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RANKS ON THE BOUNDARIES OF SECANT VARIETIES

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Abstract. In many cases (e.g. for many Segre or Segre-Veronese embeddings of multiprojective spaces) we prove (in characteristic 0) that a hypersurface of the b-secant variety of $X \subset \mathbb{P}^r$ has X-rank $> b$. We prove it proving that the X-rank of a general point of the join of $b-2$ copies of X and the tangential variety of X is $> b$.

1. Introduction

Let $X \subset \mathbb{P}^r$ be an integral and non-degenerate variety defined over an algebraically closed field. For any $q \in X$ the X-rank $r_X(q)$ of X is the minimal cardinality of a set $S \subset X$ such that $q \in \langle S\rangle$, where $\langle\ \rangle$ denote the linear span. For any $q \in \mathbb{P}^r$ let $\mathcal{S}(X,q)$ denote the set of all finite subsets $S \subset X$ such that $q \in \langle S\rangle$ and $\sharp(S) = r_X(q)$. For any integer $s > 0$ let $\sigma_s(X) \subseteq \mathbb{P}^r$ be the s-secant variety of X, i.e. the closure of the union of all linear space $\langle S\rangle$ with $S \subset X$ and $\sharp(S) = s$. See [**17**] for many applications of X-ranks (e.g. the tensor rank) and secant varieties (a.k.a. the border rank). The algebraic set $\sigma_s(X)$ is an integral variety of dimension $\le s(1+\dim X)-1$ and $\sigma_s(X)$ is said to be non-defective if it has dimension $\min\{r, s(1+\dim X)-1\}$. Every secant variety of a curve is non-defective ([**3**, Corollary 4]). Let $\tau(X) \subseteq \mathbb{P}^r$ be the tangential variety of X, i.e. the closure in $\mathbb{P}^r$ of the union of all tangent spaces T_pX, $p \in X_{\mathrm{reg}}$. The algebraic set $\tau(X)$ is an integral variety of dimension $\le 2(\dim X)$ and $\tau(X) \subseteq \sigma_2(X)$. For any integer $b \ge 2$ let $\tau(X,b)$ denote the join of one copy of $\tau(X)$ and $b-2$ copies of X. If X is a curve, then $\dim \tau(X,b) = \min\{r, s(1+\dim X)-2\}$ (use $b-2$ times [**3**, part 2) of Proposition 1.3] and that $\dim \tau(X) = 2$) and hence $\tau(X,b)$ is a non-empty codimension 1 subset of $\sigma_b(X)$ if X is a curve and $r > 2b$. For a variety X of arbitrary dimensional usually $\tau(X,b)$ is a hypersurface of $\sigma_b(X)$, but this is not always true. For instance, if $\sigma_2(X)$ has not the expected dimension one expects that $\tau(X,b) = \sigma_b(X)$ and this is the case if X is smooth ([**14**, Corollary 4]). A general $q \in \tau(X)$ has $r_X(q) = 2$ (and hence for any $b \ge 2$ a general $o \in \tau(X,b)$ has X-rank $\le b$) if a general tangent line to X_{reg} meets X at another point of X, i.e. if X is *tangentially degenerate* in the sense of [**15**]. It is easy to check that X is tangentially degenerate if and only if the curve $X \cap M \subset M$ is tangentially degenerate, where M is a general codimension $n-1$ linear subspace of X. H. Kaji proved that in characteristic zero a smooth curve in $\mathbb{P}^m$, $t \ge 3$, is not tangentially degenerate ([**15**, Theorem 3.1]) and this is true also if the normalization map of

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$X \cap M$ is unramified ([**15**, Remark 3.8]) or if $X \cap M$ has only toric singularities ([**11**]). See [**16**] for the state of the art (at that time) on tangentially degenerate curves and a list to the examples known in positive characteristic.

In [**7**] we raised the following question and gave a positive answer (in characteristic zero) when X is a curve.

Question 1.1. *Assume $b \geq 2$, $r \geq b(1 + \dim X) - 2$, and that $\sigma_s(X)$ has the expected dimension. Is $r_X(q) > b$ for a non-empty subset of $\sigma_b(X)$ of codimension 1 in $\sigma_b(X)$? Is $r_X(q) > b$ for a general point of $\tau(X, b)$?*

Our aim is to refine this question for $n := \dim X > 1$ and get (in some cases) a positive answer. Take a general $q \in \tau(X, b)$. There is $o \in X_{\mathrm{reg}}$, a degree 2 connected zero-dimensional scheme v with $v_{\mathrm{red}} = \{o\}$ and $p_1, \ldots, p_{b-2} \in X$ such that $p_i \neq p_j$ for all $i \neq j$, $p_i \neq o$ for all i and $q \in \langle v \cup \{p_1, \ldots, p_{b-2}\}\rangle$. For a general $q \in \tau(x, b)$ the set $(o, p_1, \ldots, p_{b-2})$ is general in X^{b-1} and v is a general tangent vector to X at o. Let $\mathcal{Z}(X, b)$ be the set of all degree b schemes $v \cup \{p_1, \ldots, p_{b-2}\}$ with $p_i \neq p_j$ for all $i \neq j$ and $o := v_{\mathrm{red}} \in X_{\mathrm{reg}} \setminus \{p_1, \ldots, p_{b-2}\}$. Let $\tau(X, b)'$ be the union of all $q \in \tau(X, b)$ such that there is $Z \in \mathcal{Z}(X, b)$ with $q \in \langle Z\rangle$. For any $q \in \tau(X, b)'$ let $\mathcal{Z}(X, b, q)$ be the set of all $Z \in \mathcal{Z}(X, b)$ such that $q \in \langle Z\rangle$.

(i) Is $\dim \tau(X, b) = b(n + 1) - 2 = \dim \sigma_b(X) - 1$?

(ii) Is $\sharp(\mathcal{Z}(X, b, q)) = 1$ for a general $q \in \tau(X, b)'$?

(iii) Is $r_X(q) > b$ for a general $q \in \tau(X, b)$?

If (i) and (iii) are true, then the set of all $q \in \sigma_b(X)$ with $r_X(q) > b$ has dimension $b(n + 1) - 2$ (i.e. if the base field is the complex number field $\mathbb{C}$ it has Hausdorff dimension $2b(n + 1) - 4$). To get a positive answer the first part of Question 1.1 for X and b it is not necessary to prove that (i) and (iii) hold and probably (ii) never will be used to prove (i) and (iii), but (ii) is a nice question, similar to ask if $\sharp(\mathcal{S}(X, o)) = 1$ for a general $o \in \sigma_b(X)$ (this is called the identifiability of $\sigma_b(X)$). The way we prove (iii) in the next theorem we get with a very similar proof also (ii), while (i) comes for free.

We prove the following result (in characteristic zero).

Theorem 1.2. *Take $b \geq 2$. Let $X \subset \mathbb{P}^r$, be a an integral and non-degenerate variety, which is non-singular in codimension 1. Set $n := \dim X$. Assume $\mathcal{O}_X(1) = \mathcal{L} \otimes \mathcal{R}$ and the existence of base point free linear spaces $V \subseteq H^0(\mathcal{L})$, $W \subseteq H^0(\mathcal{R})$ such that $v := \dim V \geq n + b + 2$, $w := \dim W \geq n + b + 2$, the morphisms $u_V : X \to \mathbb{P}^{v-1}$ and $u_W : X \to \mathbb{P}^{w-1}$, are birational onto their images, that the closures of their images X_V and X_W have singular locus of dimension $\leq n-1$, and that $\dim \sigma_2(X_V) = 2n + 1$. Assume that the image of the multiplication map $V \otimes W \to H^0(\mathcal{O}_X(1))$ is contained in the image of the restriction map $H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{O}_X(1))$ and it induces an embedding. Then $\dim \tau(X, b) = b(n + 1) - 2 = \dim \sigma_b(X) - 1$ and a general $q \in \tau(X, b)$ has $r_X(q) > b$ and $\sharp(\mathcal{Z}(X, b, q)) = 1$.*

If u_v and u_W are embeddings the assumptions on the singularities of X_V and X_W are satisfied if and only if X is non-singular in codimension 1.

We apply Theorem 1.2 to the case of certain Segre-Veronese embeddings of multiprojective spaces (see Example 2.6), but since we assumed that both $\mathcal{L}$ and $\mathcal{R}$ are birationally very ample, we cannot use Theorem 1.2 for the most important case: tensors, i.e. the Segre embedding of a multiprojective space. For tensors we prove the following result.

Theorem 1.3. *Let $X \subset \mathbb{P}^r$, $r+1 = \prod_{i=1}^{s}(n_i+1)$, be the Segre embedding of the multiprojective space $\mathbb{P}^{n_1}\times\cdots\times\mathbb{P}^{n_s}$. Fix an integer $b \ge 2$ and assume the existence of a decomposition $E \sqcup F = \{1,\dots,s\}$ such that $\prod_{i\in E}(n_i+1) > b+3+\sum_{i\in E} n_i$ and $\prod_{i\in E}(n_i+1) > b+3+\sum_{i\in F} n_i$. Then $\dim \sigma_b(X) = b(n+1)-1$, $\dim \tau(X,b) = b(n+1)-2$ and $r_X(q) > b$ for a general $q \in \tau(X,b)$.*

The assumptions of Theorem 1.3 imply $\sharp(E) \ge 2$ and $\sharp(F) \ge 2$ and hence they exclude the case $s = 2,3$. The exclusion of the case $s = 2$ is not a fault of our too restrictive assumptions. If $s = 2$ every $q \in \tau(X)\setminus X$ has X-rank 2 ([**8**], [**12**, Proposition 1.1]) and hence a general $q \in \tau(X,b)$ has rank at most b. The paper [**10**] contains 3 results related to Theorem 1.3 ([**10**, Theorems 3.1, 4.6 and 4.10]), but none of them covers Theorem 1.3.

For a better description of the X-ranks of $\sigma_3(X)$ for $s = 3$ see [**12**]. In this case $\tau(X)$ is not contained in the singular locus of $\sigma_2(X)$ ([**12**, Theorem 1.3]. We expect that the same holds for $\tau(X,b)$ for certain very positively embedded X. For the case $b = 2$, see [**19**].

We work over an algebraically closed field $\mathbb{K}$ with characteristic zero.

2. Proof of Theorem 1.2

For any integer $b > 0$ let $\mathcal{A}(X,b)$ denote the set of all subsets of X with cardinality b. For any zero-dimensional scheme $Z \subset X$ and any effective Cartier divisor D of X the residual scheme of Z with respect to D is the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(Z\cap D)+\deg(\mathrm{Res}_D(Z))$. For any line bundle $\mathcal{L}$ on X we have an exact sequence (the residual sequence of $\mathcal{I}_Z\otimes\mathcal{L}$ with respect to D):

$$0 \to \mathcal{I}_{\mathrm{Res}_D(Z)}\otimes\mathcal{L}(-D) \to \mathcal{I}_Z\otimes\mathcal{L} \to \mathcal{I}_{Z\cap D,D}\otimes\mathcal{L}_{|D} \to 0 \tag{1}$$

For any $\mathcal{L} \in \mathrm{Pic}(X)$, any linear subspace $V \subseteq H^0(X,\mathcal{L})$ and any zero-dimensional scheme $Z \subset X$ set $V(-Z) := V \cap H^0(X, \mathcal{I}_Z\otimes\mathcal{L})$.

For any integral variety M and any $o \in M_{\mathrm{reg}}$ let $(2o, M)$ be the first infinitesimal neighborhood of o in M, i.e. the closed subscheme of M with $(\mathcal{I}_{o,M})^2$ as its ideal sheaf.

Lemma 2.1. *Let $X \subsetneq \mathbb{P}^r$, be an integral and non-degenerate variety, which is scheme-theoretically cut out by quadrics. Then X is not tangentially degenerate.*

Proof. Take a general $q \in X_{\mathrm{reg}}$ and a general line $L \subset \mathbb{P}^r$ tangent to X_{reg} at q and assume that $(L\cap X)_{\mathrm{red}}$ contains a point $o \ne q$. Since the connected component of $L \cap X$ containing q contains the divisor $2q$ of L and X is scheme-theoretically cut out by quadrics, we have $L \subset X$. Since L is general, we get $\tau(X) \subseteq X$ and so $\tau(X) = X$. Let $M \subset \mathbb{P}^r$ be a general linear space with codimension $n-1$. The scheme $X \cap M$ is an integral curve spanning M and we get $\tau(X\cap M) = X \cap M$, contradicting the assumption $X \subsetneq \mathbb{P}^r$. □

Remark 2.2. The homogeneous ideal of a Segre-Veronese variety $X \subset \mathbb{P}^r$ is generated by the 2×2 minors of flattenings ([**17**, Theorem 6.10.6.5]) and in particular (unless $X = \mathbb{P}^r$) it is not tangentially degenerate by Lemma 2.1. Just to know that X is scheme-theoretically cut out by quadrics (to be able to apply Lemma 2.1) is easier, since this is easily seen to be true if it is true for the Segre embedding of X.

Lemma 2.3. *Let $X \subset \mathbb{P}^r$, $r \geq 3+n$, be an integral and non-degenerate n-dimensional variety, which is non-singular in codimension 1. Let $L \subset \mathbb{P}^3$ be a general tangent line to X_{reg}. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ denote the linear projection from L. Then $\ell_{L|X\setminus X\cap L}$ is birational onto its image.*

Proof. Since we are in characteristic zero, it is sufficient to prove that $\ell_{L|X\setminus X\cap L}$ is generically injective, i.e. that for a general $q \in X$ the plane $\langle L \cup \{q\}\rangle$ intersects X only in q and the set $(X \cap L)_{\mathrm{red}}$. If $n = 1$, then this is true by [**7**, Lemma 2.5]. Now assume $n > 1$ and that the lemma is true for varieties of dimension $< n$. Since $n > 1$, for a general hyperplane $M \subset \mathbb{P}^n$ the scheme $X \cap M$ is an integral variety non-singular in codimension 1 and spanning M. Since $\dim X \cap M > 1$, some tangent line of X_{reg} is contained in M and it is tangent to $(X \cap M)_{\mathrm{reg}}$. Since L is a general tangent line of X_{reg}, we get that for a general hyperplane $H \supset L$ the scheme $X \cap H$ is integral and spans M. Since L is a general tangent line, the set $X \cap L$ is finite. Take $p \in X_{\mathrm{reg}} \cap L$ such that $L \subset T_pX$. Since $\dim T_pX > 1$ a general $H \supset L$ does not contain T_pX, i.e. $X \cap H$ is smooth at p. We move L among the tangent lines of $(X \cap H)_{\mathrm{reg}}$ and apply the inductive assumption to $X \cap H$. We get that for a general $q \in X \cap H$ the plane $\langle L \cup \{q\}\rangle$ intersects $X \cap H$ (and hence X) only in q and the set $(X \cap L)_{\mathrm{red}}$. Moving H among the hyperplanes containing L we get the lemma. □

Lemma 2.4. *Fix an integer $b \geq 2$. Let $X \subset \mathbb{P}^r$, $r \geq 4+n$, be an integral and non-degenerate n-dimensional variety which is non-singular in codimension 1 and take a general $Z \in \mathcal{Z}(X,b)$. Write $Z = v \sqcup \{p_1,\dots,p_{b-2}\}$ with $\deg(v) = 2$ and v connected. Set $L := \langle v\rangle$ and $M := \langle Z\rangle$. Then $\dim M = b-1$ and $X \cap M = \{p_1,\dots,p_{b-2}\} \cup (X \cap L)$ (as schemes).*

Proof. If $b = 2$, then $L = M$ and the lemma is trivial. Now assume $b > 2$. We have $\dim M = b-1$, because $p_1,\dots,p_{b-2}$ are general, X is non-degenerate and $n+1 \leq r$. Let $\ell_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-2}$ denote the linear projection from L. Let $Y \subset \mathbb{P}^{r-2}$ be the closure of $\ell_L(X \setminus X \cap L)$. By Lemma 2.3 ℓ_L sends $X \setminus L \cap X$ birational into Y and $\dim Y = n$. Since Z is general, we have $p_i \notin L$ for all i and hence the points $q_i := \ell_L(p_i)$ are well-defined. For a general Z the b-tuple $(q_1,\dots,q_{b-2})$ is general in Y^{b-2}. Hence $N := \langle\{q_1,\dots,q_{b-2}\}\rangle$ has dimension $b-3$. Since we are in characteristic zero, the trisecant lemma (also known as the uniform position principle) ([**6**, p. 109]) implies that $N \cap Y = \{q_1,\dots,q_{b-2}\}$ (as schemes). Since $p_1,\dots,p_{b-2}$ are general and ℓ_L is birational onto its image, we get the lemma. □

Remark 2.5. Let $X \in \mathbb{P}^r$ be an integral and non-degenerate variety. Set $n := \dim X$. Let $\tau(X) \subseteq \mathbb{P}^r$ be the tangential variety of X. In characteristic zero if $\tau(X) \neq \mathbb{P}^r$ we have $X \subseteq \mathrm{Sing}(\tau(X))$. For a general $x \in \tau(X)$ there is $o \in X_{\mathrm{reg}}$ and a line $L \subseteq T_oX$ with $x \in L \setminus \{o\}$. The tangent space of $\tau(X)$ is constant at all points of $\tau(X)_{\mathrm{reg}} \cap L$. L is uniquely determined by a degree 2 zero-dimensional scheme $v \subset M$ such that $v_{\mathrm{red}} = \{o\}$. Let $Z(o,v)$ denote the following zero-dimensional scheme of X (and hence of $\mathbb{P}^r$) with $Z(o,v)_{\mathrm{red}} = \{o\}$ and $\deg(Z(o,v)) = 2n+1$. It is sufficient to define the ideal $\mathcal{J}$ of $Z(o,v)$ in the local ring $\mathcal{O}_{X,o}$. Since $\mathcal{O}_{X,o}$ is an n-dimensional regular local ring, there is a regular system of parameters $x_1,\dots,x_r$ such that $x_1^2, x_2,\dots,x_n$ generate the ideal sheaf of v in $\mathbb{P}^r$. Take as $\mathcal{J}$ the ideal generated by all $x_ix_jx_k$, $i,j,k \in \{1,\dots,,n\}$ and all x_1x_i, $i = 1,\dots,x_n$. Now we check that this definition depends only on X, o and v, but not on the choice of $x_1,\dots,x_r$. Let μ be the maximal ideal of $\mathcal{O}_{X,o}$. Take another regular system of parameters $y_1,\dots,y_n$ of $\mathcal{O}_{X,o}$ with $y_1^2,y_2,\dots,y_n$ generating the ideal sheaf of v in X. Since $\mathcal{O}_{X,o}$ is regular, the completion $\hat{\mathcal{O}}_{X,o}$ of $\mathcal{O}_{X,o}$ with respect to its maximal ideal is isomorphic to $\mathbb{K}[[x_1,\dots,x_n]]$. In $\mathbb{K}[[x_1,\dots,x_n]]$ we have $y_i = L_i + M_i$, with M_i a power series with no constant and no linear term, $L_1,\dots,L_n$ linear forms in $x_1,\dots,x_n$ with invertible Jacobian with respect to $x_1,\dots,x_n$ and there is a non-zero constant c such that $x_1 - cy_1 \in (x_2,\dots,x_n) + \mu^2$. Thus $y_1,\dots,y_n$ gives the same ideal. We have $T_x\tau(X) \supset Z(o,v)$. Now assume that the scheme $Z(o,v)$ is linearly independent in $\mathbb{P}^r$, i.e. that $\dim\langle Z(o,v)\rangle = 2n$. Since $\dim T_o\tau(X) = 2n$ and $T_o\tau(X) \supset Z(o,v)$, we get $\langle Z(o,v)\rangle = T_o\tau(X)$.

Proof of Theorem 1.2: Taking a linear projection we reduce to prove the theorem when the map $V \otimes W \to H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ is surjective. Let I_V (resp. I_W) be proper closed subschemes of X such that u_V (resp. u_W) is an embedding over $X \setminus I_V$ (resp. $X \setminus I_W$) and $u_V^{-1}(u_V(X \setminus I_V)) = X \setminus I_V$ (resp. $u_W^{-1}(u_W(X \setminus I_W)) = X \setminus I_W$).

(a) In this step we prove that $\dim \sigma_b(X) = b(n+1)-1$. Fix a general $S \subset X$ with $\sharp(S) = b$. Set $Z := \cup_{o\in S}(2o,X)$. By [**3**, Corollary 1.10] it is sufficient to prove that $h^1(\mathcal{I}_Z(1)) = 0$. We use induction on the integer b, starting the induction here with the obvious case $b = 1$. Fix $o \in S$ and set $S' := S\setminus\{o\}$ and $B := \cup_{o\in S'}(2o,X)$. By the inductive assumption we may assume $h^1(\mathcal{I}_B(1)) = 0$. Thus it is sufficient to prove that $(2o,X)$ gives $n+1$ independent conditions to $H^0(\mathcal{I}_B(1))$. Since we may take o general after fixing S', o is not in the base locus of $H^0(\mathcal{I}_B(1))$. Take $N \subseteq T_oX$ with $0 \le \dim N \le n$ and maximal giving independent conditions to $H^0(\mathcal{I}_{B\cup\{o\}}(1))$ and let $N' \subseteq (2o,X)$ the corresponding zero-dimensional scheme with $\deg(N') = 1+\dim N$. Assume $N' \ne (2o,X)$ and fix $N'' \subseteq (2o,X)$ with $\deg(N'') = \deg(N')+1$. To get a contradiction it is sufficient to prove that $H^0(\mathcal{I}_{N''}(1)) \subsetneq H^0(\mathcal{I}_{N'}(1))$. Since S is general, we may assume $S \cap I_V = S \cap I_W = \emptyset$. Since u_V and u_W are embedding at o, we have $\dim V(-N'') = \dim V(-N') - 1$. Since we may take S' general after fixing o, we have $\dim V(-N'') = \dim V(-N') - 1$. Take $f \in V(-N'-S') \setminus V(-N''-S')$. Since we may take o general after fixing S', we have $W(-S) \ne W(-S')$. Take $g \in W(-S') \setminus W(-S)$. The image of $f \otimes g$ shows that $H^0(\mathcal{I}_{N''}(1)) \subsetneq H^0(\mathcal{I}_{N'}(1))$.

(b) In this step we prove that $\dim \tau(X,b) = b(n+1)-2$. Fix a general $(o,o_1,\dots,o_{b-2}) \in X_{\mathrm{reg}}^{b-1}$ and a general tangent vector v to X at o. Let Z' be the degree $2n+1$ scheme associated to o and v as in Remark 2.5. Set $Z'' := (2o_1,X)\cup\cdots\cup(2o_{b-2},X)$ and $Z := Z' \cup Z''$. Since $\tau(X,b)$ is the join of $\tau(X)$ and $b-2$ copies of X, by Terracini lemma it is sufficient to prove that $h^1(\mathcal{I}_Z(1)) = 0$. For a general $(o,o_1,\dots,o_{b-2})$ we may assume $o \notin (I_V \cup I_W)$. Since u_V and u_W are embeddings at o, we have $\dim V(-(2o,X)) = v-n-1$ and $\dim W(-(2o,X)) = w-n-1$. Using $V \otimes W$ we see that $H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ separates the 2-jets of X at o and in particular $h^1(\mathcal{I}_{Z'}(1)) = 0$, concluding the proof of the case $b = 2$. We proved also that $(3o,X)$ is linearly independent in $\mathbb{P}^r$, where $(3o,X)$ is the closed subscheme of X with $(\mathcal{I}_{o,X})^3$ as its ideal sheaf. Now assume $b > 2$ and that the last assertion is true for the integer $b-1$, i.e. assume that the zero-dimensional scheme $E := (3o,X)\cup(2o_1,X)\cup\cdots\cup(2o_{b-3},X)$ is linearly independent. To prove that $\dim\tau(X,b) = b(n+1)-2$ it is sufficient to prove that $(3o,X)\cup(2o_1,X)\cup\cdots\cup(2o_{b-2},X)$ is linearly independent. Fix an integer $a \in \{0,\dots,n\}$ and schemes $A_1 \subset$

$A_2 \subseteq (2o_{b-2}, X)$ with $\deg(A_1) = \deg(A_2) - 1 = a$. By induction on a it is sufficient to prove that $H^0(\mathbb{P}^{v-1}, \mathcal{I}_{E\cup A_2}(1)) \subsetneq H^0(\mathbb{P}^{v-1}, \mathcal{I}_{E\cup A_1}(1))$. Since $\dim \sigma_2(X_V) = 2n+1$, Terracini's lemma gives $h^1(\mathbb{P}^{v-1}, \mathcal{I}_{(2u_V(o),X_V)\cup(2u_V(o_{b-2}),X_V)}(1)) = 0$ and hence $h^0(\mathbb{P}^{v-1}, \mathcal{I}_{(2u_V(o),X_V)\cup u_V(A_2)}(1)) < h^0(\mathbb{P}^{v-1}, \mathcal{I}_{(2u_V(o),X_V)\cup u_V(A_f)}(1))$. Take $f \in V(-(2o) - A_2))$ with $f \notin V(-(2o) - A_1))$. Since W is a local embedding at o, $o_1, \dots, o_{b-2}$ are general and $\dim W \ge n+b-1$, there is $g \in W(-2(o,X))$ such that $g(o_i) = 0$ if and only if $i \ne b-2$. Use the image of $f \otimes g$.

(c) In this step we prove that $\sharp(\mathcal{Z}(X,b,q)) = 1$ for a general $q \in \tau(X,b)$. Fix a general $q \in \tau(X,b)$ and assume $\sharp(\mathcal{Z}(X,b)) > 1$ and so there are $Z, A \in \mathcal{Z}(X,b)$ with $Z \ne A$. Since $\dim \tau(X,q) = b(n+1)-2$, a dimensional count shows that $\mathcal{Z}(X,b,q)$ is finite for a general $q \in \tau(X,b)$. Hence we may assume that $\mathcal{Z}(X,b,q)$ is finite. A dimensional count gives that Z and A are general in $\mathcal{Z}(X,b)$, but of course we do not assume any generality for $Z\cup A$. In particular we may assume $Z\cap(I_V\cup I_W) = \emptyset$ and $A\cap(I_V\cup I_W) = \emptyset$. Since $\dim V > b$, we get $\dim V(-Z) = \dim V - b > 0$. Let $D \subset X$ be the hypersurface whose equation is a general element of $V(-Z)$. Let E denote the residual scheme $\mathrm{Res}_D(Z\cup A)$ of $Z\cup A$ with respect to the effective Cartier divisor $D \subset X$. Since $Z \subset D$, we have $E = \mathrm{Res}_D(A)$. Thus E is a closed subscheme of A and $E = \emptyset$ if and only if $A \subset D$. Note that each element of $\mathcal{Z}(X,b)$ has only finitely many subschemes. Since $\dim \tau(X,b) = b(n+1)-2$ and q is general in $\tau(X,b)$, we have $q \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$ and $q \notin \langle A'\rangle$ for any $A' \subsetneq A$. Since $q \in \langle Z\rangle \cap \langle A\rangle$, $A \ne Z$, $q \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$ and $q \notin \langle A'\rangle$ for any $A' \subsetneq A$, we have $h^1(\mathbb{P}^r, \mathcal{I}_{Z\cup A}(1)) > 0$.

Since $u_V(X)$ is not singular in codimension 1 and it is embedded in a projective space of dimension $\ge n+2$, $u_V(X)$ is not tangentially degenerate ([**15**, Theorem 3.1]). By Lemma 2.4 applied to $X_V \subset \mathbb{P}^{v-1}$ the scheme $u_V(Z)$ is the scheme-theoretic theoretical base locus of $X_V \cap \langle u_V(Z)\rangle$. Since $A\cap I_V = \emptyset$ and $A \ne Z$, A is not contained in the base locus of $V(-Z)$. Since D is a general element of $V(-Z)$, we get $A \nsubseteq D$, i.e. $E \ne \emptyset$. Since A is general in $\mathcal{Z}(X,b)$ we have $\dim W(-A) = \dim W - \deg(A)$. Since $E \subseteq A$, we have $\dim W(-E) = w - \deg(E)$, a contradiction. The surjection $V\otimes W \to H^0(\mathcal{O}_{\mathbb{P}^r}(1)_{|X})$ gives $h^1(\mathbb{P}^r, \mathcal{I}_{Z\cup A}(1)) = 0$, a contradiction.

(d) In this step we prove that $r_X(q) > b$ for a general $q \in \tau(X,b)$. Since $\dim \tau(X,b) > \dim \sigma_{b-1}(X)$ by step (b), we have $r_X(q) = b$. Take $Z \in \mathcal{Z}(X,b,q)$ and $S \in \mathcal{S}(X,q)$. Since $S \in \mathcal{S}(X,q)$, there is no $S' \subsetneq S$ with $q \in \langle S'\rangle$. Since $\dim \tau(X,b) = b(n+1)-2$, we may assume that Z is general in $\mathcal{Z}(X,b)$ and that $q \notin \langle Z'\rangle$ for any $Z' \subsetneq Z$. Since Z is not reduced, we have $Z \ne S$. Hence $h^1(\mathcal{I}_{Z\cup S}(1)) > 0$. As in step (c) we see that Z is the intersection of the open set $X \setminus I_V$ with the scheme-theoretic base locus of $V(-Z)$. Fix a general $q \in \tau(X,b)$ and assume $r_X(q) \le b$. Take $Z \in \mathcal{Z}(X,b,q)$. If we have S with $S\cap(I_V\cup I_W) = \emptyset$ and $\dim W(-S) = w-b$, then we may apply verbatim the proof in step (c) with S instead of A. If $S\cap(I_V\cup I_W) = \emptyset$ and $\dim V(-S) = v-b$, then we may apply the proof in step (c) taking (W,Z) instead of (V,Z) and (V,S) instead of (W,A). Call $\tau\tau$ a non-empty open subset of $\tau(X,b)$ such that for each $q \in \tau\tau$ we have $r_X(q) = b$ and $q \in \langle Z\rangle$ with Z sufficiently general in $\mathcal{Z}(X,b)$ (we need $\dim W(-Z) = w-b$, $\dim V(-Z) = v-b$, $Z\cap(I_V\cup I_W) = \emptyset$ and that $(X_V, u_V(Z))$ and $(X_W, u_W(Z))$ satisfy the thesis of Lemma 2.4). The set $\mathcal{S}(X,q)$ is constructible and hence it makes sense to speak about the irreducible component of $\mathcal{S}(X,q)$ and of their dimension. Let $\sigma_b(X)'$ denote the set of all $a \in \sigma_b(X)$ such that there is a finite set $B \subset X$ with $\sharp(B) = b$, $a \in \langle B\rangle$ and $a \notin \langle B'\rangle$ for any $B' \subsetneq B$. The set $\sigma_b(X)'$ is constructible (it is the image of a an open subset of the abstract join of b copies of X). Hence $\tau := \tau\tau \cap \sigma_b(X)'$ is constructible. By assumption τ contains a non-empty open subset of $\tau(X, b)$ and hence it is irreducible and of dimension $b(n + 1) - 2$. Let $\Gamma_b \subseteq X^{(b)}$ be the set of all $S \in \mathcal{S}(X, q)$ with $q \in \tau$. Since $\dim \tau = n(b + 1) - 2$, we have $\dim \Gamma_b \geq nb - 1$. If $\dim \Gamma_b = nb$, then for a general $q \in \tau$ we may take as S a general subset of X with cardinality b, concluding the proof in this case. Thus we may assume that each irreducible component of the constructible set Γ_b has dimension $nb - 1$. Thus there is a non-empty open subset τ' of τ such that $\mathcal{S}(X, q)$ is finite for all $q \in \tau'$. Restricting τ' if necessary we may assume that the positive integer $\sharp(\mathcal{S}(X, q))$ is the same for all $q \in \tau'$. Let $X^{(b)}$ denote the symmetric product of b copies of X and let $m : X^b \to X^{(b)}$ be the quotient map. Let $\mathcal{D}$ be an irreducible component of $m^{-1}(\Gamma_b)$ with dimension $nb-1$. For any $i = 1, \dots, b$ let $\eta_i : X^b \to X^{b-1}$ denote the projection onto the factors with indices in $\{1, \dots, b\} \setminus \{i\}$. Since $\dim \mathcal{D} = nb - 1$ either $\eta_i(\mathcal{D})$ contains a non-empty open subset of X^{b-1} or the closure of $\eta_i(\mathcal{D})$ is a hypersurface Δ of X_{b-1} and $\mathcal{D}$ contains a non-empty open subset of $X \times \Delta$. Thus there is $j \in \{1, \dots, s\}$ such that $\eta_{j|\mathcal{D}}$ is dominant. Thus for a general $q \in \tau$ we may find $S = \{p_1, \dots, p_b\} \in \mathcal{S}(X, q)$ with $(p_1, \dots, p_{b-1})$ general in X^{b-1}. Thus $\dim V(-S') = v - b + 1$ and $\dim W(-S') = w - b + 1$, where $S' := \{p_1, \dots, p_{b-1}\}$. Hence p_b is both in the base locus of $V(-S')$ and in the base locus of $W(-S)$. This is impossible, since X is embedded in $\mathbb{P}^r$ and (by our reduction at the beginning of the proof) the image of the map $\rho : V \otimes W \to H^0(\mathcal{O}_{\mathbb{P}^r}(1))$ is surjective. □

Example 2.6. Fix integer $s \geq 1$, $n_i > 0$, d_i, c_i, $1 \leq i \leq s$, such that $0 < c_i < d_i$ for all i. Let $X \subset \mathbb{P}^r$, $r + 1 = \prod_{i=1}^{s} \binom{n_i+d_i}{n_i}$, be the Segre-Veronese embedding of the multiprojective space $\mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$. Set $V := H^0(\mathcal{O}_X(c_1, \dots, c_s))$, $W := H^0(\mathcal{O}_X(d_1 - c_1, \dots, d_s - c_s))$ and $n := n_1 + \dots + n_s$. Fix an integer $b \geq 2$ such that $\prod_{i=1}^{s} \binom{n_i+c_i}{n_i} \geq b + n + 2$, $\prod_{i=1}^{s} \binom{n_i+d_i-c_i}{n_i} \geq n + b + 2$ and either $s \geq 3$ or $s = 2$ and $(c_1, c_2) \neq (1, 1)$ or $s = 1$ and $c_1 \geq 3$. We claim that $\dim \tau(X, b) = b(n + 1) - 2$, $\dim \sigma_b(X) = b(n + 1) - 1$ and $r_X(q) > b$ and $\sharp(\mathcal{Z}(X, b, q)) = 1$ for a general $q \in \tau(X, b)$. By Remark 2.2 to apply Theorem 2.6 it is sufficient to observe that the variety $\sigma_2(X_V)$ has dimension $2n + 1$, where X_V is the Segre-Veronese embedding of X by the complete linear system $|\mathcal{O}_X(c_1, \dots, c_s)|$, by [**1**, Theorem 4.2].

3. Proof of Theorem 1.3

In this section $X = \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$. For any $i \in \{1, \dots, s\}$ let $\pi_i : X \to \mathbb{P}^{n_i}$ denote the projection onto the i-th factor of X. Set $\mathcal{O}_X(\varepsilon_i) := \pi_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1))$. For any $E \subseteq \{1, \dots, s\}$ set $\mathcal{O}_X(E) := \otimes_{i\in E}\mathcal{O}_X(\varepsilon_i) \in \mathrm{Pic}(X)$ and let $\pi_E : X \to \prod_{i\in E} \mathbb{P}^{n_i}$ denote the projection onto the factors of X with label in E.

Proof of Theorem 1.3: There are many papers, which could be used to see that $\dim \sigma_b(X) = b(n + 1) - 1$ ([**2**], [**13**], and if $n_i = n$ for all i, [**18**] (case $s = 3$) and [**4**], any s); this is also a consequence of [**10**, Corollary 4.15], which implies that $\mathcal{S}(X, o)$ is finite for a general $o \in \sigma_b(X)$. Take a general $q \in \tau(X, b)$. Thus there is $Z \in \mathcal{Z}(X, b)$ with q general in $\langle Z\rangle$. Since q is general in $\tau(X, b)$, Z is general in

$\mathcal{Z}(X,b)$ and q is general in $\langle Z\rangle$. In particular $q\notin\langle Z'\rangle$ for any $Z'\subsetneq Z$. Take an integer $c\le b$ and assume the existence of $W\in(\mathcal{Z}(X,c)\cup\mathcal{A}(X,c))$ with $q\in\langle W\rangle$ and $q\notin\langle W'\rangle$ for any $W'\subsetneq W$ and $W\ne Z$. Since $q\in(\langle Z\rangle\cap\langle W\rangle\setminus\langle Z\cap W\rangle)$, we have $h^1(\mathcal{I}_{Z\cup W}(1))>0$.

(a) Fix a decomposition $E\sqcup F$ with $\prod_{i\in E}(n_i+1)>b$ and $\prod_{i\in F}(n_i+1)>b$. In this step we prove that $c=b$ and that $h^1(\mathcal{I}_W(E))=0$ and $H^0(\mathcal{I}_W(E))=H^0(\mathcal{I}_Z(E))$; note that this would also imply that $\pi_{E|G}:G\to X_E$ is an embedding. Since $c\le b<\prod_{i\in E}(n_i+1)$, there is $D\in|\mathcal{O}_X(E)|$ with $D\supset W$. Thus $\mathrm{Res}_D(Z\cup W)=\mathrm{Res}_D(Z)\subseteq Z$. Since $\prod_{i\in E}(n_i+1)>b$ and Z is general in $\mathcal{Z}(X,b)$ we have $h^1(\mathcal{I}_Z(F))=0$ and hence $h^1(\mathcal{I}_{\mathrm{Res}_D(Z)}(F))=0$. The residual exact sequence (1) of $\mathcal{I}_{Z\cup W}(1)$ with respect D gives $h^1(D,\mathcal{I}_{(Z\cup W)\cap D,D}(E))>0$.

(a1) Assume $(Z\cup W)\cap D\ne Z\cup W$. Since $W\subset D$, we have $Z':=Z\cap D\subsetneq Z$. Since $\mathrm{Res}_D(W)=\emptyset$ and $h^1(X,\mathcal{I}_{\mathrm{Res}_D(Z\cup W)}\otimes\mathcal{O}_X(F))=0$, the residual sequence of $\mathcal{I}_{Z\cup W}(1)$ with respect to D gives $\langle Z\rangle\cap\langle W\rangle=\langle Z'\rangle\cap\langle W\rangle$. Thus $q\in\langle Z'\rangle$, a contradiction.

(a2) Assume $(Z\cup W)\cap D=Z\cup W$, i.e. $Z\cup W\subset D$. By step (a1) we may assume that this is true for all $D\in|\mathcal{I}_W(E)|$. Since $\deg(W)\le\deg(Z)$ and $h^1(\mathcal{I}_Z(E))=0$, we get $\deg(W)=b$, $h^1(\mathcal{I}_W(E))=0$ (and in particular $\pi_{E|W}$ is an embedding) and that $H^0(\mathcal{I}_Z(E))=H^0(\mathcal{I}_W(E))$.

(a3) Exchanging the role of E and F we also get $h^1(\mathcal{I}_W(F))=0$, that $\pi_{F|W}$ is an embedding and that $H^0(\mathcal{I}_W(F))=H^0(\mathcal{I}_Z(F))$.

(b) Take E and F as in step (a). Since Z is general in $\mathcal{Z}(X,b)$, the scheme $\pi_E(Z)$ is general in $\mathcal{Z}(X_E,b)$. Since $H^0(\mathcal{I}_Z(E))=H^0(\mathcal{I}_W(E))$, Lemma 2.4 applied to X_E gives $\pi_E(W)\subseteq\pi_E(Z)$. Since $\pi_{E|W}$ is an embedding, we first get $\pi_E(W)=\pi_E(Z)$ and then that $W\in\mathcal{Z}(X,q)$ and $W\notin\mathcal{A}(X,q)$. This is sufficient to see that $r_X(q)>b$. By step (a3) we also get $\pi_F(W)=\pi_F(Z)$. Hence $\pi_i(W)=\pi_i(Z)$ for all $i\in\{1,\dots,s\}$. This is not enough to say that W has only finitely many possibilities (obviously W_{red} has only finitely many possibilities) and so to prove that $\dim\tau(X,b)=b(n+1)-2$ we need to work more. Fix again a general $q\in\tau(X,b)$ and assume that $\dim\tau(X,b)<b(n+1)-2$, i.e. assume that $\mathcal{Z}(X,b,q)$ is infinite. The set $\mathcal{Z}(X,b,q)$ is constructible and hence it makes sense to speak about the dimensions of the irreducible components of $\mathcal{Z}(X,b,q)$. Since $\dim\tau(X,b)<b(n+1)-2$, each of the irreducible components of $\mathcal{Z}(X,b,q)$ has positive dimension. Let Γ be the irreducible component of $\mathcal{Z}(X,b,q)$ containing Z. A general $U\in\Gamma$ may be considered as a general element of $\mathcal{Z}(X,b)$ and hence we may apply Lemma 2.4 for X_V and $u_V(U)$ and for X_W and $u_W(U)$. Since there are only finitely many sets W_{red}, $W\in\Gamma$, for a general $U\in\Gamma\setminus\{Z\}$ we have $U_{\mathrm{red}}=Z_{\mathrm{red}}$ and so $\deg(U\cap Z)=b-1$. Since $q\in(\langle Z\rangle\cap\langle U\rangle\setminus\langle Z\cap U\rangle)$, we get $\langle U\rangle=\langle Z\rangle$ and hence $U\subset\langle Z\rangle$, contradicting Lemma 2.4. □

References

[1] H. Abo and M. C. Brambilla, *On the dimensions of secant varieties of Segre-Veronese varieties*, Ann. Mat. Pura Appl., **192** (1) (2013), 61–92.

[2] H. Abo, G. Ottaviani, and C. Peterson, *Induction for secant varieties of Segre varieties*, Trans. Amer. Math. Soc., **361** (2) (2009), 767–792.

[3] B. Ådlandsvik, *Joins and higher secant varieties*, Math. Scand., **62** (1987), 213–222.

[4] T. Aladpoosh, and H. Haghighi, *On the dimension of higher secant varieties of Segre varieties* $\mathbb{P}^n \times \cdots \times \mathbb{P}^n$, J. Pure Appl. Algebra, **215** (5) (2011), 1040–1052.

[5] C. Araujo, A. Massarenti, and R. Rischter, *On nonsecant defectivity of Segre-Veronese varieties*, Preprint. arXiv:1611.01674.

[6] E. Arbarello, M. Cornalba, P. Griffiths, and J. Harris, *Geometry of Algebraic Curves, Vol. I*, Springer, Berlin - Heidelberg - New York, 1985.

[7] E. Ballico, *The b-secant variety of a smooth curve has a codimension* 1 *locally closed subset whose points have rank at least* $b+1$, Rivista Mat. Univ. Parma, **8** (2017), 345–351. arXiv: 1706.03633.

[8] E. Ballico and A. Bernardi, *Tensor ranks on tangent developable of Segre varieties*, Linear Multilinear Algebra, **61** (2013), 881–894.

[9] E. Ballico, A. Bernardi and L. Chiantini, *On the dimension of contact loci and the identifiability of tensors*, Ark. Mat. (to appear). arXiv: 1706.02746.

[10] E. Ballico, A. Bernardi, L. Chiantini and E. Guardo, *Bounds on the tensor rank*, Ann. Mat. Pura Appl., **197** (6) (2018), 1771–1785. arXiv: 1705.02299.

[11] M. Bolognesi and G. Pirola, *Osculating spaces and diophantine equations* (with an Appendix by P. Corvaja and U. Zannier), Math. Nachr., **284** (2011), 960–972.

[12] J. Buczyński and J.M. Landsberg, *On the third secant variety*, J. Algebraic Combin., **40** (2014), 475–502.

[13] M. V. Catalisano, A. V. Geramita, and A. Gimigliano, *Secant varieties of* $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ *(n-times) are not defective for* $n \geq 5$, J. Alg. Geom., **20** (2011), 295–327.

[14] W. Fulton and J. Hansen, *A connectedness theorem for projective varieties, with applications to intersections and singularities of mappings*, Ann. Math., **110** (1) (1979), 159–166.

[15] H. Kaji, *On the tangentially degenerate curves*, J. London Math. Soc., **33** (2) (1986), 430–440.

[16] H. Kaji, *On the tangentially degenerate curves, II*, Bull. Braz. Math. Soc., New Series, **45** (4) (2014), 745–752.

[17] J.M. Landsberg, *Tensors: Geometry and Applications*, Graduate Studies in Mathematics, 128, Amer. Math. Soc. Providence, 2012.

[18] T. Lickteig, *Typical tensorial rank,* Linear Algebra Appl., **69** (1985), 95–120.

[19] B. Ullery, *On the normality of secant varieties*, Adv. Math., **288** (2016), 631–647.

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