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ON THE CAUCHY INTEGRAL THEOREM AND POLISH SPACES

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Abstract. We prove that if a continuous function f in an open subset $U \subset \mathbb{C}$ is analytic in $U \setminus X$, where $X \subset U$ is a Polish space having characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$, then the complex line integral of f along the boundary of any triangle in U vanishes.

1. Introduction

In 1814 Augustin Louis Cauchy presented to the *French Academy of Sciences* the document *Mémoire sur les integrales définies*, containing his contributions to the development of the theory of complex functions [**2**]. In his memoirs, Cauchy proved that if a complex function f is analytic within a closed curve γ and also on the curve itself, then the complex line integral of f around that curve is equal to zero. In [**3, 4**], Edouard Goursat proved that

$$\int_{\gamma} f(z)dz = \mathbf{0},$$

that is the Cauchy Integral Theorem, this without assuming the continuity of the derivative $f'(z)$ on the closed region U bounded by the curve of integration γ. For a short summary of these works, refer to [**1**, p. 427–429]. In 1900 Eliakim Hastings Moore wrote and published his proof of the Cauchy Integral Theorem [**7**]. One year later, Alfred Pringsheim also wrote his version of the proof [**9**]. For more details about of the historical development of its proof see [**10**].

Motivated by the Cauchy Integral Theorem, one might ask: *What are the weakest set of the assumptions for what the complex line integral of an analytic function along a closed curve vanishes?*

In the classical case, when U is an open subset of $\mathbb{C}$, $\triangle(z_1, z_2, z_3) \subset U$ is a closed triangle with vertices z_1, z_2 and z_3 (throughout this manuscript, it is assumed that the points z_1, z_2 and z_3 are not collinear), and if f is an analytic function in U; then the Cauchy Integral Theorem implies that

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0},$$

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where $[z_1, z_2, z_3, z_1]$ is the polygonal closed curve parameterizing $\partial\triangle(z_1, z_2, z_3)$ the boundary of $\triangle(z_1, z_2, z_3)$. It is possible to replace the assumption on f to be analytic in the whole U by a weaker one. More precisely,

Lemma 1.1 ([**8**, Chapter V, Lemma 1.2, p. 143])**.** *If a function f is continuous in an open subset $U \subset \mathbb{C}$ and analytic in $U \setminus \{w\}$ for some point w of U, then*

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0},$$

for every triangle $\triangle(z_1, z_2, z_3) \subset U$.

We provide a more general form to the previous lemma when one removes a Polish space. In this manuscript, we shall consider the sets $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. A *Polish space* is a separable completely metrizable topological space. Given a Polish space X, the *Cantor-Bendixson derivative of* X is the set

$$X' := \{x \in X : x \text{ is a limit point of } X\}.$$

For every $k \in \mathbb{N}_0$, the *k-th Cantor-Bendixson derivative of* X is defined as $X^k := (X^{k-1})'$, where $X^0 = X$ (for details see [**5**, p. 34]). A countable Polish space is said to have *characteristic system* $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$ if $X^k \neq \emptyset$, $X^{k+1} = \emptyset$ and the cardinality of X^k is n. Any countable Polish space with characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$ is homeomorphic to the ordinal number $\omega^k + n$, where ω is the least infinite ordinal. In particular, every two countable Polish spaces with characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$ are homeomorphic. For further details, see [**6**].

Our main contribution states that the complex line integral also vanishes, when f is analytic in U, removing a Polish space X having characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$. More precisely,

Theorem 1.2. *If a function f is continuous in an open subset $U \subset \mathbb{C}$ and analytic in $U \setminus X$, where $X \subset U$ is a Polish space having characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$, then*

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0}, \tag{1.1}$$

for every triangle $\triangle(z_1, z_2, z_3) \subset U$. Hence f is analytic on U.

We shall prove Theorem 1.2 by induction to $k \in \mathbb{N}_0$.

2. Proof of Theorem 1.2

The following lemma is required for the proof of the theorem.

Lemma 2.1. *If a function f is continuous in an open subset $U \subset \mathbb{C}$ and analytic in $U \setminus X$, where $X \subset U$ is a Polish space having characteristic system $(0, n)$ for some $n \in \mathbb{N}$, then*

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0},$$

for every triangle $\triangle(z_1, z_2, z_3) \subset U$.

Proof of Lemma 2.1. We assume f is a continuous map in the open subset $U \subset \mathbb{C}$ and analytic in $U \setminus X$, where $X \subset U$ is a finite Polish space. We shall proceed by induction on the cardinality of X.

The Polish space X has only one point. In other words, X has characteristic system $(0, 1)$. From Lemma 1.1 we obtain that the complex line integral of f along the boundary of any triangle in U vanishes.

The Polish space X has cardinality n. It means, X has characteristic system $(0, n)$. Let $w_1, \dots, w_n$ be the points of X. We shall prove that

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0},$$

for any $\triangle(z_1, z_2, z_3) \subset U$. The induction hypothesis implies that the above complex line integral vanishes, when at least one of the points of X lies outside of $\triangle(z_1, z_2, z_3)$. We now consider the case when all points of X are in $\triangle(z_1, z_2, z_3)$. Without loss of generality, we take the points w_1 and w_2, and draw the straight line ℓ passing through them. Since the points z_1, z_2 and z_3 are not collinear, we can suppose without loss of generality that z_1 is not in ℓ. Thus, we draw the straight line ℓ' passing through z_1 and the middle point v of the straight line segment with endpoints w_1 and w_2, see Figure 1.

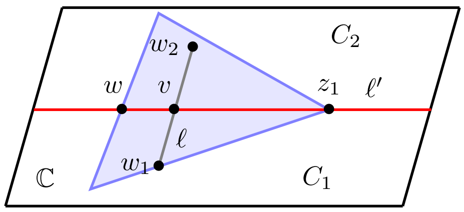


FIGURE 1. *Straight lines ℓ and ℓ'.*

Remark 2.2. The straight line ℓ' satisfies the following properties:

(a) It decomposes the complex plane $\mathbb{C}$ into two open connected subsets C_1 and C_2, such that the vertex z_2 belongs to one of these open connected, and the vertex z_3 belongs to the other open connected.

(b) If we choose a point $v_i \in C_i$, with $i \in \{1, 2\}$, then the straight line segment having endpoints v_1 and v_2, intersects the straight line ℓ' in a unique point. In particular, the points w_1 and w_2 do not belong to ℓ'.

The items **(a)** and **(b)** described above imply that ℓ' must intersect the straight line segment with endpoints z_2 and z_3, the unique common point is denoted by w. We then decompose $\triangle(z_1, z_2, z_3)$ into the triangles $\triangle(z_1, z_2, w)$ and $\triangle(z_1, w, z_3)$, which satisfy the following properties:

(1) They have the straight line segment L with endpoints z_1 and w in common. Moreover, $L \subset \ell'$.

(2) Each of the triangles $\triangle(z_1, z_2, w)$ and $\triangle(z_1, w, z_3)$ has less than n points of X.

Therefore, we have the following equality between complex line integral

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \int_{[z_1,z_2,w,z_1]} f(z)dz + \int_{[z_1,w,z_3,z_1]} f(z)dz. \tag{2.1}$$

By the property **(2)** and the induction hypothesis, one obtains

$$\int_{[z_1,z_2,w,z_1]} f(z)dz = \int_{[z_1,w,z_3,z_1]} f(z)dz = \mathbf{0}. \tag{2.2}$$

Now, substituting (2.2) into (2.1), we conclude

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0}.$$

□

We now provide the proof of our Theorem 1.2. Let f be a continuous map in the open subset $U \subset \mathbb{C}$ and analytic in $U \setminus X$, such that $X \subset U$ is a Polish space having characteristic system $(k, n) \in \mathbb{N}_0 \times \mathbb{N}$. We take a triangle $\triangle(z_1, z_2, z_3) \subset U$. We shall proceed by induction on the k-th Cantor-Bendixson derivative X^k of X.

The base case $k = 0$ is Lemma 2.1.

Fix $k \in \mathbb{N}_0$ and suppose that the complex line integral in (1.1) vanishes, for any $X \subset U$ a Polish space such that $X^k \neq \emptyset$ and $X^{k+1} = \emptyset$. We now consider $X \subset U$ a Polish space such that $X^{k+1} \neq \emptyset$ and $X^{k+2} = \emptyset$. In this instance, we will proceed by induction on the cardinality of the set X^{k+1}.

The set X^{k+1} has only one point. In other words, X has characteristic system $(k + 1, 1)$. Let w be the unique point of X^{k+1}. We must consider the following cases.

Case 1. The point w lies outside of $\triangle(z_1, z_2, z_3)$. There exists an open subset $V \subset U$, such that $\triangle(z_1, z_2, z_3) \subset V$ and the closure set $\overline{V}$ does not contain w. It implies that the intersection $\tilde{X} := \overline{V} \cap X$ is a Polish space such that $\tilde{X}^k \neq \emptyset$ and $\tilde{X}^{k+1} = \emptyset$. Without loss of generality, we may assume that $\tilde{X}$ is a subset of U. Applying the induction hypothesis on the open set U for the Polish space $\tilde{X}$, we obtain the expected value for the complex line integral.

Case 2. The point w is a vertex of $\triangle(z_1, z_2, z_3)$. Without loss of generality, we may assume that $w = z_1$. We now take the points $\boldsymbol{w}_2(t) = (1 - t)z_1 + tz_2$ and $\boldsymbol{w}_3(t) = (1 - t)z_1 + tz_3$ for each $t \in (0, 1)$ (see Figure 2-**(a)**.) In this stage, we obtain

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \int_{[z_1,\boldsymbol{w}_2(t),\boldsymbol{w}_3(t),z_1]} f(z)dz + \int_{[\boldsymbol{w}_3(t),\boldsymbol{w}_2(t),z_3,\boldsymbol{w}_3(t)]} f(z)dz + \int_{[\boldsymbol{w}_2(t),z_2,z_3,\boldsymbol{w}_2(t)]} f(z)dz.$$

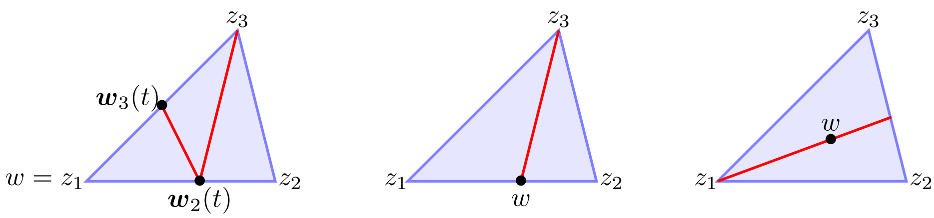
We note that the last two complex line integrals vanish by the previous *Case 1*. We now take the modulus of the complex line integral, and consider the upper bound

$$\left| \int\limits_{[z_1,z_2,z_3,z_1]} f(z)dz \right| = \left| \int\limits_{[z_1,\boldsymbol{w}_2(t),\boldsymbol{w}_3(t),z_1]} f(z)dz \right| \leq m\,t(|z_1-z_2|+|z_2-z_3|+|z_1-z_3|),$$

where $m = \max\{|f(w)| : w \in \triangle(z_1,z_2,z_3)\}$. As the above inequality is for each $t \in (0,1)$, we conclude

$$\int\limits_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0}.$$

Case 3. The point w is on an edge of $\triangle(z_1,z_2,z_3)$. Without loss of generality, we may assume that w is on the edge having endpoints z_1 and z_2. We consider the triangles $\triangle(w,z_2,z_3)$ and $\triangle(w,z_3,z_1)$ as shown in Figure 2-**(b)**.



(a) *w is a vertex.* **(b)** *w is on an edge.* **(c)** *w is in the interior.*

FIGURE 2. *Triangle $\triangle(z_1,z_2,z_3)$.*

We then obtain

$$\int\limits_{[z_1,z_2,z_3,z_1]} f(z)dz = \int\limits_{[z_1,w,z_3,z_1]} f(z)dz + \int\limits_{[w,z_2,z_3,w]} f(z)dz.$$

Both complex line integrals of the right side vanish because they are of the form described in the previous *Case 2*. This implies that

$$\int\limits_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0}.$$

Case 4. The point w is in the interior set of $\triangle(z_1,z_2,z_3)$. We consider the straight line segment such that one of its endpoints is z_1 and the other endpoints are on the opposite edge of this vertex. We note that this straight line segment passes through the point w as shown in Figure 2-**(c)**. We then decompose $\triangle(z_1,z_2,z_3)$ into two triangles, which are in the previous *Case 3*. Thus we conclude that the value of the complex line integral described in (1.1) vanishes.

For a fixed $n \in \mathbb{N}$ with $n \geq 2$ and for all $0 < l < n$, suppose that the complex line integral in (1.1) vanishes, for any $X \subset U$ a Polish space having characteristic system $(k+1,l)$.

The set X^{k+1} has cardinality n. It means, X has characteristic system $(k+1,n)$. We must consider the following cases.

Case 1. Suppose that at least a point w of X^{k+1} that lies outside of $\triangle(z_1, z_2, z_3)$. There exists an open subset $V \subset U$, such that $\triangle(z_1, z_2, z_3) \subset V$ and the closure set $\overline{V}$ does not contain the point w. This implies that the intersection $\tilde{X} := \overline{V} \cap X$ is a Polish space having characteristic system $(p, q) \in \mathbb{N}_0 \times \mathbb{N}$ such that $p \leq k+1$ and $q < n$. Without loss of generality, we may assume that $\tilde{X}$ is a subset of U. Applying the induction hypothesis on the open set U for the Polish space $\tilde{X}$, we obtain the expected value for the complex line integral.

Case 2. Otherwise, we now suppose that $X^{k+1} = \{w_1, \ldots, w_n\} \subset \triangle(z_1, z_2, z_3)$. We then take the straight line ℓ passing through two different points of X^{k+1}. We may assume that these points are w_1 and w_2. One of the vertices of $\triangle(z_1, z_2, z_3)$ is not on ℓ. Without loss of generality, we may assume that such vertex is z_1. We draw the straight line ℓ' passing through z_1 and the middle point v of the straight line segment with endpoints w_1 and w_2 as shown in Figure 3.

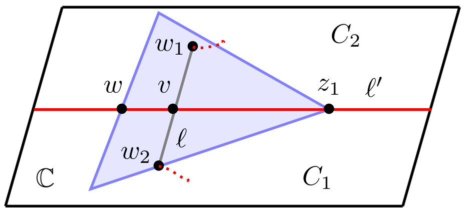


FIGURE 3. *Connected component C_1 and C_2.*

We note that ℓ' satisfies the properties described in Remark 2.2. According to the paragraphs **(a)** and **(b)** in this remark, we have that ℓ' must intersect the side of $\triangle(z_1, z_2, z_3)$ with endpoints z_2 and z_3, the unique common point is denoted by w. In this stage, we obtain

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \int_{[z_1,z_2,w,z_1]} f(z)dz + \int_{[z_1,w,z_3,z_1]} f(z)dz.$$

Both complex line integrals of the right side vanish because they are of the form described in the previous *Case 1.* This implies that

$$\int_{[z_1,z_2,z_3,z_1]} f(z)dz = \mathbf{0}.$$

□

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