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# RECENT RESULTS ON THE POWER DOMINATION NUMBERS OF GRAPH PRODUCTS

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Abstract. We review recent results on the power domination problem of graph products and establish improved results for some families of graph products, namely, $C_n \times C_m$, $P_n \times C_m$, $P_n \boxtimes P_m$, $P_n \boxtimes C_m$ and $C_n \boxtimes C_m$. We also characterize graphs $G$ and $H$ for which the power domination number of the Cartesian product of $G$ and $H$, which is denoted as $\gamma_p(G \,\Box\, H)$, is 1.

## 1. Introduction

The notion of power domination in graphs originates from an optimization problem faced by the electrical power system industry. Electrical power companies need to continually monitor their system's state as defined by a set of variables, for example, the voltage magnitude at loads and the machine phase angle at generators [**6, 10**]. These variables can be monitored by placing phase measurement units (PMUs) at selected locations in the system. Due to the high cost of a PMU, it is desirable to monitor (observe) the entire system using the least possible number of PMUs.

To model this optimization problem, we use a graph to represent an electrical network. A vertex denotes a possible location where PMU can be placed, and an edge denotes a current carrying wire. A PMU measures the state variable (voltage and phase angle) for the vertex at which it is placed and its incident edges and their ends. These vertices and edges are said to be observed by the PMU. We can apply Ohm's law and Kirchhoff's current law to deduce the other three observation rules:

(1) Any vertex that is incident to an observed edge is observed.
(2) Any edge joining two observed vertices is observed.
(3) For $k \geq 2$, if a vertex is incident to $k$ edges such that $k-1$ of these edges are observed, then all $k$ of these edges are observed.

We consider only graphs without loops or multiple edges. Unless specified otherwise in this paper, all graphs are connected. Let $G = \big(V(G), E(G)\big)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. A *trivial* graph is a graph with one vertex. A *nontrivial* graph is a graph that is not trivial. An *induced subgraph* $F$ of a graph $G$ is a graph such that whenever $u, v \in V(F)$ and $uv \in E(G)$, then $uv \in E(F)$. The *distance* between $u$ and $v$ is the length of a shortest $u-v$ path, and is denoted by $d(u,v)$. The *neighborhood* $N_G(v)$ of a vertex $v \in V(G)$ is the set

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of all vertices adjacent to $v$. The subscript may be dropped if there is no confusion about the vertex set that $v$ belongs to. The *degree* of a vertex $v$, denoted by $deg(v)$, is the cardinality of the set $N(v)$. The *maximum degree* of $G$ is defined as $\Delta(G) = \max\{deg(v) \mid v \in V(G)\}$. We denote $N[v]$ for the set $N(v) \cup \{v\}$. For a set $S \subseteq V(G)$, we write $N(S) = \bigcup_{v \in S} N(v)$ and $N[S] = \bigcup_{v \in S} N[v]$.

A set $S \subseteq V(G)$ is said to be a *dominating set* if every vertex in $V(G) \backslash S$ has at least one neighbor in $S$. A dominating set of minimum cardinality is called a *minimum dominating set*. The *domination number* $\gamma(G)$ is the cardinality of a minimum dominating set of $G$. If $\gamma(G) = 1$, then there exists a vertex in $G$ that is adjacent to all other vertices of $G$. Such a vertex is called a *universal vertex*. For the power system monitoring problem, a set $S$ is defined to be a *power dominating set* (PDS) if every vertex and every edge in $G$ are observed by $S$ after applying the observation rules. The *power domination number* $\gamma_p(G)$ is the minimum cardinality of a power dominating set of $G$. We will call a power dominating set with minimum cardinality a $\gamma_p(G)$-set. The following algorithm is an alternative approach to the observation rules.

**Algorithm 1.** [**4**] *Let $S \subseteq V(G)$ be the set of vertices where the PMUs are placed.*
(1) *(Domination)*
*Set $M(S) \leftarrow S \cup N(S)$.*
(2) *(Propagation)*
*As long as there exist $v \in M(S)$ and $w \notin M(S)$, such that $N(v) \cap (V(G) - M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.*

It is easy to see that for any PDS, applying the three observation rules to determine the set of all observed vertices yields the same result as invoking Algorithm 1.

A *zero forcing set* for a graph $G$ is a subset of vertices $B$ such that when initially the vertices in $B$ are colored black and the vertices in $V(G) \backslash B$ are colored white, all the vertices of $G$ eventually become black by the repeated application of the following color-change rule: "If $u$ is a black vertex and exactly one neighbor $w$ of $u$ is white, then change the color of $w$ to black". The *zero forcing number* of $G$, denoted by $Z(G)$, is the minimum cardinality of a zero forcing set of $G$. In [**2**], a lower bound for the power domination number of $G$ is determined in terms of $Z(G)$ and $\Delta(G)$.

**Theorem 2.** [**2**] *Let $G$ be a graph that may not be connected. If $G$ has an edge, then $\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil \leq \gamma_p(G)$.*

Let $G \star H$ be a graph product with vertex set $V(G) \times V(H) = \{(g, h) \mid g \in V(G)$ and $h \in V(H)\}$. There are four standard graph products, namely, the Cartesian, the strong, the direct, and the lexicographic product. Their respective graph products are denoted by $\square$, $\boxtimes$, $\times$, and $\bullet$. Two vertices $(g_1, h_1)$ and $(g_2, h_2)$ of $G \star H$ are adjacent if and only if
(1) $\star = \square$, and either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$,
(2) $\star = \boxtimes$, and either $g_1 = g_2$ and $h_1h_2 \in E(H)$, or $h_1 = h_2$ and $g_1g_2 \in E(G)$, or $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$,
(3) $\star = \times$, $g_1g_2 \in E(G)$ and $h_1h_2 \in E(H)$, or
(4) $\star = \bullet$, and either $g_1g_2 \in E(G)$, or $g_1 = g_2$ and $h_1h_2 \in E(H)$.

The subgraph of $G \star H$ induced by $\{g\} \times V(H)$ is called a *H-fiber*, which is denoted by ${}^{g}H$, and the subgraph induced by $V(G) \times \{h\}$ is called a *G-fiber*, which is denoted by $G^h$. All the four graph products are associative. Except for the lexicographic product, the other graph products are also commutative. When both $G$ and $H$ are paths or cycles, we make the following definitions. A *column* is a set of the form $\{k\} \times \{1, 2, ..., m\}$, and a *row* is a set of the form $\{1, 2, ..., n\} \times \{k\}$. We say that a set $S \subseteq V(G \star H)$ *covers* a column (or row) $D$ if $S \cap D \neq \emptyset$.

A set $S \subseteq V(G)$ is a *total dominating set* if each vertex in $V(G)$ is adjacent to at least one vertex of $S$. The *total domination number* of $G$, which is denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set. In [**4**], the power domination problem for any lexicographic product of two graphs is determined in terms of the domination number and the total domination number of its factors.

**Theorem 3.** [**4**] *Let $G$ be a nontrivial graph without isolated vertices. Then for any nontrivial graph $H$,*

$$\gamma_p(G \bullet H) = \begin{cases} \gamma(G) & \text{if } \gamma_p(H) = 1, \\ \gamma_t(G) & \text{if } \gamma_p(H) \geq 2. \end{cases}$$

For each of the remaining three graph products of two graphs, the approach to the power domination problem is to determine the power domination number for each family of graphs. As such, the remainder of this paper is organized as follows. In Sections 2 to 4, we give a brief survey of existing results involving the Cartesian product, direct product and the strong product. Each new result is also presented within its appropriate section.

## 2. Cartesian Product

Let $P_n$, $C_n$, $K_n$ and $W_n$ denote, respectively, the path, cycle, complete graph and wheel of order $n$; $K_{1,n}$ denotes the star with $n + 1$ vertices such that $n$ of them are end-vertices. For the Cartesian product, the power domination number was first studied in [**5**] for grid graphs $P_n \,\square\, P_m$, and in [**1**] for cylinders $P_n \,\square\, C_m$ and tori $C_n \,\square\, C_m$. Koh and Soh [**7**] extended the study of the power domination problem to the Cartesian product of any two of the following graphs: $P_n$, $C_n$, $K_n$, $W_n$ and $K_{1,n}$. The study was completed in a subsequent paper and all the 15 exact formulas are summarized in [**8**].

Let $Q_n$ denote the $n$-dimensional hypercube, which is defined as

$$\underbrace{P_2 \,\square\, P_2 \,\square\, ... \,\square\, P_2}_{n \text{ terms}}.$$

Dean et al. [**3**] gave a lower bound and an upper bound of $\gamma_p(Q_n)$. Pai and Chiu [**11**] evaluated the power domination numbers for $Q_n$, where $1 \leq n \leq 7$. We present their results as follows.

**Theorem 4.** [**3**] *For the $n$-dimensional hypercube,*

$$\frac{2^{n-1}}{n} \leq \gamma_p(Q_n) \leq 2^{n - \lfloor log\ n \rfloor - 1}.$$

**Theorem 5.** [**11**] *For the n-dimensional hypercube,*

$$\gamma_p(Q_n) = \begin{cases} \lceil \frac{n}{2} \rceil & \text{if } n \in \{1,2,3,4\}, \\ 4 & \text{if } n = 5, \\ 6 & \text{if } n = 6, \\ 10 & \text{if } n = 7. \end{cases}$$

For the power domination problem that involves the Cartesian product of two general graphs, we have the following results.

**Theorem 6.** [**13**] *For any two nontrivial graphs $G$ and $H$,*

$$\gamma_p(G \,\square\, H) \leq min\big\{\gamma_p(G)|V(H)|, \gamma_p(H)|V(G)|\big\}.$$

**Theorem 7.** [**13**] *Let $G$ and $H$ be two nontrivial graphs. If $H$ has a universal vertex, then $\gamma_p(G \,\square\, H) \leq Z(G)$.*

**Theorem 8.** [**13**] *For $n \geq 2$ and any nontrivial graph $G$, $\gamma_p(P_n \,\square\, G) \leq \gamma(G)$.*

We remark that the bounds in Theorems 6 - 8 can be sharp. For Theorem 6, an example is $G = P_3 \,\square\, P_3$ and $H = P_3$. It can be shown by exhaustion that $\gamma_p(P_3 \,\square\, P_3 \,\square\, P_3) = 3 = \gamma_p(P_3 \,\square\, P_3)|V(P_3)|$. Examples for Theorems 7 and 8 are given in [**13**].

We present a recent result on the Cartesian product of two general graphs that has small power domination numbers.

**Lemma 9.** [**13**] *For graphs $G$ and $H$ each with order at least four,*

$$\gamma_p(G \,\square\, H) = 1$$

*if and only if one of the graphs is a path and the other has a universal vertex.*

While a complete classification of graphs $G$ for which $\gamma_p(G) = 1$ is not known yet, we are able to do this for the Cartesian product of two graphs. Before we give the result, we define a graph operation. The graph obtained from $G$ and $H$ by *amalgamating* two vertices $g \in V(G)$ and $h \in V(H)$ has vertex set $V(G) \cup V(H) \backslash \{h\}$ such that the subgraphs induced by $V(G)$ and $\big(V(H)\backslash\{h\}\big) \cup \{g\}$ are $G$ and $H$ respectively.

**Theorem 10.** *Let $G$ and $H$ be two nontrivial graphs. Then $\gamma_p(G \,\square\, H) = 1$ if and only if either*

(i) *$G$ and $H$ each has order at least four, one of the graphs is a path and the other has a universal vertex, or*

(ii) *one of the graphs is either $P_2$ or $P_3$ and the other can be obtained by amalgamating any vertex of a graph, say $D$, with $\gamma(D) = 1$ and an end vertex of $P_n$ with $n \geq 1$, or*

(iii) *one of the graphs is $C_3$ and the other is a path.*

**Proof.** By Lemma 9, it remains to consider graphs $G$ and $H$ such that at least one of them, say $G$, has order either two or three. In other words, $G$ must be $P_2$, $P_3$ or $C_3$. For $G = P_m$ with $m \in \{2,3\}$ and any graph $H$ with $\gamma(H) = 1$, result holds by Theorem 8. We therefore consider three cases for all the remaining possible graphs $G$ and $H$. To prove the necessary condition, we suppose that $\gamma_p(G \,\square\, H) = 1$ and $\{(g,h)\}$ is a PDS of $G \,\square\, H$.

**Case 1:** $P_2 \,\square\, H$ with $\gamma(H) \geq 2$.

Let $g'$ be the neighbor of $g$ and $A = \{h' \in N_H(h) \mid N_H[h'] \not\subseteq N_H[h]\}$. Since $\gamma(H) \geq 2$, $A$ is nonempty. We claim that $|A| = 1$; for otherwise, propagation is not possible from any vertex $a \in A$ in the ${}^gH$-fiber or the vertex $(g', h)$ to observe $(g', a)$ in the ${}^{g'}H$-fiber. It follows from our claim that the set of vertices $\{(g', b) \mid b \in N_H[h] \backslash A\}$ in the ${}^{g'}H$-fiber is observed after applying the first propagation step. Since $\{(g', a) \mid a \in A\}$ is now the only neighbor of $(g', h)$ that is not observed, the vertex $(g', a)$ is observed in the second propagation step. This in turn implies that $N_H(a) \backslash N_H[h]$ is also a singleton. In fact, the subgraph induced by the remaining unobserved vertices, if any, in each $H$-fiber must be a path in order for further propagation to take place. Hence result follows from our construction of $H$.

**Case 2:** $P_3 \,\square\, H$ with $\gamma(H) \geq 2$.

Let $g$, $g'$ and $g''$ be the vertices of $P_3$. If both $g$ and $h$ each have degree at least two or both $g$ and $h$ are end vertices, then no more vertices get observed after the domination step. If $g$ has degree two and $h$ has degree one, then by applying the proof method presented in Case 1, $H$ must be a path $P_n$ with $n \geq 4$. The remaining possibility is $g$ has degree one and $h$ has degree at least two. We assume WLOG that $h$ has the largest degree. The proof method presented in Case 1 can be applied successfully as long as $d(h') \leq d(h)$ for all $h' \in N_H(h)$, even though it takes more propagation steps to get vertices $(g', a)$ and $(g'', a)$ observed.

**Case 3:** $C_3 \,\square\, H$.

When $H$ is a path, we have $\gamma(C_3 \,\square\, H) = 1$ by [**1**]. Also by [**8**], $\gamma_p(C_3 \,\square\, C_n) \neq 1$ for $n \geq 3$. Therefore it remains to show that $\gamma_p(C_3 \,\square\, H) \neq 1$ when $H$ is any graph that is not a path or cycle. In other words, $H$ has a vertex, say $h'$ with degree at least three. However this contradicts the fact that $\{(g, h)\}$ is a PDS of $G \,\square\, H$. This is because if $h = h'$, then the propagation step cannot take place, and if $h \neq h'$, propagation cannot proceed beyond $h'$ in each $H$-fiber.

To complete the proof for the sufficient condition, it can be seen easily that $\{(g, h)\}$ is a PDS of $G \,\square\, H$, where $g$ is an end vertex of $G$ and $h$ is a universal vertex of $D$. □

In [**7**], it was conjectured that the Vizing-like inequality "$\gamma_p(G)\gamma_p(H) \leq \gamma_p(G \,\square\, H)$" holds for any two graphs $G$ and $H$. It was further shown that this inequality is true when one of the graphs is a tree.

**Theorem 11.** [**7**] *For any graph $G$ and any tree $T$,*

$$\gamma_p(G)\gamma_p(T) \leq \gamma_p(G \,\square\, T).$$

## 3. Direct Product

The power domination problem for the direct product of two graphs was first studied in [**4**]. In that paper, they considered each component of $P_n \times P_m$ and denoted the component containing the vertex $(1, 1)$ as the *even component.* Otherwise the component is called the *odd component.* Except for the odd component of $P_n \times P_m$, where both $n$ and $m$ are odd, the power domination numbers are exactly determined.

**Theorem 12.** [**4**] *For $G = P_m \times P_n$, where $n, m \geq 2$,*

(i) *if both $n$ and $m$ are even with $n \geq m$, then $\gamma_p(G) = 2\lceil \frac{m}{4} \rceil$;*

(ii) *if $n$ is odd and $m$ is even, then* $\gamma_p(G) = 2\lceil \frac{m}{4} \rceil$;
(iii) *if both $n$ and $m$ are odd with $n \geq m$, then* $\gamma_p(E) = max\{\lceil \frac{n}{4} \rceil, \lceil \frac{m+n}{6} \rceil\}$ *and* $\gamma_p(O) \leq max\{\lceil \frac{n-2}{4} \rceil, \lceil \frac{m+n-2}{6} \rceil\}$, *where $E$ and $O$ are the even and odd components of $G$ respectively.*

In [**9**], the power domination problem for the direct product of two cycles is studied. While the final result is correct in that paper, we point out that the authors have erred in the proof of their lower bounds in Lemma 3. In particular and using the definition of $B(M)$ as presented in their proof, it was not shown whether the resulting "maximum" $|B(M)|$ can be attained after a vertex is removed from $S$. To illustrate this, we consider a component of $C_4 \times C_6$, which has power domination number equal to 2. After we remove one vertex from $S$, it is easy to see that the maximum $|B(M)|$ is 4, which is less than $m + n - 4$. Subsequently the inequality $4(|S| - 1) \geq (m + n - 4)$ as given in Case 1 does not hold.

In what follows, we shall correct the proof for the lower bound of the power domination number for $C_n \times C_m$, where $n, m \geq 3$.

**Lemma 13.** [**9**] *Every three consecutive columns or rows of $C_n \times C_m$ contains at least one vertex in a PDS, so that* $\gamma_p(C_n \times C_m) \geq max\{\lceil \frac{n}{3} \rceil, \lceil \frac{m}{3} \rceil\}$.

We remark that if $C_n \times C_m$ is not connected, then Lemma 13 holds for each of its connected component.

**Lemma 14.** *If $G = C_n \times C_m$, where $n \geq m \geq 4$ with $n$ and $m$ both even, then* $\gamma_p(G) \geq max\{2\lceil \frac{m+n-2}{4} \rceil, 2\lceil \frac{n}{3} \rceil\}$.

**Proof.** Observe that $G$ consists of two isomorphic connected components, so it suffices to consider one of the components. Let $S$ be a PDS of a component of $G$. Suppose that we remove a vertex from $S$ and denote the resulting set as $S'$. By Lemma 13, $|S| \neq 1$ so that $S'$ is nonempty. Let $M$ be a set of vertices that is observed by $S'$, and let $B(M) \subseteq M$ be the set of vertices on the boundary of $M$, that is, the set of vertices of $M$ that has at least a neighbor not in $M$. We make the following claims.

**Claim 1:** $|B(M)|$ does not increase when any vertex is added to $M$ during the propagation step of Algorithm 1.

It is easy to see that if propagation occurs from vertex $v$ to vertex $w$ such that $w$ is added to $M$, then at most one vertex, $w$, may be added to $B(M)$. However vertex $v$ and possibly other vertices are removed from $B(M)$.

**Claim 2:** Up to symmetry, all vertices in the set $\{2, 3, ..., n-1\} \times \{2, 3, ..., m-1\}$ are observed by $S'$.

Suppose on the contrary that such a set does not exist. Since the set of observed vertices by $S'$ must be bounded by a rectangular region by the propagation step of Algorithm 1 (see Figure 1(left)), we can find three consecutive rows and two consecutive columns (here the words "rows" and "columns" can be interchanged in this paragraph without affecting our proof) that is not observed by $S'$. Denote the three rows as $p - 1$, $p$ and $p + 1$. As there is no vertex in $S'$ that is in row $p - 2$ or $p + 2$, by Lemma 13, we must have exactly one vertex in $S$ that is in row $p$ (see Figure 1(right)). However, this vertex can only monitor some but not all of the remaining vertices that are not observed by $S'$. Therefore $S$ does not observe all vertices in this component of $G$, which contradicts the fact that $S$ is a PDS.

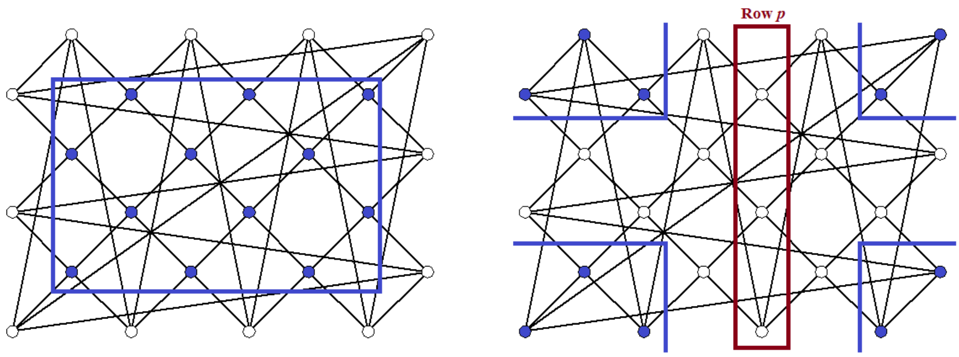


FIGURE 1. Illustration of Claim 2 (left) and row $p$ (right) on a component of $C_8 \times C_6$ in Lemma 14.

From Claim 2, $|B(M)| = 2\left(\frac{m}{2} - 1\right) + 2\left(\frac{n}{2} - 1\right) - 2 = m + n - 6$. Using Claim 1 and the fact that $G$ is 4-regular, we have $m + n - 6 \leq 4\big(|S| - 1\big)$. Since $|S|$ must be an integer, it follows that $|S| \geq \left\lceil \frac{m+n-2}{4} \right\rceil$. □

**Lemma 15.** *If $G = C_n \times C_m$, where $n$ is odd and $m$ is even, then $\gamma_p(G) \geq max\left\{\left\lceil \frac{m+2n-2}{4} \right\rceil, \left\lceil \frac{2n}{3} \right\rceil, \left\lceil \frac{m}{3} \right\rceil\right\}$.*

**Proof.** Result follows from Lemma 14 after we observe that $G$ is isomorphic to a component of $C_{2n} \times C_m$. □

In the proof of the following lemma, we shall denote the vertices of $C_n \times C_m$ as $(i, j)$. If $i > n$ (or respectively, $j > m$), then $i$ (or respectively, $j$) is taken modulo $n$ (or respectively, $m$). Let $V_{odd}$ and $V_{even}$ be the vertex sets such that $V_{odd} = \{(i, j) \mid i+j \equiv 1 (\text{mod } 2)\}$ and $V_{even} = \{(i, j) \mid i+j \equiv 0 (\text{mod } 2)\}$ respectively (see Figure 2). Observe that when either $n$ or $m$ is odd, then there are vertices in $V_{even}$ that are adjacent to some vertices in $V_{odd}$. For any two vertices in $S$, we denote $|\Delta i|$ and $|\Delta j|$ to be the difference in the row positions and column positions respectively. Note that the word "position" should not be replaced with the word "number". To illustrate this, we note that the difference in the row positions of two vertices, one in row 1 and the other in row $n$, is equal to one (because vertices in row $n$ are adjacent to vertices in row 1), even though the difference in their row numbers is $n - 1$.

**Lemma 16.** *If $G = C_n \times C_m$, where $n \geq m \geq 3$ with $n$ and $m$ both odd, then $\gamma_p(G) \geq max\left\{\left\lceil \frac{m+n}{4} \right\rceil, \left\lceil \frac{n}{3} \right\rceil\right\}$.*

**Proof.** Let $S$ be a PDS of $G$. WLOG, let $u = (2, 2) \in S \cap V_{even}$. Since the smallest possible graph $G$, which is $C_3 \times C_3$, has power domination number equal to 2, it is clear that $|S| \geq 2$. We initialize $S' = \{u\}$. For the propagation step in Algorithm 1 to take place, there must be another vertex $v = (i, j) \in S$ such that $i, j > 2$ (up to symmetry), $d(u, v) \leq 3$ and $|\Delta i| + |\Delta j| \leq 4$. We then update $S' \leftarrow S' \cup \{v\}$. If $N[v] \subset V_{even}$, then vertices in $V_{odd}$ are not observed by $S'$. Therefore we can apply the above steps for a finite number of times and add more vertices with

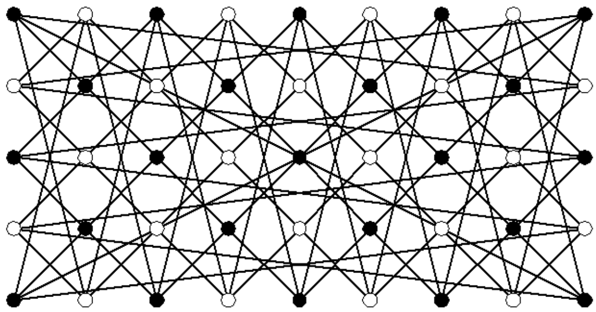

FIGURE 2. Vertices in $V_{even} \subset V(C_9 \times C_5)$ are colored black.

non-decreasing row and column numbers to $S'$ until there exists $w = (a, b) \in S'$ such that $a > n$, $b > m$ and at least two vertices in $N[w]$ belongs to $V_{odd}$. Since $S' \subseteq S$, it follows that $m + n \leq \sum(|\Delta i| + |\Delta j|) \leq 4|S'| \leq 4|S|$. Therefore we have $|S| \geq \lceil \frac{m+n}{4} \rceil$. □

By Lemmas 14 to 16 and the constructions of PDS in [**9**], we present the power domination numbers of $C_n \times C_m$ in the following theorem.

**Theorem 17.** *For $G = C_n \times C_m$, where $n, m \geq 3$,*

(i) *if $n$ and $m$ are both even, $n \geq m$ and $m \leq \frac{1}{3}(n+6)$, then $\gamma_p(G) = 2\lceil \frac{n}{3} \rceil$;*
(ii) *if $n$ and $m$ are both even, $n \geq m$ and $m > \frac{1}{3}(n+6)$, then $\gamma_p(G) = 2\lfloor \frac{m+n}{4} \rfloor$ if $n + m \not\equiv 6$ (mod 8), and $2\lfloor \frac{m+n}{4} \rfloor \leq \gamma_p(G) \leq 2\lceil \frac{m+n}{4} \rceil$ if $n + m \equiv 6$ (mod 8);*
(iii) *if $n$ is odd, $m$ is even and $m \leq \frac{1}{3}(2n+6)$, then $\gamma_p(G) = \lceil \frac{2n}{3} \rceil$;*
(iv) *if $n$ is odd, $m$ is even and $m \geq 6n - 6$, then $\gamma_p(G) = \lceil \frac{m}{3} \rceil$;*
(v) *if $n$ is odd, $m$ is even and $\frac{1}{3}(2n+6) < m < 6n - 6$, then $\gamma_p(G) = \lfloor \frac{m+2n}{4} \rfloor$ if $2n + m \not\equiv 6$ (mod 8), and $\lfloor \frac{m+2n}{4} \rfloor \leq \gamma_p(G) \leq \lceil \frac{m+2n}{4} \rceil$ if $2n + m \equiv 6$ (mod 8);*
(vi) *if $n$ and $m$ are both odd, $n \geq m$ and $m \leq \frac{1}{3}(n+5)$, then $\gamma_p(G) = \lceil \frac{n}{3} \rceil$;*
(vii) *if $n$ and $m$ are both odd, $n \geq m$ and $m > \frac{1}{3}(n+5)$, then $\gamma_p(G) = \lceil \frac{m+n}{4} \rceil$.*

In [**12**], the power domination number for the direct product of a path $P_n$ and a cycle $C_m$, was established for an even integer $n$. In what follows, we present their results and proceed to partially solve the power domination problem for $P_n \times C_m$ when $n$ is odd.

**Lemma 18.** [**12**] *Let $G$ be a connected component of $P_n \times C_m$. Then $\gamma_p(G) \geq \lceil \frac{n}{3} \rceil$.*

**Theorem 19.** [**12**] *Let $G = P_n \times C_m$, where $n$ is even and $m \geq 3$. Then*

$$\gamma_p(G) = \begin{cases} 2\lceil \frac{n}{3} \rceil & \text{if } m \text{ is even,} \\ \lceil \frac{n}{3} \rceil & \text{if } m \text{ is odd.} \end{cases}$$

**Lemma 20.** *Let $G$ be a connected component of $P_n \times C_m$, where $n$ is odd and $m$ is an even integer greater than or equal to 4. Then $\gamma_p(G) \geq \lceil \frac{m}{4} \rceil$.*

**Proof.** Let $S$ be a PDS of $G$, and let $U$ be the vertices of any four consecutive columns of $G$. We label these four columns sequentially from 1 to 4, and WLOG suppose that columns 1 and 3 each has more vertices than that of column 2 or 4. We claim that $S \cap U \neq \emptyset$; for otherwise, vertices in column 3 are not observed during the domination step of Algorithm 1. Furthermore, propagation cannot proceed from any vertex in column 2 or 4 to vertices in column 3. This contradicts our assumption that $S$ is a PDS. Therefore we have $S \geq \frac{m}{4}$. □

**Theorem 21.** *Suppose that $n, m \geq 3$ with $n$ odd. If $m$ is even, then*

$$2\ max\Big\{\Big\lceil\frac{n}{3}\Big\rceil, \Big\lceil\frac{m}{4}\Big\rceil\Big\} \leq \gamma_p(P_n \times C_m) \leq 2\ max\Big\{\Big\lceil\frac{n}{3}\Big\rceil, \Big\lceil\frac{3n+m-4}{10}\Big\rceil, \Big\lceil\frac{m}{4}\Big\rceil\Big\}.$$

*Otherwise $m$ is odd, and we have*

$$max\Big\{\Big\lceil\frac{n}{3}\Big\rceil, \Big\lceil\frac{m}{2}\Big\rceil\Big\} \leq \gamma_p(P_n \times C_m) \leq max\Big\{\Big\lceil\frac{n}{3}\Big\rceil, \Big\lceil\frac{3n+2m-4}{10}\Big\rceil, \Big\lceil\frac{m}{2}\Big\rceil\Big\}.$$

**Proof.** We denote the vertices of $P_n \times C_m$ as $(i, j)$ with $i$ and $j$ taken modulo $n$ and $m$ respectively. If $m$ is even, then we observe that $P_n \times C_m$ comprises of two isomorphic connected components. Let $G$ be the component that contains the vertex $(0, 0)$. By Lemmas 18 and 20, $\gamma_p(G) \geq \max\big\{\lceil\frac{n}{3}\rceil, \lceil\frac{m}{4}\rceil\big\}$. To prove the upper bound, we need to construct a PDS of $G$. When $m \leq 2n-4$, we denote $\alpha = \max\big\{\lceil\frac{n}{3}\rceil, \lceil\frac{3n+m-4}{10}\rceil\big\}$ and let $S = \big\{(3k-2, k) \mid k = 1, 2, ..., n-2\alpha\big\} \cup \big\{(3n-6\alpha-2+2k, n-2\alpha+4k) \mid k = 1, 2, ..., 3\alpha-n\big\}$. When $m \geq 2n-2$, we let $S = \big\{\big(\min\{2k-1, n-2\}, 4k-3\big) \mid k = 1, 2, ..., \lceil\frac{m}{4}\rceil\big\}$. It can be verified that the set $S$ is a PDS of $G$. Finally for the case where $m$ is odd, result follows after observing that $P_n \times C_m$ is isomorphic to a connected component of $P_n \times C_{2m}$. □

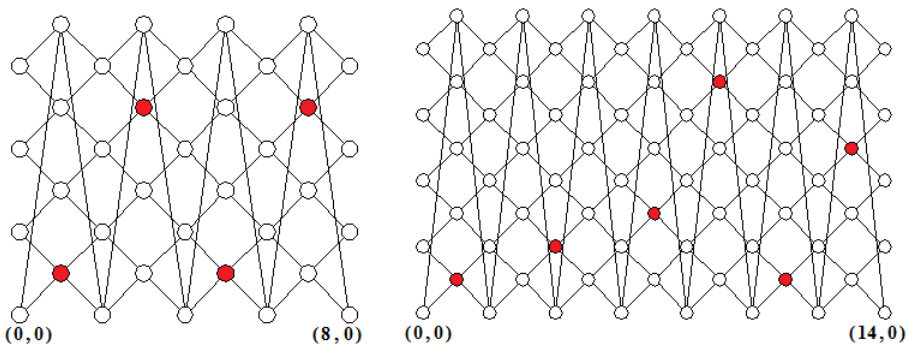


FIGURE 3. A PDS on a component of $P_9 \times C_8$ (left) and $P_{15} \times C_{10}$ (right).

We believe that the upper bounds in Theorem 21 are optimal. However we are not able to show that $\gamma_p(P_n \times C_m) \geq \lceil\frac{3n+m-4}{10}\rceil$ for odd $n$ and even $m$.

**Theorem 22.** [12] *For $m, n \geq 2$,*

$$\gamma_p(K_n \times K_m) = \begin{cases} 2 & \text{if } m+n \geq 6 \text{ or } m+n=4, \\ 1 & \text{otherwise.} \end{cases}$$

**Theorem 23.** [**12**] *For $n \geq 3$ and $m \geq 4$,*

$$\gamma_p(K_n \times C_m) = \begin{cases} \lceil \frac{m+1}{2} \rceil & \text{if } m \equiv 2 \pmod 4, \\ \lceil \frac{m}{2} \rceil & \text{otherwise.} \end{cases}$$

For Theorem 22, we include the condition "$m + n = 4$", which was missing in [**12**]. We also remark that Theorem 23 holds when $m = 3$, that is, $\gamma_p(K_n \times C_3) = 2$ for $n \geq 3$.

**Theorem 24.** [**12**] *If $G$ and $H$ each has at least two universal vertices, then $\gamma_p(G \times H) \leq 2$.*

## 4. Strong Product

The authors in [**4**] studied the set of observed vertices of $P_n \boxtimes P_m$ during a propagation step of Algorithm 1 and subsequently obtained its power domination number.

**Theorem 25.** [**4**] *Let $n \geq m \geq 2$. Then*

$$\gamma_p(P_n \boxtimes P_m) = max\{\lceil \tfrac{n}{3} \rceil, \lceil \tfrac{n+m-2}{4} \rceil\}$$

*unless $3m - n - 6 \equiv 4$ (mod 8), in which case*

$$max\{\lceil \tfrac{n}{3} \rceil, \lceil \tfrac{n+m-2}{4} \rceil\} \leq \gamma_p(P_n \boxtimes P_m) \leq max\{\lceil \tfrac{n}{3} \rceil, \lceil \tfrac{n+m-2}{4} \rceil + 1\}.$$

In what follows, we shall prove equality for the last remaining case in Theorem 25.

**Theorem 26.** *For $n \geq m \geq 2$,*

$$\gamma_p(P_n \boxtimes P_m) = \begin{cases} max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil + 1\} & \text{if } 3m - n \equiv 2 \pmod 8, \\ max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil\} & \text{otherwise.} \end{cases}$$

**Proof.** It remains to prove that $\gamma_p(P_n \boxtimes P_m) \geq \lceil \frac{n+m-2}{4} \rceil + 1$ when $3m - n \equiv 2$ (mod 8). Let $S$ be a PDS of $P_n \boxtimes P_m$. As in [**4**], we let $M$ be the set of observed vertices of $P_n \boxtimes P_m$ during a propagation step of Algorithm 1, and let $B(M)$ be the set of vertices of $M$ that have less than eight neighbors in $M$. It was shown in that paper that $|B(M)|$ is non-increasing during the propagation step, and $2(m + n - 2) \leq B(M) \leq 8|S|$.

Suppose on the contrary that $|S| = \lceil \frac{n+m-2}{4} \rceil$. Notice that $m$ and $n$ must both be either odd or even, so that $4(n - m)$ is a multiple of 8. Then $3n - m = (3m - n) + 4(n - m) \equiv 2$ (mod 8) and $m + n = \frac{1}{2}\big[(3m - n) + (3n - m)\big] \equiv 2$ (mod 4), so that $|S| = \frac{n+m-2}{4}$. It follows that $2(m + n - 2) = B(M) = 8|S|$, which implies that $N(S) = 8|S|$ and $B(M)$ remains the same during a propagation step. We claim that both $m$ and $n$ must therefore satisfy the following conditions:

$$m = 3 + p + 3q, \qquad n = 3 + 3p + q,$$

$p$ and $q$ are non-negative integers that satisfies $p + q = |S| - 1$.

We prove this claim using induction on $|S|$. For the base case when $|S| = 1$, $m = n = 3$ in order to have $B(M) = N(S) = 8$. Notice that the vertices in $N[S]$ are confined in a rectangular region. To prove the inductive step, we assume for some integer $k$ that when $|S| = k$, both $m = m_k$ and $n = n_k$ satisfy the conditions, and the vertices in $N[S]$ are confined in a rectangular region. We also assume

WLOG that there exists a vertex $u \in S$ that belongs to row $n_k - 1$ and column $m_k - 1$. If a vertex $v$ is added to $S$ such that $|S| = k + 1$, then $v$ lies in or beyond either row $n_k + 2$ or column $m_k + 2$ so that the domination step of Algorithm 1 adds eight vertices, excluding $v$. Then for propagation to take place such that $S$ is a PDS of $P_n \boxtimes P_m$ and $B(M)$ remains the same, it can be verified that $v$ must belong to either (a) row $n_k + 2$ and column $m_k$, which increases only the value of $p$ by one, or (b) row $n_k$ and column $m_k + 2$, which increases only the value of $q$ by one (see Figure 4). It follows from cases (a) and (b) that both $m$ and $n$ satisfy the required conditions.

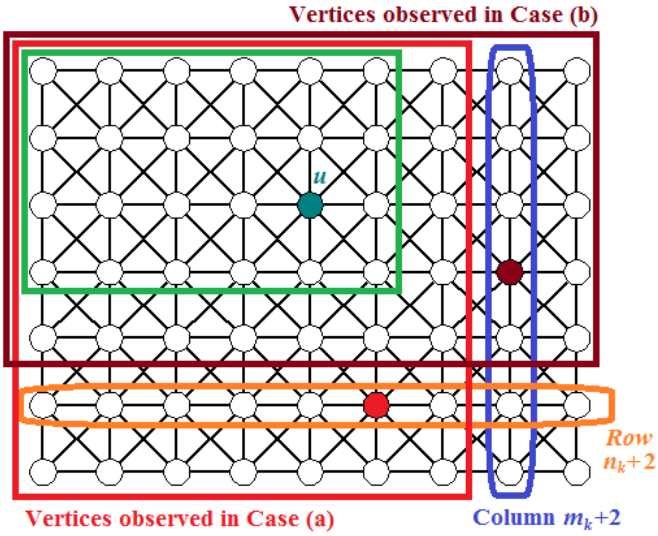


FIGURE 4. Illustration of the inductive step when $k$ is increased from 2 to 3.

Finally, we substitute $q = |S| - 1 - p = \frac{n+m-6}{4} - p$ into either $m = 3 + p + 3q$ or $n = 3 + 3p + q$ to obtain the contradiction that $3n - m = 6 + 8p \not\equiv 2 \pmod 8$. □

**Lemma 27.** *For $n \geq m \geq 3$, $\gamma_p(C_n \boxtimes C_m) \geq \lceil \frac{n}{3} \rceil$.*

**Proof.** Proof is identical to that of [**4**] (Lemma 3.2) and is therefore omitted. □

**Lemma 28.** *For $n \geq m \geq 3$, $\gamma_p(C_n \boxtimes C_m) \geq \lceil \frac{n+m-2}{4} \rceil$.*

**Proof.** Let $S$ be a PDS of $C_n \boxtimes C_m$. It is easy to see that $|S| = 1$ when $n = m = 3$. When $n \geq 4$, we have $|S| \geq 2$, so the set $S'$ obtained by removing a vertex from $S$ is nonempty. By following the proof similar to that of Lemma 14, it can be shown that up to symmetry, all vertices in either one of the sets $\{1, 2, ..., n-3\} \times \{1, 2, ..., m-1\}$ or $\{1, 2, ..., n-1\} \times \{1, 2, ..., m-3\}$ are observed by $S'$. We then obtain the inequality $2n + 2m - 12 \leq |B(M)| \leq 8(|S| - 1)$, and result follows. □

We remark that for the PDS constructed in [**4**] for $P_n \boxtimes P_m$, the propagation step in Algorithm 1 is not affected when edges are added to connect vertices in row

1 to vertices in row $n$. The same can also be said for vertices in columns 1 and $m$. Hence the construction for the PDS of $P_n \boxtimes P_m$ is valid for $C_n \boxtimes C_m$. Finally by following the proof for Theorem 26 for the case when $3m - n \equiv 2 \pmod 8$, we have the following result.

**Theorem 29.** *For $n \geq m \geq 3$,*

$$\gamma_p(C_n \boxtimes C_m) = \begin{cases} max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil + 1\} & \text{if } 3m - n \equiv 2 \text{ (mod 8)}, \\ max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil\} & \text{otherwise.} \end{cases}$$

The result for the power domination problem of the strong product of a path and a cycle can similarly be shown by following the proof of $\gamma_p(C_n \boxtimes C_m)$ closely.

**Theorem 30.** *Let $G$ be the graph $P_n \boxtimes C_m$ for $n \geq 2$ and $m \geq 3$, or the graph $C_n \boxtimes P_m$ for $n \geq 3$ and $m \geq 2$. If $n \geq m$, then*

$$\gamma_p(G) = \begin{cases} max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil + 1\} & \text{if } 3m - n \equiv 2 \text{ (mod 8)}, \\ max\{\lceil \frac{n}{3} \rceil, \lceil \frac{n+m-2}{4} \rceil\} & \text{otherwise.} \end{cases}$$

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