde{U} v_j, \overline{\tilde{c}_k}\tilde{U} v_k\rangle, \qquad \forall j,k,$$

which gives

$$\frac{c_k}{c_j}\langle v_j, v_k\rangle = \frac{\tilde{c}_k}{\tilde{c}_j}\langle v_j, v_k\rangle, \qquad \forall j,k.$$

In particular,

$$\langle P_\Phi e_j, P_\Phi e_k\rangle = \langle v_j, v_k\rangle \neq 0 \quad \Longrightarrow \quad \frac{c_j}{\tilde{c}_j} = \frac{c_k}{\tilde{c}_k}. \tag{6.16}$$

Since the frame graph of Φ is connected, for any j and k, there exists a sequence of vectors $v_j, v_{\ell_1}, v_{\ell_2}, \ldots, v_{\ell_m}, v_k$ with

$$\langle v_j, v_{\ell_1}\rangle \neq 0, \quad \langle v_{\ell_1}, v_{\ell_2}\rangle \neq 0, \quad \ldots \quad \langle v_{\ell_{m-1}}, v_{\ell_m}\rangle \neq 0, \quad \langle v_{\ell_m}, v_k\rangle \neq 0,$$

and so by (6.16), we have

$$\frac{c_j}{\tilde{c}_j} = \frac{c_{\ell_1}}{\tilde{c}_{\ell_1}} = \frac{c_{\ell_2}}{\tilde{c}_{\ell_2}} = \cdots = \frac{c_{\ell_m}}{\tilde{c}_{\ell_m}} = \frac{c_k}{\tilde{c}_k}.$$

Thus $\tilde{c} = \alpha c$ for some unit scalar α, and (6.15) gives

$$\tilde{U} v_j = \tilde{c}_j v_{\sigma j} = \alpha c_j v_{\sigma j} = \alpha U v_j, \qquad \forall j,$$

i.e., $\tilde{U} = \alpha U$. Finally, we consider the general case. Using (2.9), $Lv_j = c_j v_{\sigma j}$ can be written as

$$\big(R_\Phi L R_\Phi^{-1}\big) P_\Phi e_j = c_j(P_\Phi e_{\sigma j}), \qquad \forall j,$$

where the inverse of the injective linear map R_Φ is defined on its image (the range of P_Φ). Since $(P_\Phi e_j)$ is a normalised tight frame, $U = R_\Phi L R_\Phi^{-1}$ is unitary, and we may apply the result just proved. □

Theorem 6.6. *Let $\Phi = (v_j)$ be a frame of n vectors for $\mathcal{H}$, for which*
(1) No vector in Φ is repeated (up to a unit scalar multiple).
(2) The frame graph of $(P_\Phi e_j)$ is connected.
Then Φ is a group frame for G/Z (up to scalar multiples of its vectors) if and only if $\mathrm{Sym}_P(\Phi)$ *has a transitive subgroup which is isomorphic to G/Z.*

Proof. In view of Corollary 6.4, it suffices to prove that given a transitive subgroup of H of $\mathrm{Sym}_P(\Phi)$, there is a group G of linear maps for which the orbit of any vector $v \in \Phi$ is Φ (with possible repeats). Let d be the dimension of $\mathcal{H}$, and $\omega := e^{\frac{2\pi i}{d}}$. For each $\sigma \in H$, choose some L_σ as in Lemma 6.5, with

$$\det(L_\sigma) = 1.$$

There are d scalar multiples of L_σ with determinant 1, namely $L_\sigma, \omega L_\sigma, \ldots, \omega^{d-1} L_\sigma$. Let

$$G := \{\omega^j L_\sigma : j = 0, \ldots, d-1, \sigma \in H\}, \qquad Z := \{\omega^j I\}_{j=0}^{d-1}.$$

Since the vectors in Φ are not repeated, G has $d|H|$ elements. Moreover, G is a group with $G/Z \approx H$, since

$$L_{\sigma\tau} v_j = c_j^{\sigma\tau} v_{\sigma\tau j}, \qquad L_\sigma L_\tau v_j = L_\sigma c_j^\tau v_{\tau j} = c_j^\tau c_{\tau j}^\sigma v_{\sigma\tau j},$$

and the uniqueness of $L_{\sigma\tau}$ gives

$$\begin{aligned} L_\sigma L_\tau = \alpha L_{\sigma\tau}, \quad |\alpha| = 1 \quad &\Longrightarrow \quad 1 = \det(L_\sigma L_\tau) = \det(\alpha L_{\sigma\tau}) = \alpha^d \\ &\Longrightarrow \quad \alpha \in \{1, \omega, \ldots, \omega^{d-1}\}, \quad L_\sigma L_\tau \in G. \end{aligned}$$

Since H is transitive, Φ is the G–orbit of any one of its vectors (up to repeats). □

As an example, we consider the generalisation of Example 4.2.

Definition 6.7. *A family $\mathcal{B}_1, \ldots \mathcal{B}_m$ of orthonormal bases for $\mathbb{C}^d$ is* **mutually unbiased** *if*

$$|\langle v, w\rangle| = \frac{1}{\sqrt{d}}, \qquad v \in \mathcal{B}_j,\ w \in \mathcal{B}_k, \quad j \neq k.$$

We call $\mathcal{B}_1, \ldots \mathcal{B}_m$ a sequence of m **mutually unbiased bases***, or* **MUBs** *for short.*

The frame graph of two or more MUBs is connected. There is considerable interest in finding $\mathcal{M}(d)$ the maximal number of MUBs in $\mathbb{C}^d$ (cf. [**26**], [**11**], [**5**]). For d a prime power, $\mathcal{M}(d) = d+1$, and for $d = 6$, it is only known that $3 \leq \mathcal{M}(6) \leq 7$.

Example 6.8. *One can add a third orthonormal basis to those of Example 4.2, to obtain a maximal set of MUBs for $\mathbb{C}^2$*

$$\Phi = (v_j) = \left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix} \right).$$

The projective symmetry group $\mathrm{Sym}_P(\Phi)$ has order 24, and is generated by the permutations

$$(13)(24)(56), \quad (3645).$$

It has three transitive subgroups

$$\begin{aligned} &H_1 = \langle (12)(36)(45), (164)(253)\rangle \approx S_3, \qquad |H_1| = 6, \\ &H_2 = \langle (164)(253), (34)(56), (12)(56)\rangle, \qquad |H_2| = 12, \\ &H_3 = \mathrm{Sym}_P(\Phi), \qquad |H_3| = 24. \end{aligned}$$

Thus the three MUBs are a group frame for $G/Z \approx H_j$. For $G/Z \approx S_3$, it is even possible to choose G so that $Z = \{1\}$, e.g., take

$$G = \langle A, B\rangle, \qquad A := \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad B := \frac{1}{\sqrt{2}}\begin{pmatrix}0 & -1-i\\ -1+i & 0\end{pmatrix}.$$

For d a prime power, there is a standard construction for Φ a maximal set of $d+1$ MUBs in $\mathbb{C}^d$ [**1**], [**26**], as the eigenvectors of elements of a Weyl–Heisenberg group. This Φ can be interpreted as a group frame [**27**], [**3**], [**5**], where the group is a subgroup of the Clifford group (the normaliser of the Weyl–Heisenberg group in the unitary matrices). For d a prime, the appropriate Weyl–Heisenberg group is

d	m	n	$\mathcal{C}(\mathbb{Z}_d)/Z$	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	transitive subgroups of $\mathrm{Sym}_P(\Phi)$
2	3	6	<24,12>	<24,12>	<48,48>	<6,1>, <12,3>
3	4	12	<216,153>	<216,153>	<432,734>	<72,41>
4	5	20	<768,1088659>	1920	3840	<20,3>,<60,5>, <80,49>, <120,34>, <160,234>, <320,1635>, <960,11357>,
5	6	30	3000	3000	6000	<600,150>
7	8	56	16464			

TABLE 1. The symmetry groups $\mathrm{Sym}_P(\Phi)$ and $\mathrm{Sym}_{EP}(\Phi)$ for Φ the tight frame of n vectors given by m MUBs in $\mathbb{C}^d$, including the proper transitive subgroups of $\mathrm{Sym}_P(\Phi)$.

that generated by S and Ω of (6.14), and the Clifford group is generated by these, the unitary scalar matrices, the Fourier matrix F and the diagonal matrix R, where

$$F_{jk} := \frac{1}{\sqrt{d}}\omega^{-jk}, \qquad R_{jk} := \mu^{j(j+d)}\delta_{jk}, \qquad \mu := e^{\frac{2\pi i}{2d}}.$$

We denote this Clifford group, factored by its centre (the unit scalar matrices) by $\mathcal{C}(\mathbb{Z}_d)/Z$. Here we have

$$\Phi = (ge_j)_{g\in\mathcal{C}(\mathbb{Z}_d)/Z},$$

where e_j is any standard basis vector. Our calculations for $d \leq 5$ suggest (see Table 1), the Clifford group accounts for all the projective symmetries, and the extended Clifford group [**2**] for all extended projective symmetries.

Conjecture 6.9. *Let d be a prime. The standard construction of $d+1$ MUBs in $\mathbb{C}^d$ has no projective symmetries other than those given by the (generating) Clifford group.*

For d a prime power p^m, $m > 1$, the appropriate Clifford group is the *Galoisian Clifford group* of [**3**], [**15**]. We consider the first such example.

Example 6.10. *Let Φ be the five MUBs in $\mathbb{C}^4$ (these are unique up to projective similarity). We calculated that the projective symmetry group $\mathrm{Sym}_P(\Phi)$. It is transitive, with order*

$$1920 = 2^7 \cdot 3 \cdot 5.$$

*The Galoisian Clifford group has order $11520 = 2^8 \cdot 3^2 \cdot 5$ (see [**15**]). Since Φ is known to be a group frame for a subgroup of the Galoisian Clifford group (see [**27**]), we conclude that Φ is not a group frame for the full Galoisian Clifford group. It is a group frame for groups with orders 20, 60, 80, 120, 160, 320, 960, 1920 (see Table 1). It is not known whether all these groups appear as subgroups of the Galoisian Clifford group.*

7. Harmonic Frames

Tight group frames for abelian groups are called **harmonic frames**. Of particular interest (cf. [**13**], [**12**]) are those for the cyclic group $G=\mathbb{Z}_n$, which are said to be **cyclic**. All of these can be given explicitly (up to unitary equivalence) by subsets J of $\mathbb{Z}_n$ (cf. [**20**], [**8**]), as follows. Let $\omega := e^{\frac{2\pi i}{n}}$. For $J=\{j_1,\ldots,j_d\}$ a subset of $\mathbb{Z}_n$, a group frame for $\mathbb{C}^d$ is given by

$$\Phi_J=(v_k)_{k\in\mathbb{Z}_n},\qquad v_k := \begin{pmatrix}\omega^{j_1} & & \\ & \ddots & \\ & & \omega^{j_d}\end{pmatrix}^k \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \begin{pmatrix}\omega^{kj_1}\\ \vdots\\ \omega^{kj_d}\end{pmatrix}.$$

Let $\mathrm{Aut}(G)$ be the automorphisms of G. For cyclic harmonic frames Φ_J and Φ_K, it was shown in [**8**], [**9**] that

- Φ_J and Φ_K are similar (up to a reordering) if $K=\sigma(J)$, for some $\sigma\in\mathrm{Aut}(G)$.
- Φ_J and Φ_K are projectively similar (up to a reordering) if $K=\sigma(J)-b$, for some $\sigma\in\mathrm{Aut}(G)$, $b\in G$.

We say that frame is **projectively real** if all its m–products are real, and hence it is projectively similar to a frame in $\mathbb{R}^d$. Otherwise it is said to be **projectively complex**. We say that $\Phi=(v_j)$ has **projectively distinct vectors** if no v_j is a unit scalar multiple of another, i.e., none of the lines corresponding to vectors of equal length are equal. For example, the harmonic frame

$$\Phi=\Phi_{\{0,1\}}=\{v_0,v_1,v_2\}=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}1\\ \omega\end{pmatrix},\begin{pmatrix}1\\ \omega^2\end{pmatrix}\right\},\qquad \omega=e^{\frac{2\pi}{3}},$$

for $\mathbb{C}^2$ is not similar to real frame, but is projectively similar to a real frame, since

$$\mathrm{Gram}(\Phi)=\begin{pmatrix}2 & 1+\omega & 1+\omega^2\\ 1+\omega^2 & 2 & 1+\omega\\ 1+\omega & 1+\omega^2 & 2\end{pmatrix},\qquad \langle v_0,v_1\rangle\langle v_1,v_2\rangle\langle v_2,v_0\rangle=(1+\omega^2)^3=-1.$$

The classes of projectively equivalent cyclic harmonic frames (up to reordering) were calculated in [**9**]. For those with projectively distinct vectors, we used our algorithm to calculate the projective symmetry group and the extended projective symmetry group (in the case it was projectively complex). The results for $d=2,\ldots,7$ are summarised in the tables of the Appendix. We observe that for projectively complex cyclic harmonic frames the conjugation map gives an anti projective symmetry $\sigma: k\mapsto -k$, since

$$\overline{v_k}=\overline{(\omega^{kj_1},\ldots,\omega^{kj_d})}=(\omega^{-kj_1},\ldots,\omega^{-kj_d})=v_{-k},\qquad k\in\mathbb{Z}_n.$$

Here $\mathrm{Sym}_P(\Phi)$ is an index 2 subgroup of $\mathrm{Sym}_{EP}(\Phi)$ (and so is normal), but in all the examples considered (including Example 5.2), $\mathrm{Sym}_{EP}(\Phi)$ is not isomorphic to the direct product $\mathrm{Sym}_P(\Phi)\times\mathbb{Z}_2$. We conclude with some examples of interest.

Example 7.1. *(Four vectors in $\mathbb{C}^2$). There are two projective equivalence classes, given by $\{0,1\}$ and $\{1,3\}$. The first has projectively distinct vectors, is projectively real, and requires the 4–products to calculate its projective symmetry group* `<8,3>`*. The second does not have vectors which are projectively distinct.*

Example 7.2. *(Six vectors in $\mathbb{C}^3$). There are three projective equivalence classes. The first $\{1,3,5\}$ does not have projectively distinct vectors, and the second $\{1,2,3\}$*

has projectively distinct vectors, is projectively real, and requires the 4*–products to calculate its projective symmetry group* <12,4>*, The third* $\{0,1,4\}$ *is projectively complex, with projectively distinct vectors, none of which are orthogonal. Its projective and extended projective symmetry groups are symmetry groups* <18,3> *and* <36,10>*. This is the first example of a projectively complex cyclic harmonic frame. For* $d=3$*, they also exist for* $n=7,8,9,10,11,12$*.*

Example 7.3. *(Eight vectors in* $\mathbb{C}^3$*). The harmonic frame* $\Phi=\Phi_J$ *of eight vectors for* $\mathbb{C}^3$ *given by* $J=\{0,1,4\}$ *is projectively complex, with*

$$\mathrm{Sym}_P(\Phi) = \texttt{<32,11>}, \qquad \mathrm{Sym}_{EP}(\Phi) = \texttt{<64,134>}.$$

The projective symmetry group has transitive subgroups of order eight isomorphic to $\mathbb{Z}_8$*,* $\mathbb{Z}_4\times\mathbb{Z}_2$*,* D_4 *(dihedral group),* Q_8 *(quaternian group), and so* Φ *is a group frame for each of these groups. The only other group of order eight is* $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$*, which occurs as a transitive subgroup of* $\mathrm{Sym}_{EP}(\Phi)$*.*

Example 7.4. *(Projectively real frames) In 2 dimensions all cyclic harmonic frames of* n *vectors are projectively real. This follows by showing for all* $n>4$*, they are determined by their* m*–products,* $m\le 3$*, and then showing the* 3*–products are all real by direct computation. For* $d\ge 3$*, it appears there always exist projectively complex cyclic harmonic frames of* n *projectively distinct vectors, with the smallest* n *being* $n=d+3$*.*

Example 7.5. *(Erasures) The* **erasures** *of a frame* Φ *of* n *vectors for* $\mathbb{C}^d$ *is the number of vectors that can be removed from* Φ *so that those remaining still span* $\mathbb{C}^d$*. Frames with a large number of erasures are useful for robust data transmission* [**13**]*,* [**12**]*. For the cyclic harmonic frame given by* $J=\{0,1,2,\ldots,d-1\}$ *the number of erasures is* $n-d$*. Our calculations show this is not true in general, e.g., for six vectors in* $\mathbb{C}^2$ *the erasures of a cyclic harmonic frame can be* $2,3,4$*. It does seem however, that for* n *a prime the erasures are always* $n-d$*.*

8. Appendix

The tables here summarise our calculation of the projective symmetry groups of the cyclic harmonic frames $\Phi=\Phi_J$ of n vectors for $\mathbb{C}^d$, $d=2,\ldots,7$. The columns are as follows:

(1) d – the dimension $\mathbb{C}^d$.
(2) n – the number of vectors in Φ.
(3) real – whether Φ is projectively real.
(4) orth – whether any vectors in Φ are orthogonal.
(5) reps – whether the vectors in Φ projectively repeated, i.e., are not projectively distinct.
(6) $\mathrm{Sym}_P(\Phi)$ – the projective symmetry group of Φ, when the vectors of Φ are projectively distinct.
(7) $\mathrm{Sym}_{EP}(\Phi)$ – the extended projective symmetry group of Φ, when Φ is projectively complex with projectively distinct vectors.
(8) J – a subset of $\mathbb{Z}_n$ giving Φ_J (up to projective equivalence and reordering).
(9) 4-cycle – whether 4–cycles were needed to define the projective equivalence class.
(10) erasures – the number of vectors that can be removed from Φ so that those remaining still span $\mathbb{C}^d$.

d	n	real	orth	reps	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
2	2	yes	y		<2,1>		$\{0,1\}$		0
2	3	y			<6,1>		$\{0,1\}$		1
2	4	y	y		<8,3>		$\{0,1\}$	y	2
		y	y	y			$\{1,3\}$		1
2	5	y			<10,1>		$\{0,1\}$		3
2	6	y		y			$\{1,3\}$		3
		y	y		<12,4>		$\{1,2\}$		4
		y	y	y			$\{1,4\}$		2
2	7	y			<14,1>		$\{1,4\}$		5
2	8	y	y		<16,7>		$\{1,6\}$		6
		y	y	y			$\{1,3\}$		5
		y	y	y			$\{1,5\}$		3
2	9	y			<18,1>		$\{1,3\}$		7
		y		y			$\{1,4\}$		5
2	10	y		y			$\{1,5\}$		7
		y	y		<20,4>		$\{1,4\}$		8
		y	y	y			$\{1,6\}$		4
2	11	y			<22,1>		$\{1,10\}$		9
2	12	y	y	y			$\{1,10\}$		8
		y	y		<24,6>		$\{1,8\}$		10
		y		y			$\{1,9\}$		7
		y	y	y			$\{1,7\}$		5
		y	y	y			$\{1,3\}$		9
2	13	y			<26,1>		$\{1,4\}$		11
2	14	y	y		<28,3>		$\{1,2\}$		12
		y		y			$\{1,5\}$		11
		y	y	y			$\{1,8\}$		6
2	15	y			<30,3>		$\{0,1\}$		13
		y		y			$\{1,10\}$		11
		y		y			$\{1,6\}$		9

d	n	real	orth	reps	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
3	3	y	y		<6,1>		$\{0,1,2\}$		0
3	4	y			<24,12>		$\{1,2,3\}$		1
3	5	y			<10,1>		$\{0,1,3\}$		2
3	6				<18,3>	<36,10>	$\{0,1,4\}$		2
		y	y		<12,4>		$\{1,2,3\}$	y	3
		y	y	y			$\{1,3,5\}$		1
3	7				<21,1>	<42,1>	$\{1,2,6\}$		4
		y			<14,1>		$\{1,3,5\}$		4
3	8				<16,8>	<32,43>	$\{1,3,4\}$		5
					<32,11>	<64,134>	$\{0,1,4\}$		3
		y			<16,7>		$\{0,1,2\}$		5
		y		y			$\{1,3,5\}$		3
3	9				<9,1>	<18,1>	$\{1,4,6\}$		5
		y	y		<18,1>		$\{0,1,2\}$		6
		y	y	y			$\{1,4,7\}$		2
3	10				<50,3>	<100,13>	$\{0,1,5\}$		4
		y			<20,4>		$\{0,1,9\}$		7
					<10,2>	<20,4>	$\{0,1,8\}$		7
		y		y			$\{1,5,7\}$		5
3	11				<11,1>	<22,1>	$\{0,1,3\}$		8
		y			<22,1>		$\{1,2,3\}$		8
3	12				<12,2>	<24,6>	$\{1,2,11\}$		8
		y	y		<24,6>		$\{1,2,3\}$		9
		y		y			$\{1,4,10\}$		5
					<12,2>	<24,6>	$\{0,3,4\}$		7
				y			$\{1,5,7\}$		5
			y		<24,5>	<48,38>	$\{0,1,8\}$		7
		y	y	y			$\{1,3,5\}$		7
					<72,30>	<144,154>	$\{2,3,8\}$		5
		y	y	y			$\{1,5,9\}$		3
3	13	y			<26,1>		$\{0,1,12\}$		10
					<13,1>	<26,1>	$\{0,1,3\}$		10
					<39,1>	<78,1>	$\{1,2,11\}$		10

d	n	real	orth	reps	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
4	4	y	y		<24,12>		$\{0,1,2,3\}$		0
4	5	y			<120,34>		$\{0,1,2,3\}$		1
4	6	y	y		<12,4>		$\{1,2,3,4\}$		2
		y			<48,48>		$\{1,2,3,5\}$		1
		y	y		<72,40>		$\{0,1,3,4\}$		1
4	7				<21,1>	<42,1>	$\{0,1,2,4\}$		3
		y			<14,1>		$\{1,2,3,4\}$		3
4	8	y	y		<16,7>		$\{1,2,4,7\}$	y	4
					<8,1>	<16,7>	$\{1,2,3,5\}$		3
		y	y		<128,928>		$\{0,1,4,5\}$	y	2
		y	y		<32,43>		$\{0,1,3,4\}$		3
		y	y	y			$\{1,3,5,7\}$		1
4	9				<9,1>	<18,1>	$\{1,2,3,8\}$		5
					<162,10>	<324,39>	$\{0,1,4,7\}$		2
		y			<18,1>		$\{1,2,4,5\}$		4
		y			<18,1>		$\{0,1,2,3\}$		5
4	10				<10,2>	<20,4>	$\{0,1,3,7\}$		5
		y	y		<20,4>		$\{1,2,4,9\}$		4
		y	y		<20,4>		$\{0,1,2,3\}$		6
		y	y		<40,12>		$\{0,1,3,4\}$		5
					<10,2>	<20,4>	$\{1,2,4,6\}$		4
					<10,2>	<20,4>	$\{0,1,5,8\}$		4
		y			<20,4>		$\{0,2,5,8\}$		4
		y	y		<200,43>		$\{1,2,6,7\}$		3
		y		y			$\{1,3,5,7\}$		3
4	11	y			<22,1>		$\{1,2,5,9\}$		7
					<11,1>	<22,1>	$\{0,1,2,7\}$		7
					<11,1>	<22,1>	$\{0,1,2,8\}$		7
		y			<22,1>		$\{1,3,7,8\}$		7
4	12				<12,2>	<24,6>	$\{1,3,6,9\}$		5
					<384,5557>	<768,1088009>	$\{1,2,6,10\}$		3
					<24,5>	<48,38>	$\{3,4,6,8\}$		7
		y	y		<48,38>		$\{2,3,9,10\}$		5
		y	y		<24,6>		$\{1,4,8,9\}$		6
					<12,2>	<24,6>	$\{0,1,2,8\}$		5
					<36,6>	<72,23>	$\{0,2,3,8\}$		5
		y	y		<24,6>		$\{1,4,8,11\}$		7
		y	y		<24,6>		$\{1,4,6,11\}$		8
		y			<12,2>	<24,6>	$\{1,3,6,11\}$		7
			y		<24,10>	<48,38>	$\{2,3,6,9\}$		5
			y		<12,2>	<24,6>	$\{0,1,4,11\}$		7
			y		<72,42>	<144,183>	$\{1,2,8,11\}$		5
		y	y		<48,38>		$\{0,1,4,9\}$		5
		y	y		<48,38>		$\{1,2,4,11\}$		6
		y	y		<48,38>		$\{0,1,6,11\}$		5
		y	y	y			$\{1,4,7,10\}$		2
		y	y	y			$\{1,5,7,11\}$		3
		y	y	y			$\{1,3,7,11\}$		3
		y	y		<288,889>		$\{0,1,6,7\}$		4
		y	y	y			$\{1,3,5,7\}$		5

d	n	real	orth	reps	$\text{Sym}_P(\Phi)$	$\text{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
5	5	y	y		<120,34>		$\{0,1,2,3,4\}$		0
5	6	y			<720,763>		$\{0,1,2,3,4\}$		1
5	7	y			<14,1>		$\{0,1,2,3,5\}$		2
5	8				<32,11>	<64, 134>	$\{1,2,3,5,6\}$		2
		y			<16,7>		$\{0,1,2,6,7\}$		3
					<16,8>	<32,43>	$\{0,1,2,3,6\}$		3
		y			<384,5602>		$\{1,2,3,5,7\}$		1
5	9				<9,1>	<18,1>	$\{0,1,2,3,7\}$		4
					<162,10>	<324,39>	$\{0,1,3,4,7\}$		2
		y			<18,1>		$\{0,1,2,3,8\}$		4
		y			<18,1>		$\{1,2,3,4,7\}$		2
5	10				<10,2>	<20,4>	$\{1,2,5,6,9\}$		4
					<10,2>	<20,4>	$\{1,2,4,8,9\}$		4
		y	y		<20,4>		$\{1,2,4,5,8\}$	y	5
		y			<200,43>		$\{1,2,3,6,8\}$		3
					<10,2>	<204,>	$\{1,3,4,5,8\}$		4
					<50,3>	<100,13>	$\{1,2,3,7,8\}$		3
					<10,2>	<20,4>	$\{0,1,3,5,9\}$		3
		y	y		<240,189>		$\{0,1,2,4,8\}$		3
		y	y	y			$\{1,3,5,7,9\}$		1
5	11				<11,1>	<22,1>	$\{0,1,2,4,5\}$		6
					<11,1>	<22,1>	$\{0,1,2,5,9\}$		6
					<55,1>	<110,1>	$\{1,2,3,5,8\}$		6
					<11,1>	<22,1>	$\{1,2,3,4,10\}$		6
		y			<22,1>		$\{0,1,2,3,10\}$		6
		y			<22,1>		$\{0,1,2,4,9\}$		6

d	n	real	orth	reps	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
6	6	y	y		<720,763>		$\{0,1,2,3,4,5\}$		0
6	7	y			5040		$\{0,1,2,3,4,5\}$		1
6	8	y	y		<16,7>		$\{0,1,2,3,5,6\}$		
		y	y		<128,928>		$\{0,1,2,4,6,7\}$		
		y	y		1152		$\{0,1,2,4,5,6\}$		
6	9				<9,1>	<18,1>	$\{0,1,2,3,4,6\}$		
		y	y		<18,1>		$\{0,1,2,3,4,8\}$		
		y	y		<1296,3490>		$\{0,1,3,4,6,7\}$		
6	10				<10,2>	<20,4>	$\{0,1,2,5,7,9\}$		
			y		<10,2>	<20,4>	$\{0,1,3,4,5,8\}$		
		y			3840		$\{1,2,3,5,7,9\}$		
		y	y		<20,4>		$\{0,1,2,3,6,7\}$		
					<10,2>	<20,4>	$\{1,2,3,4,5,9\}$		
		y	y		<200,43>		$\{0,1,4,5,6,9\}$		
		y	y		<20,4>		$\{0,1,3,4,7,8\}$		
		y	y		<40,12>		$\{0,1,2,5,8,9\}$		
		y			<20,4>		$\{1,2,3,5,6,7\}$		
6	11				<11,1>	<22,1>	$\{1,2,3,4,5,9\}$		
					<11,1>	<22,1>	$\{0,1,2,3,5,10\}$		
		y			<22,1>		$\{0,1,2,8,9,10\}$		
		y			<22,1>		$\{1,2,3,8,9,10\}$		
					<11,1>	<22,1>	$\{1,2,3,5,6,10\}$		
					<55,1>	<110,1>	$\{0,1,3,4,5,9\}$		

d	n	real	orth	reps	$\mathrm{Sym}_P(\Phi)$	$\mathrm{Sym}_{EP}(\Phi)$	J	4-cycle	erasures
7	7	y	y		5040		$\{0,1,2,3,4,5,6\}$		0
7	8	y			40320		$\{0,1,2,3,4,5,7\}$		1
7	9	y			<1296,3490>		$\{0,1,2,4,5,7,8\}$		
		y			<18,1>		$\{0,1,2,3,5,7,8\}$		
7	10				<10,2>	<20,4>	$\{1,2,3,4,5,6,8\}$		
					<50,3>	<100,13>	$\{0,1,2,5,6,7,8\}$		
		y			<320,1636>		$\{1,2,3,5,6,7,9\}$		
		y			<20,4>		$\{1,2,4,5,6,8,9\}$		
	11				<11,1>	<22,1>	$\{0,1,2,3,5,6,9\}$		
					<11,1>	<22,1>	$\{0,1,2,6,7,8,9\}$		
		y			<22,1>		$\{1,2,3,4,5,6,8,10\}$		
		y			<22,1>		$\{1,2,3,4,5,8,9\}$		
7	12	y	y		<24,6>		$\{0,1,2,4,8,10,11\}$		
					<12,2>	<24,6>	$\{1,2,3,4,6,7,10\}$		
					<72,30>	<144,154>	$\{1,2,3,4,7,8,10\}$		
					<12,2>	<24,6>	$\{1,3,4,5,7,8,9\}$		
					<12,2>	<24,6>	$\{0,1,2,4,6,10,11\}$		
					<12,2>	<24,6>	$\{0,1,2,3,6,10,11\}$		
					<12,2>	<24,6>	$\{0,1,2,3,4,8,9\}$		
					<12,2>	<24,6>	$\{1,2,3,4,5,6,11\}$		
		y			<24,6>		$\{0,1,2,3,4,10,11\}$		
					<12,2>	<24,6>	$\{1,2,3,4,6,8,9\}$		
		y			<96,187>		$\{0,1,2,4,5,6,9\}$		
					<72,30>	<144,154>	$\{1,2,3,5,8,9,11\}$		
					<12,2>	<24,6>	$\{1,2,4,5,6,10,11\}$		
					<72,30>	<144,154>	$\{1,2,3,7,8,9,10\}$		
					<12,2>	<24,6>	$\{1,2,3,4,5,7,10\}$		
					<24,10>	<48,38>	$\{1,2,3,4,6,7,9\}$		
					<12,2>	<24,6>	$\{0,1,2,3,5,6,11\}$		
		y			<144,183>		$\{0,1,2,3,4,7,9\}$		
		y			46080		$\{1,3,4,5,7,9,11\}$		
					<24,5>	<48,38>	$\{1,2,3,4,6,7,11\}$		
			y		<24,5>	<48,38>	$\{1,2,3,5,7,10,11\}$		
		y			<288,889>		$\{1,2,3,5,7,8,9\}$		
					<384,5557>	<768, 1088009>	$\{1,2,3,6,7,10,11\}$		
		y	y		<192, 1472>		$\{0,1,3,4,7,8,9\}$	y	
					<1944,3876>	3888	$\{0,1,3,4,6,7,9\}$		

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