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# A ZERO-FREE REGION FOR THE FRACTIONAL DERIVATIVES OF THE RIEMANN ZETA FUNCTION

Ricky E. Farr, Sebastian Pauli, and Filip Saidak

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Abstract. For any $\alpha \in \mathbb{R}$, we denote by $D_s^\alpha[\zeta(s)]$ the $\alpha$-th Grünwald-Letnikov fractional derivative of the Riemann zeta function $\zeta(s)$. For these derivatives we show:

$$D_s^\alpha[\zeta(s)] \neq 0$$

inside the region $|s-1| < 1$. This result, the first of its kind, is proved by a careful analysis of integrals involving Bernoulli polynomials and bounds for fractional Stieltjes constants.

## 1. Introduction

The Riemann zeta function $\zeta(s)$ and its derivatives $\zeta^{(k)}(s)$ are given by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \text{and} \quad \zeta^{(k)}(s) := (-1)^k \sum_{n=2}^{\infty} \frac{(\log n)^k}{n^s},$$

for all $k \in \mathbb{N}$, everywhere in the half-plane $\Re(s) > 1$. Each of them can, by a process of analytic continuation, be extended to a meromorphic function with a single pole at $s = 1$.

In 2003, Skorokhodov [**13**] observed that discretely increasing $k$ moves the non-trivial zeros of $\zeta^{(k)}(s)$, in a one-to-one fashion, to the right. Investigating the zero-free regions of higher derivatives $\zeta^{(k)}(s)$, in [**2**] the authors proved this phenomenon for sufficiently large $k$, and have discovered that, for integers $k \geq 0$, all of these derivatives have *identical* zero counts in the region $\Re(s) > 1/2$. Unfortunately, due to increasing densities of the zeros of derivatives of $\zeta(s)$ in the vertical direction, this simple bijective idea is very difficult to state quantitatively (e.g. in terms of counting functions such as $N_k(T) := \sum_{\zeta^{(k)}(\rho)=0, \Im(\rho) \leq T} 1$). However, the existence of a visible "flow" of the zeros suggests that perhaps an indepth study of the *fractional* derivatives could provide a missing link needed to establish this fascinating but currently little-understood property. Despite an incredible amount of research concerning the theory of $\zeta(s)$ and its $k$-th derivatives (for integer values of $k$), the problems of *fractional* derivatives have been largely neglected so far ([**10**], [**9**], [**3**] being a few rare exceptions).

In this paper we will not try to prove the audacious one-to-one conjecture stated above, but instead we will take a first step towards it by establishing a new general zero-free region for (arbitrary) fractional derivatives of $\zeta(s)$, a result that should be of an independent interest. In particular, we will show that no integral or fractional

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| Function | Zero | Distance from 1 |
|---|---|---|
| $\zeta$ | $s = -2$ | $\lvert s-1\rvert = 3$ |
| $\zeta'$ | $s \approx -2.7173$ | $\lvert s-1\rvert < 3.7174$ |
| $D_s^{(1.4677)}\zeta(s)$ | $s \approx -1.5249 + 2.6383i$ | $\lvert s-1\rvert < 3.6519$ |
| $\zeta''$ | $s \approx -0.3551 + 3.5908i$ | $\lvert s-1\rvert < 3.838$ |

FIGURE 1. Zeros of selected derivatives of $\zeta$ close to $s = 1$.

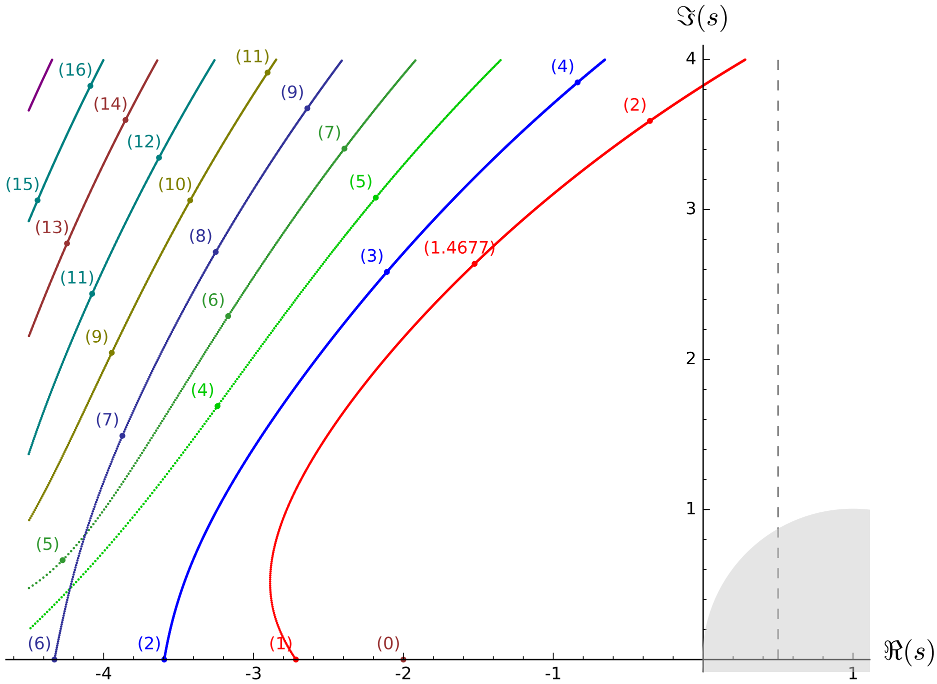


FIGURE 2. Zeros of integral and fractional derivatives $D_s^\alpha[\zeta(s)]$ of the Riemann zeta function $\zeta$ in the neighbourhood of our zero free region $|s-1| < 1$. Selected zeros of $D_s^\alpha[\zeta(s)]$ are denoted by ${}^{(\alpha)}\bullet$.

derivative of $\zeta(s)$ has a zero inside the disk $|s-1| < 1$. And while the result is not the sharpest possible, it is not too far removed from it. The Figure 1 above gives the list of closest zeros to the pole at $s = 1$, and the Figure 2 depicts the distribution (and the flow) of these zeros in the left half-plane. Here, as it was noted in [**5**], the same phenomenon of translation of zeros continues; however, the linear and periodic movement found in the right half-plane is deformed into curves that terminate in the "trivial" zeros of derivatives of $\zeta(s)$ found on the negative real axis.

The structure of the remainder of the paper is as follows. In Section 2 we begin with the definition of the Grünwald-Letnikov fractional derivatives and state some of their properties, then in Section 3 we recall some basic results concerning fractional

Stieltjes constants needed in our proof. We derive bounds for these constants and state a couple of other useful auxiliary results in Section 4. Finally, in Section 5, we prove our main result.

## 2. Fractional Derivatives

Fractional derivative operators are natural generalizations of the standard differentiation operator $D^\alpha$ to arbitrary (integer, rational, or complex) values of $\alpha$. We have found that, among the multitude of existing definitions of fractional derivatives, the reverse Grünwald-Letnikov derivative works best for situations dealing with $\zeta(s)$ and its derivatives. In fact, in [**6**] we have applied it successfully in a proof of a conjecture by Kreminski [**10**]. Here we follow suit: we write $\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}$, and for any $\alpha \in \mathbb{C}$, recall the definition of the so-called "reverse $\alpha^{th}$ Grünwald-Letnikov derivative" of a function $f(z)$ (see [**8**]):

$$D_z^\alpha [f(z)] = \lim_{h\to 0^+} \frac{\Delta_h^\alpha f(z)}{h^\alpha} = \lim_{h\to 0^+} \frac{(-1)^\alpha \sum\limits_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(z+kh)}{h^\alpha},$$

whenever the limit exists. Thus defined, $D_z^\alpha[f(z)]$ coincide with the standard derivatives for all $\alpha \in \mathbb{N}$. Furthermore, we have $D_z^0[f(z)] = f(z)$ and $D_z^\alpha\left[D_z^\beta[f(z)]\right] = D_z^{\alpha+\beta}[f(z)]$.

In [**11**] it was shown that for $z \in \mathbb{C}$ one has $D_z^\alpha[e^{-az}] = (-1)^\alpha a^\alpha e^{-az}$, and that $D_z^\alpha[1] = 0$. For the Riemann zeta function this implies that, if $\alpha > 0$ and $\Re(s) > 1$, then we can write

$$D_s^\alpha[\zeta(s)] = (-1)^\alpha \sum_{n=1}^{\infty} \frac{\log^\alpha(n+1)}{n^s}. \tag{1}$$

Note that the Grünwald-Letnikov derivative of the Riemann zeta function is defined for all real $\alpha > 0$, and $D_s^\alpha[\zeta(s)]$ is analytic in $s$; and what matters to us most is that analytic continuation will yield the Grünwald-Letnikov derivative for all $s \in \mathbb{C}$ with $\Re(s) \leq 1$.

## 3. Fractional Stieltjes Constants

We start by recalling some basics. First, note that $\zeta(s)$ can be extended to a meromorphic function with a simple pole at $s = 1$, with residue 1, and has a Laurent series expansion:

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n, \tag{2}$$

where $\gamma_n$ are the Stieltjes constants [**14**]. The *fractional Stieltjes constants* $\gamma_\alpha$, with $\alpha \in \mathbb{R}^{>0}$, were introduced by Kreminski [**10**] and can be defined as the coefficients of the Laurent expansion of the $\alpha$-th Grünwald-Letnikov fractional derivative of $\zeta(s) - 1$, for all $s \neq 1$:

$$D_s^\alpha[\zeta(s)] = (-1)^{-\alpha} \frac{\Gamma(\alpha+1)}{(s-1)^{\alpha+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}}{n!}(s-1)^n. \tag{3}$$

In [**6**] we have proved the following generalization of a result of Williams and Zhang [**15**]: Let $\lfloor x \rfloor$ denote the integer part of $x$. Then, for $\alpha > 0$ and $m \in \mathbb{N}$, we have

$$\gamma_\alpha = \sum_{k=1}^{m} \frac{\log^\alpha(k+1)}{k+1} - \frac{\log^{\alpha+1}(k+1)}{\alpha+1} - \frac{\log^\alpha(m+1)}{2(m+1)} + \int\limits_m^\infty P_1(x) f_\alpha'(x) dx, \tag{4}$$

where $P_1(x) = x - \lfloor x \rfloor - \frac{1}{2}$ and $f_\alpha(x) = \dfrac{\log^\alpha x + 1}{x+1}$. Integrating (4) by parts $m$ times yields

$$\begin{aligned}\int_m^\infty P_1(x) f_\alpha'(x) dx &= \sum_{j=1}^{v} \Big[ P_j(x) f_\alpha^{(j-1)}(x) \Big]_{x=m}^{\infty} + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx \\ &= -\sum_{j=1}^{v} P_j(m) f_\alpha^{(j-1)}(m) + (-1)^{v-1} \int\limits_m^\infty P_v(x) f_\alpha^{(v)}(x) dx, \end{aligned} \tag{5}$$

where for $k \in \mathbb{N}$, $P_k(x) = \frac{B_k(x - \lfloor x \rfloor)}{k!}$ is the $k^{th}$ periodic Bernoulli polynomial and $B_k$ is the $k^{th}$ Bernoulli number. These ideas can also be used to obtain an upper bound for the fractional Stieltjes constants: For any integer $n$, with $1 \le n < \alpha$, we have (see [**7**]):

$$|\gamma_\alpha| \le \frac{(3 + (-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!}. \tag{6}$$

The expression (3) for the fractional dertivatives of the Riemann zeta function will be the starting point of our proof of their zero-free regions. In order to establish the non-vanishing result, bounds on Stieltjes constants will be needed, plus a careful estimation of the behavior of the periodic Bernoulli polynomials $P_k(x)$, defined in (5). This is done in the next section.

## 4. Four Auxiliary Lemmas

**Lemma 4.1.** *Let $0 < \alpha \le 1$ and $f_\alpha(x) = \frac{\log^\alpha(x+1)}{x+1}$. Then* $\left| \int\limits_1^\infty P_3(x) f_\alpha'''(x) \right| < 0.013.$

**Proof.** Ostrowski [**12**] observed that, for odd integers $n > 1$, one always has: $|P_n(x)| < \frac{2}{(2\pi)^n}$. Combining this with the triangle inequality (and the change of variables for the integral), we are able to write:

$$\begin{aligned}\left| \int\limits_1^\infty P_3(x) f_\alpha'''(x) \right| &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} |s(4, i+1)(\alpha)_i| \int\limits_1^\infty \frac{\log^{\alpha-i}(x+1)}{(x+1)^4} dx \\ &< \frac{2}{(2\pi)^3} \sum_{i=0}^{3} \frac{|s(4, i+1)(\alpha)_i|}{3^{\alpha-i+1}} \int\limits_{3\log(2)}^{\infty} x^{\alpha-i} e^{-x} dx. \end{aligned} \tag{7}$$

In what follows, we will estimate each of the four summands on the right side of this inequality separately. We start with $i = 0$. Since $x^\alpha \le x$ in the interval

$[3\log(2), \infty)$, we can write

$$\frac{|s(4,1)(\alpha)_0|}{3^{\alpha+1}} \int_{3\log(2)}^{\infty} x^{\alpha} e^{-x} dx \leq \frac{6}{3^{\alpha+1}} \int_{3\log(2)}^{\infty} x e^{-x} dx \tag{8}$$

$$= \frac{1}{4} \frac{3\log(2)+1}{3^{\alpha}}. \tag{9}$$

For $i = 1$, in the interval $[3\log(2), \infty)$ we have $x^{\alpha-1} \leq 3^{\alpha-1}\log^{\alpha-1}(2)$, for all $\alpha \leq 1$; thus

$$\frac{|s(4,2)(\alpha)_1|}{3^{\alpha}} \int_{3\log(2)}^{\infty} x^{\alpha-1} e^{-x} dx \leq \frac{11\alpha}{3^{\alpha}} 3^{\alpha-1}\log^{\alpha-1}(2) \int_{3\log(2)}^{\infty} e^{-x} dx \tag{10}$$

$$\leq \frac{11\log^{\alpha-1}(2)}{24}. \tag{11}$$

Now, for the summand corresponding to $i = 2$ we have

$$\frac{|s(4,3)(\alpha)_2|}{3^{\alpha-1}} \int_{3\log(2)}^{\infty} x^{\alpha-2} e^{-x} dx = \frac{6|\alpha(\alpha-1)|}{3^{\alpha-1}} \int_{3\log(2)}^{\infty} x^{\alpha-2} e^{-x} dx \tag{12}$$

$$\leq \frac{3}{2} \frac{1}{3^{\alpha-1}} 3^{\alpha-2}\log^{\alpha-2}(2) \int_{3\log(2)}^{\infty} e^{-x} dx$$

$$= \frac{\log^{\alpha-2}(2)}{16}, \tag{13}$$

since for $0 < \alpha \leq 1$ we have $|\alpha(\alpha-1)| \leq \frac{1}{4}$ and for $x \in [3\log(2), \infty)$: $x^{\alpha-2} \leq 3^{\alpha-2}\log^{\alpha-2}(2)$.

Finally, for $i = 3$ we can write

$$\frac{|s(4,4)(\alpha)_3|}{3^{\alpha-2}} \int_{3\log(2)}^{\infty} x^{\alpha-3} e^{-x} dx = \frac{|\alpha(\alpha-1)(\alpha-2)|}{3^{\alpha-2}} \int_{3\log(2)}^{\infty} x^{\alpha-3} e^{-x} dx \tag{14}$$

$$\leq \frac{2\sqrt{3}}{9} \frac{3^{\alpha-3}\log^{\alpha-3}(2)}{3^{\alpha-2}} \int_{3\log(2)}^{\infty} e^{-x} dx$$

$$= \frac{\sqrt{3}\log^{\alpha-3}(2)}{108}, \tag{15}$$

since $|\alpha(\alpha-1)(\alpha-2)| \leq \frac{2}{9}\sqrt{3}$ for $\alpha \in (0,1]$ and $x^{\alpha-3} \leq 3^{\alpha-3}\log^{\alpha-3}(2)$ for $x \in [3\log(2), \infty)$.

Combining these four bounds, we conclude:

$$\left| \int_{1}^{\infty} P_3(x) f_{\alpha}'''(x) \right| < \frac{2}{(2\pi)^3} \left[ \frac{1}{4}\frac{3\log(2)+1}{3^{\alpha}} + \frac{11\log^{\alpha-1}(2)}{24} + \frac{\log^{\alpha-2}(2)}{16} + \frac{\sqrt{3}\log^{\alpha-3}(2)}{108} \right] < 0.013, \tag{16}$$

as desired. □

**Lemma 4.2.** *If* $0 < \alpha < 1$*, then* $|\gamma_\alpha| < 0.436$.

**Proof.** First, note that letting m = 1 in (4), we get

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} + \int_1^\infty P_1(x) f'_\alpha(x)dx.$$

Second, observe that from (5) we also know

$$\gamma_\alpha = \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - P_2(1)f'_\alpha(1) + P_3(1)f''_\alpha(1) + \int_1^\infty P_3(x) f'''_\alpha(x)dx.$$

Therefore, recalling that $P_2(1) = \frac{B_2}{2!} = \frac{1}{12}$ and $P_3(1) = \frac{B_3}{3!} = 0$ and also noting that $f'_\alpha(x) = \alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}$, we obtain

$$\begin{aligned}\gamma_\alpha &= \frac{\log^\alpha(2)}{4} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{1}{12}\left[\alpha\frac{\log^{\alpha-1}(2)}{4} - \frac{\log^\alpha(2)}{4}\right] + \int_1^\infty P_3(x) f'''_\alpha(x)dx \\ &= \frac{13\log^\alpha(2)}{48} - \frac{\log^{\alpha+1}(2)}{\alpha+1} - \frac{\alpha\log^{\alpha-1}(2)}{48} + \int_1^\infty P_3(x) f'''_\alpha(x)dx.\end{aligned}$$

Here, the maxima of the sum of the first three terms is attained when $\alpha = 0$. Combining this with the bound on the integral in Lemma 4.1, we get the wanted bound: $|\gamma_\alpha| \leq 0.436$. □

**Lemma 4.3.** *For all* $\alpha > 0$*, we have*

$$(i)\ \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < 0.348 \quad \text{and} \quad (ii)\ \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \leq 0.323.$$

**Proof.** Combining the bound for $|\gamma_\alpha|$ proved in Lemma 4.2 and the fact that $\Gamma(\alpha+1) \geq \Gamma(3/2) = \frac{\sqrt{2\pi}}{2}$, for $0 < \alpha \leq 1$, we deduce that $\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} < \frac{0.436}{\frac{\sqrt{2\pi}}{2}} < 0.348$ in the region $0 < \alpha \leq 1$.

Now, in the complementary region $\alpha > 1$, one can apply the bound (6), and for all natural numbers $n$ that satisfy $1 \leq n < \alpha$ one can compute

$$\begin{aligned}\frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} &\leq \frac{4}{(2\pi)^{n+1}(n+1)^{\alpha+1}}\frac{(2(n+1))!}{(n+1)!} \\ &\leq \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha+1}}\left(\frac{4(n+1)}{e}\right)^{n+1} \\ &= \frac{4\sqrt{2}}{(2\pi)^{n+1}(n+1)^{\alpha-n}}\left(\frac{4}{e}\right)^{n+1} \\ &\leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^{n+1} \leq 4\sqrt{2}\left(\frac{2}{\pi e}\right)^{2} \leq 0.311,\end{aligned}$$

which is an even sharper bound. Together, these two bounds prove (i) for all $\alpha > 0$.

Similarly, to justify (ii), note that since $\alpha+1>1$, the equation (6) with $n=1$ yields

$$\frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} \le \frac{4\Gamma(\alpha+2)4!}{(2\pi)^2 2^{\alpha+2}2!\Gamma(\alpha+1)} = \frac{12(\alpha+1)}{(2\pi)^2 2^\alpha}. \tag{17}$$

As is easy to check, the maximum of $g(\alpha)=\frac{\alpha+1}{2^\alpha}$ is attained at $\alpha=\frac{1}{\log(2)}-1$, and this immediately yields the result (ii). $\square$

We need one more technical lemma before we can prove our main theorem.

**Lemma 4.4.** *For all $\alpha>0$ and $n\in\mathbb{N}\cup\{0\}$,*

$$\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha} < \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2} < 1.036,$$

*where*

$$\alpha_1 = \frac{\sqrt{5\log^2(3)+4}}{2\log(3)} + \frac{1}{\log(3)} - \frac{3}{2}.$$

**Proof.** We proceed by induction on $n$. For $n=0$ we have

$$\frac{\Gamma(\alpha+3)}{\Gamma(\alpha+1)2!3^\alpha} = \frac{\alpha^2+3\alpha+2}{3^\alpha 2}.$$

The maximum of $g(\alpha)=\frac{\alpha^2+3\alpha+2}{3^\alpha 2}=\frac{(\alpha^2+3\alpha+2)e^{-\alpha\log(3)}}{2}$ is at $\alpha_1=\frac{\sqrt{5\log^2(3)+4}}{2\log(3)}+\frac{1}{\log(3)}-\frac{3}{2}$, with $g(\alpha_1)=1.0356$. Now, let us assume that, for all integers $j$ with $1\le j\le n$, we have

$$\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} \le \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.$$

We will show the assertion is true for $j=n+1$. Applying the induction hypothesis gives

$$\begin{aligned}
\frac{\Gamma(\alpha+j+3)}{\Gamma(\alpha+1)(j+2)!2^j(j+3)^\alpha} &= \frac{\Gamma(\alpha+n+4)}{\Gamma(\alpha+1)(n+3)!2^{n+1}(n+4)^\alpha}\\
&= \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^\alpha}\\
&\le \frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha \frac{\alpha+n+3}{n+3}\frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}.
\end{aligned} \tag{18}$$

Hence, all we need to show is that $\frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha\frac{\alpha+n+3}{n+3}\le 1$. However, notice that the function $g(\alpha)=\frac{1}{2}\left(\frac{n+3}{n+4}\right)^\alpha\frac{\alpha+n+3}{n+3}$ is positive for all $\alpha>0$; and taking the logarithmic derivative we get

$$\frac{g'(\alpha)}{g(\alpha)} = \log\left(\frac{n+3}{n+4}\right) + \frac{1}{\alpha+n+3} \le -\frac{1}{n+4} - \frac{1}{2}\left(\frac{1}{n+4}\right)^2 + \frac{1}{\alpha+n+3},$$

since (using Taylor expansion) we know that $\log(1-x)\le -x-\frac{1}{2}x^2$, in the range $0\le x<1$. Moreover, $\frac{1}{\alpha+n+3}\le\frac{1}{n+4}$, and since $g(\alpha)>0$, we can conclude that $g'(\alpha)<0$. Therefore $g(\alpha)$ is decreasing in the interval $[1,\infty)$, with the maximum at $g(1)=\frac{1}{2}$.

On the other hand, if $0 < \alpha < 1$, the maximum of $\left(\frac{n+3}{n+4}\right)^{\alpha}$ is attained at $\alpha = 0$. And since $\frac{\alpha+n+3}{n+3} < \frac{n+4}{n+3} = 1 + \frac{1}{n+3} \le \frac{4}{3}$, we have $g(\alpha) < \frac{1}{2}\frac{4}{3} = \frac{2}{3}$, for $\alpha \in (0,1)$. Combining these two results in (18), we deduce the bound for $j = n+1$. This completes the inductive proof. □

## 5. A Zero-Free Region

Now we are ready to prove our main result.

**Theorem 5.1.** *For all* $\alpha \ge 0$,

$$D_s^{\alpha}[\zeta(s)] \neq 0,$$

*in the region* $|s-1| < 1$.

For a discussion of the special case $\alpha = 0$, see Berndt [**1**].

**Proof.** We will deduce the result by showing $\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)} D_s^{\alpha}[\zeta(s)] \neq 0$, in the region $|s-1| < 1$. Employing the expression (3), we start by writing:

$$\begin{aligned}
\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| &= \left|1 + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_{\alpha+n}(s-1)^{\alpha+n+1}}{\Gamma(\alpha+1)n!}\right| \\
&\ge 1 - \frac{|\gamma_\alpha|}{\Gamma(\alpha+1)} - \frac{|\gamma_{\alpha+1}|}{\Gamma(\alpha+1)} - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!},
\end{aligned}$$

and then, applying the bound from Lemma 4.3, we obtain:

$$\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| > 1 - 0.492 - 0.323 - \sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}. \tag{19}$$

Now it suffices to focus on finding an upper bound for $\sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!}$. Using (6) gives:

$$\frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} \le \frac{4\Gamma(\alpha+n+1)(2(n+1))!}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}(n+1)^{\alpha+n+1}(n+1)!n!\Gamma(\alpha+1)},$$

while from Stirling's formula it follows that $\frac{(2n)!}{n!} \le \sqrt{2}\left(\frac{4n}{e}\right)^n$ for all integers $n \ge 1$. Therefore

$$\begin{aligned}
\sum_{n=2}^{\infty} \frac{|\gamma_{\alpha+n}|}{\Gamma(\alpha+1)n!} &\le \sum_{n=2}^{\infty} \frac{4\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha+n+1}n!\Gamma(\alpha+1)}\sqrt{2}\left(\frac{4(n+1)}{e}\right)^{n+1} \\
&= \sum_{n=2}^{\infty} \frac{4\sqrt{2}\Gamma(\alpha+n+1)}{(2\pi)^{n+1}(n+1)^{\alpha}n!\Gamma(\alpha+1)}\left(\frac{4}{e}\right)^{n+1} \\
&= 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n+3)}{\Gamma(\alpha+1)(n+2)!2^n(n+3)^{\alpha}}\left(\frac{4}{\pi e}\right)^n \\
&\le 4\sqrt{2}\left(\frac{2}{\pi e}\right)^3 \sum_{n=0}^{\infty} \frac{(\alpha_1+2)(\alpha_1+1)}{3^{\alpha_1}2}\left(\frac{4}{\pi e}\right)^n < 0.142,
\end{aligned}$$

by Lemma 4.4. Inserting this upper bound back into the expression (19), we obtain

$$\left|\frac{(s-1)^{\alpha+1}}{\Gamma(\alpha+1)}\zeta^{(\alpha)}(s)\right| \;>\; 1-0.492-0.323-0.142>0,$$

which implies that $D_s^{\alpha}[\zeta(s)] \neq 0$, for all $\alpha > 0$, in the region $|s-1| < 1$. □

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Ricky E. Farr
Department of Mathematics and Statistics
University of North Carolina
Greensboro, NC 27402, U.S.A.
refarr@uncg.edu

Sebastian Pauli
Department of Mathematics and Statistics
University of North Carolina
Greensboro, NC 27402, U.S.A.
s_pauli@uncg.edu

Filip Saidak
Department of Mathematics and Statistics
University of North Carolina
Greensboro, NC 27402, U.S.A.
f_saidak@uncg.edu