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k-RATIONAL HOMOTOPY FIXED POINTS, $k \in \mathbb{N}$

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Abstract. For $k \in \mathbb{N}$, we introduce the notion of k-rational homotopy fixed points and we prove, under a certain assumption, that if X is a rational elliptic space of formal dimension n, then X admits an $(n-1)$-rational homotopy fixed point.

1. Introduction

In this paper, by a space we mean a rational simply connected CW-complex X of finite type, i.e., $\dim H^n(X;\mathbb{Z}) < \infty$ for all n.

A space X is called elliptic if both the graded vector spaces $H^*(X;\mathbb{Z})$ and $\pi_*(X)$ are finite dimensional. Let us call $n = \max\{i : H^i(X,\mathbb{Z}) \neq 0\}$ the formal dimension of X.

Let $\mathcal{E}_*(X)$ denote the group of self-homotopy equivalences of a space X inducing the identity on the homology groups (see [**1, 3**]).

A space X is said to have a k-rational homotopy fixed point if there exist $[\alpha] \in \mathcal{E}_*(X^k)$ and a non-zero homotopy class $x \in \pi_k(X^k)$ such that $\alpha \not\simeq id$ and $\pi_k(\alpha)(x) = x$, where X^k is the k-skeleton of X.

The aim of this paper is to establish the existence of a k-rational homotopy fixed point for an elliptic space under a certain condition. More precisely, we use the Quillen model to construct a homomorphism of groups $\psi : \pi_n(X^n) \rightarrowtail \mathcal{E}_*(X^n)$ and we prove the following main result.

Theorem 1. *Let X be an elliptic space of formal dimension n. If ψ is not surjective, then X has an $(n-1)$-homotopy fixed point.*

The paper is organised as follows. In section 2, we recall the basic properties of the Quillen model in rational homotopy theory as well as some results regarding the group $\mathcal{E}_*(X)$. In section 3, we formulate and prove the main theorem in an algebraic setting and then we give a mere transcription of the above result in the topological context.

2. Quillen Model in Rational Homotopy Theory

We briefly recall Quillen's differential graded Lie algebras (DGL for short) frameworks for rational homotopy theory. All the materials can be founded in [**6**]. Also we mention some results regarding the group $\mathcal{E}_*(X)$, and based on them we shall prove the main theorems in this paper.

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Indeed, if X is a space, then there exists a DGL $(\mathbb{L}(W),\delta)$ called the Quillen model of X, unique up to isomorphism, which determines completely the homotopy type of the space X. Also in this setting, there is a concept of "homotopy" between differential graded Lie algebra morphisms, analogous in many respects to the topological notion of homotopy (see [**6**, pp.321]). Thus, let $\mathcal{E}_*(\mathbb{L}(W))$ denote the group of the homotopy self-equivalences of $(\mathbb{L}(W),\delta)$ inducing the identity on the graded vector space of the indecomposables W (see [**2, 4**]). By virtue of the properties of the model of Quillen, we can recover the homotopy data via the following identifications

$$\pi_*(X)\cong H_{*-1}(\mathbb{L}(W)),\qquad H_*(X,\mathbb{Z})\cong W_{*-1},\qquad \mathcal{E}_*(X)\cong\mathcal{E}_*(\mathbb{L}(W)). \tag{1}$$

Furthermore, for every n, the DGL $(\mathbb{L}(W_{\leq n}),\delta)$ can be considered as the Quillen model of X^{n+1} implying that

$$\mathcal{E}_*(X^{n+1})\cong\mathcal{E}_*(\mathbb{L}(W_{\leq n}))\quad\text{and}\quad \pi_*(X^{n+1})\cong H_{*-1}(\mathbb{L}(W_{\leq n})). \tag{2}$$

Thus, if $[\alpha]\in\mathcal{E}_*(X)$, then $[\alpha]$ induces an element $[\beta]\in\mathcal{E}_*(\mathbb{L}(W))$ such that the following diagram commutes.

$$\begin{array}{ccc} \pi_*(X) & \xrightarrow{\pi_*(\alpha)} & \pi_*(X) \\ \downarrow\cong & & \downarrow\cong \\ H_*(\mathbb{L}(W)) & \xrightarrow{H_*(\beta)} & H_*(\mathbb{L}(W)) \end{array} \tag{3}$$

2.1. A certain short exact sequence. (see [**5**] for more details)

Definition 2.1. Let $(\mathbb{L}(W_q\oplus W_{\leq k}),\delta)$ be a DGL, where $q>k$ and let $b_q: W_q\to H_{q-1}(\mathbb{L}(W_{\leq k}))$ be the linear map defined by setting

$$b_q(v)=[\delta(v)],\qquad v\in W_q. \tag{4}$$

Here $[\delta(v)]$ denotes the homology class of $\delta(v)\in\mathbb{L}_{q-1}(W_{\leq k})$.

Let $\mathcal{R}_k^q$ be the subgroup of $\mathcal{E}_*(\mathbb{L}(W_{\leq k}))$ consisting of the elements $[\gamma]$ making the following diagram commute.

$$\begin{array}{ccc} W_q & & \\ \downarrow b_q & \searrow^{b_q} & \\ H_{q-1}(\mathbb{L}(W_{\leq k})) & \xrightarrow{H_{q-1}(\gamma)} & H_{q-1}(\mathbb{L}(W_{\leq k})) \end{array} \tag{5}$$

Theorem 2.2 ([**5**], Theorem 2.6)**.** *Let $(\mathbb{L}(W_q\oplus W_{\leq k}),\delta)$ be a DGL, where $q>k$. There is a short exact sequence of groups*

$$\mathrm{Hom}\big(W_q,H_q(\mathbb{L}(W_{\leq k}))\big)\rightarrowtail\mathcal{E}_*(\mathbb{L}(W_q\oplus W_{\leq k}))\twoheadrightarrow\mathcal{R}_k^q. \tag{6}$$

2.2. Elliptic rational spaces. As was stated in the introduction, a space X is called elliptic if both the graded vector spaces $H^*(X;\mathbb{Z})$ and $\pi_*(X)$ are finite dimensional. Notice that the integer $n = \max\{i : H^i(X,\mathbb{Z}) \neq 0\}$ is called the formal dimension of X. The following theorem mentions some important properties of elliptic spaces.

Theorem 2.3. *([**6**], § 32, pp. 434). If X is elliptic of formal dimension n, then we have the following facts.*
(1) *$H^n(X;\mathbb{Z}) \cong \mathbb{Q}$ and $H^i(X;\mathbb{Z}) = 0$ for all $i \geq n+1$.*
(2) *$\pi_i(X) = 0$, for $i \geq 2n$.*
(3) *$\pi_i(X) = 0$, for $i > n$ and i is even.*
(4) *For $i > n$, there exists at most one vector space $\pi_i(X) \neq 0$. In this case, i must be odd and $\dim \pi_i(X) = 1$.*
(5) *$\sum_{i\geq 1}(2i+1) \dim \pi_{2i+1}(X) \leq 2n-1$.*
(6) *$\sum_{i\geq 1}(2i) \dim \pi_{2i}(X) \leq n$.*

Translating the above properties into the DGL's framework by using the identifications (1), we derive that the Quillen model of X has the following properties.

Proposition 2.4. *If $(\mathbb{L}(W),\delta)$ is the Quillen model of an elliptic space of formal dimension n, then we have*
(1) *$\dim W_{n-1} = 1$ and $W_i = 0$ for all $i \geq n$.*
(2) *$H_{i-1}(\mathbb{L}(W)) = 0$, for $i \geq 2n$.*
(3) *$H_{i-1}(\mathbb{L}(W)) = 0$, for $i > n$ and i is even.*
(4) *There is only one non-trivial vector space. $H_{i-1}(\mathbb{L}(W))$, for $i > n-1$ and i must be odd. Necessary, we have $\dim H_{i-1}(\mathbb{L}(W)) = 1$.*
(5) *$\sum_{i\geq 1}(2i+1) \dim H_{2i}(\mathbb{L}(W)) \leq 2n-1$.*
(6) *$\sum_{i\geq 1}(2i) \dim H_{2i-1}(\mathbb{L}(W)) \leq n$.*

The following crucial result is needed subsequently.

Theorem 2.5. *Let $(\mathbb{L}(W),\delta)$ be the Quillen model of an elliptic space X of formal dimension n. If the linear map b_{n-1}, given in the diagram (5), is nil, then X is homotopic to a sphere.*

Proof. Since $(\mathbb{L}(W),\delta)$ is the Quillen model of an elliptic space of formal dimension n, by the relation (1) of Proposition 2.4, we can write $W_{n-1} = \langle w \rangle$. Now if we assume that b_{n-1} is null, then $b_{n-1}(w) = 0$ and going back to (4), we derive that there exists $q \in \mathbb{L}_{n-1}(W_{\leq n-2})$ such that $\delta(w+q) = 0$, providing a non-zero homology class in $H_{n-1}(\mathbb{L}(W))$. Now let us distinguish the following two cases.

Case 1: Assume n is odd. If W_{n-1} is not the only non-zero vector space which form the graded vector space W, then we must have w' such that $|w'| = k < n-1$ and $\delta(w') = 0$. As a result the bracket $[w', w+q]$ is a cycle defining a non-zero homology class lying in $H_{n-1+k}(\mathbb{L}(W))$ due to the relation (1) of Proposition 2.4. Therefore, $H_{n-1+k}(\mathbb{L}(W)) \neq 0$.
Now if k is odd, then $n-1+k$ is also odd which contradicts the relation (6) of Proposition 2.4.
If k is even, then $n-1+k$ is also even which contradicts the relation (5) of Proposition 2.4, taking into account that $H_{n-1}(\mathbb{L}(W)) \neq 0$ and $n-1$ is even.

Case 2: Assume n is even. Since $H_{n-1+k}(\mathbb{L}(W)) \neq 0$, by applying the relation (6) of Proposition 2.4, we deduce that

$$H_i(\mathbb{L}(W)) = 0, \text{ for every } i < n-1 \text{ and } i \text{ is odd.} \tag{7}$$

Likewise, if W_{n-1} is not the only non-zero vector space which form the graded vector space W, then we must have a generator $w' \in W$ such that $\delta(w') = 0$, where $|w'| = k$ must be even according to (7). As a result the bracket $[w', w+q]$ is a cycle defining a non-zero homology class in $H_{n-1+k}(\mathbb{L}(W)) \neq 0$ by the relation (1) of Proposition 2.4. Since $n-1+k$ is odd we obtain a contradiction with the relation (6) of Proposition 2.4.

Consequently, in both cases, $(\mathbb{L}(W), \delta)$ has the form $(\mathbb{L}(w), 0)$ with $|w| = n-1$ which is the Quillen model of the sphere S^n. □

Corollary 2.6. *Let $(\mathbb{L}(W), \delta)$ be the Quillen model of an elliptic space X of formal dimension n. If the linear map b_{n-1} is null, then $\mathcal{E}_*(\mathbb{L}(W))$ is trivial.*

Proof. By Theorem 2.5 the DGL $(\mathbb{L}(W), \delta)$ has the form $(\mathbb{L}(w), 0)$ with $|w| = n-1$ and easy to see that $\mathcal{E}_*(\mathbb{L}(W)) = \mathcal{E}_*(\mathbb{L}(w)) = 1$. □

3. Main Results

3.1. Rational homotopy fixed points.

Definition 3.1. A space X is said to have a k-rational homotopy fixed point if there exist $[\alpha] \in \mathcal{E}_*(X^k)$ and a non-zero homotopy class $x \in \pi_k(X^k)$ such that $\alpha \not\simeq id$ and $\pi_k(\alpha)(x) = x$, where X^k denotes the k^{th} skeleton of the space X.

Remark 3.2. Let X be an elliptic space. From definition 3.1, it is clear that if the group $\mathcal{E}_*(X^k)$ is trivial, then X does not have a k-rational homotopy fixed point.

Lemma 3.3. *If $(\mathbb{L}(W), \delta)$ is the Quillen model of an elliptic space X of formal dimension n, then we have the following short exact sequence*

$$H_{n-1}(\mathbb{L}(W_{\leq n-1})) \overset{\phi}{\twoheadrightarrow} \mathcal{E}_*(\mathbb{L}(W_{\leq n-1})) \twoheadrightarrow \mathcal{R}^{n-1}_{n-2}. \tag{8}$$

Proof. It suffices to apply the short exact sequence (6), given in Theorem 2.2, for $q = n-1$, $k = n-2$ and taking into consideration that as X is elliptic of formal dimension n, its Quillen model has the form $(\mathbb{L}(W_{\leq n-1}), \delta)$ with $W_{n-1} \cong \mathbb{Q}$ according the relation (1) of Proposition 2.4. □

Remark 3.4. Using the identifications (2), the homomorphism ϕ induces a homomorphism of groups $\psi : \pi_n(X^n) \rightarrowtail \mathcal{E}_*(X^n)$.

Now we are ready to state the main theorem of this paper.

Theorem 3.5. *Let X be an elliptic space X of formal dimension n. If ψ is not surjective, then X has an $(n-1)$-rational homotopy fixed point.*

Proof. Recall that as X has formal dimension n, its Quillen model has the form $(\mathbb{L}(W_{\leq n-1}), \delta)$ with $W_{n-1} \cong \mathbb{Q}$. Now let us consider the short exact sequence (8). By hypothesis ϕ is not surjective, it follows that the two groups $\mathcal{E}_*(\mathbb{L}(W_{\leq n-1}))$ and

$\mathcal{R}_{n-2}^{n-1}$ are not trivial. Recall that $\mathcal{R}_{n-2}^{n-1}$ is the subgroup of $\mathcal{E}(\mathbb{L}(W_{\leq n-2}))$ consisting of the all the elements $[\gamma]$ making the following diagram commute.

$$\begin{array}{ccc} W_{n-1} \cong \mathbb{Q} & & \\ \downarrow{\scriptstyle b_{n-1}} & \searrow^{b_{n-1}} & \\ H_{n-2}(\mathbb{L}(W_{\leq n-2})) & \xrightarrow{H_{n-2}(\gamma)} & H_{n-2}(\mathbb{L}(W_{\leq n-2})) \end{array} \tag{9}$$

It is worth mentioning that from Corollary 2.6, we know that $b_{n-1}(w) \neq 0$ as $\mathcal{E}_*(\mathbb{L}(W_{\leq n-1}))$ is not trivial. Consequently, if we write $W_{n-1} = \langle w \rangle$, then by virtue of the commutativity of the diagram (9) we obtain $H_{n-2}(\gamma) \circ b_{n-1}(w) = b_{n-1}(w)$.

Summarizing, if we set $b_{n-1}(w) = x$, then there exist $[\gamma] \in \mathcal{E}(\mathbb{L}(W_{\leq n-2}))$ and $x \in H_{n-2}(\mathbb{L}(W_{\leq n-2}))$ such that

$$H_{n-2}(\gamma)(x) = x.$$

A mere transcription of the above result in the topological context by using the identifications (2) and the diagram (3), allows us to write that there exist $[\gamma] \in \mathcal{E}_*(X^{n-1})$ and a non-zero homotopy class $x \in \pi_{n-1}(X^{n-1})$ such that

$$\pi_{n-1}(\gamma)(x) = x.$$

Thus, the space X has a $(n-1)$-rational homotopy fixed point. □

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