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AN ESOTERIC IDENTITY WITH MANY PARAMETERS AND OTHER ELLIPTIC EXTENSIONS OF ELEMENTARY IDENTITIES

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Abstract. We provide elliptic extensions of elementary identities such as the sum of the first n odd or even numbers, the geometric sum and the sum of the first n cubes. Many such identities, and their q-analogues, are indefinite sums, and can be obtained from telescoping. So we used telescoping in our study to find elliptic extensions of these identities. In the course of our study, we obtained an identity with many parameters, which appears to be new even in the q-case. In addition, we recover some q-identities due to Warnaar.

1. Introduction

The geometric sum is a staple of high school algebra. It can be written as

$$\sum_{k=0}^{n-1} q^k = \frac{1-q^n}{1-q} =: [n]_q, \tag{1.1}$$

where $[n]_q$ denotes the q-number of n. This notation is justified because the limit as $q \to 1$ is

$$\sum_{k=0}^{n-1} 1 = n.$$

More generally, we can define $[z]_q := (1-q^z)/(1-q)$ for any complex z, where q is a complex number with $q \neq 0$, and observe that $\lim_{q\to 1}[z]_q = z$. Thus, we call $[z]_q$ the q-analogue of z.

The objective of this paper is to extend several classical and elementary identities to the so-called elliptic numbers—which are even more general than the q-numbers—defined in [**25**]. Rather surprisingly, these lead to new identities even in the q-case.

To be able to define an elliptic number, we need some notation. The **modified Jacobi theta function** of the complex number a with (fixed) nome p is defined as

$$\theta(a;p) := \prod_{j=0}^{\infty} (1-ap^j)(1-p^{j+1}/a),$$

where $a \neq 0$ and $|p| < 1$. When the nome $p = 0$, the modified theta function $\theta(a;p)$ reduces to $(1-a)$. We use the shorthand notation

$$\theta(a_1, a_2, \ldots, a_r; p) := \theta(a_1;p)\,\theta(a_2;p)\cdots\theta(a_r;p)\,.$$

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An elliptic analogue of a complex number z is defined by [**25**] as

$$[z]_{a,b;q,p} := \frac{\theta(q^z, aq^z, bq^2, a/b; p)}{\theta(q, aq, bq^{z+1}, aq^{z-1}/b; p)}. \tag{1.2a}$$

This has additional (complex) parameters a and b, in addition to the *base* q and nome p. Note that $[0]_{a,b;q,p} = 0$ and $[1]_{a,b;q,p} = 1$. Let the elliptic weight be defined by

$$W_{a,b;q,p}(k) := \frac{\theta(aq^{2k+1}, bq, bq^2, aq^{-1}/b, a/b; p)}{\theta(aq, bq^{k+1}, bq^{k+2}, aq^{k-1}/b, aq^k/b; p)} q^k, \tag{1.2b}$$

for any k. By the Weierstraß addition formula for theta functions (see (1.6b), below) we have

$$[x + y]_{a,b;q,p} = [x]_{a,b;q,p} + W_{a,b;q,p}(x)[y]_{aq^{2x},bq^x;q,p}. \tag{1.2c}$$

Note that if we set $p = 0$ and subsequently take $a = 0$ and then $b = 0$, the elliptic weight in (1.2b) reduces to q^k. In this case (1.2c) reduces to the recurrence relation

$$[x + y]_q = [x]_q + q^x[y]_q.$$

This, along with the initial conditions $[0]_q = 0$ and $[1]_q = 1$, is used to define the q-number for integers. Thus, the elliptic number in (1.2a) is indeed an extension of the q-number $[z]_q$ for any complex z.

The analogue of the geometric sum (1.1)—obtained by iterating (1.2c)—is as follows. (Here n is assumed to be a non-negative integer.)

$$1 + W_{a,b;q,p}(1) + W_{a,b;q,p}(2) + \cdots + W_{a,b;q,p}(n-1) = [n]_{a,b;q,p}. \tag{1.3}$$

The present work is in the context of elliptic hypergeometric series. Among the first results in this area were by Frenkel and Turaev [**10**]. Subsequently, Warnaar [**33**] obtained many results, and proposed several conjectures. Further summation and transformation formulas appear in [**7, 8, 13, 15, 17, 26, 32, 34, 36**]. In addition, there are results for multiple series over root systems; see [**18**]. Gasper and Rahman treat elliptic hypergeometric series identities in Chapter 11 of the second edition of [**12**].

By contrast, the results in this paper are elliptic extensions of more elementary identities. We study identities which appear often in enumeration, such as the geometric sum, the sum of the first n natural numbers, and the sum of the first odd numbers, or squares, or cubes. Many results of this nature (even in the $q = 1$ case) are examples of indefinite sums, and can be proved by a telescoping method going back to Euler (see Bhatnagar [**4**]). This motivates the study of elliptic extensions of these identities using this technique. In doing so, we naturally came across the following identity, which is somewhat esoteric, but appears to be new even in the q-case.

At this point, we would like to emphasize that the parameters in our identities should be chosen to avoid not-removable singularities and poles, so that the identities make sense.

Theorem 1.1. *For any non-negative integer n and complex numbers c, d, g and h, we have the following identity:*

$$
\begin{aligned}
&\sum_{k=0}^{n}\Bigg(\frac{\big[2(gk+c)(hk+d)\big]_{a,b;q,p}\,\big[2ghk+ch+dg\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}}{\big[2cd\big]_{a,b;q,p}\,\big[ch+dg\big]_{aq^{2(c-g)d},bq^{(c-g)d};q,p}}\\
&\qquad\times\prod_{j=0}^{k-1}\frac{\big[(gj+g+c)(hj+d)\big]_{aq^{2(gj-g+c)(hj+d)},bq^{(gj-g+c)(hj+d)};q,p}}{\big[(gj+g+c)(hj+d)\big]_{aq^{2(gj+g+c)(hj+2h+d)},bq^{(gj+g+c)(hj+2h+d)};q,p}}\\
&\qquad\times\prod_{j=0}^{k-1}W_{aq^{2(gj+c)(hj+h+d)},bq^{(gj+c)(hj+h+d)};q,p}\big(2ghj+2gh+ch+dg\big)^{-1}\Bigg)\\
&=\frac{\big[(gn+c)(hn+h+d)\big]_{a,b;q,p}\,\big[(g+c)d\big]_{aq^{2(c-g)d},bq^{(c-g)d};q,p}}{\big[2cd\big]_{a,b;q,p}\,\big[ch+dg\big]_{aq^{2(c-g)d},bq^{(c-g)d};q,p}}\\
&\qquad\times\prod_{j=1}^{n}\frac{\big[(gj+g+c)(hj+d)\big]_{aq^{2(gj-g+c)(hj+d)},bq^{(gj-g+c)(hj+d)};q,p}}{\big[(gj+c)(hj-h+d)\big]_{aq^{2(gj+c)(hj+h+d)},bq^{(gj+c)(hj+h+d)};q,p}}\\
&\qquad\times\prod_{j=1}^{n}W_{aq^{2(gj-g+c)(hj+d)},bq^{(gj-g+c)(hj+d)};q,p}\big(2ghj+ch+dg\big)^{-1}\\
&\qquad-\frac{\big[(c-g)d\big]_{a,b;q,p}\,\big[c(d-h)\big]_{aq^{2c(h+d)},bq^{c(h+d)};q,p}}{\big[2cd\big]_{a,b;q,p}\,\big[ch+dg\big]_{aq^{2(c-g)d},bq^{(c-g)d};q,p}}\,W_{aq^{2(c-g)d},bq^{(c-g)d};q,p}\big(ch+dg\big).
\end{aligned}
\tag{1.4}
$$

The "hypergeometric version" of (1.4) is given by

$$
\begin{aligned}
&\sum_{k=0}^{n}\frac{(gk+c)(hk+d)(2ghk+ch+dg)}{cd(ch+dg)}\\
&=\frac{(gn+c)(hn+h+d)(gn+g+c)(hn+d)}{2cd(ch+dg)}-\frac{(d-h)(c-g)}{2(ch+dg)}.
\end{aligned}
$$

Note that this extends the well-known formula for the sum of the first n cubes. Multiply both sides by $cd(ch+dg)/2$ and then take $c=d=0$, and $h=g=1$ to obtain

$$\sum_{k=0}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2.$$

This is indeed an elementary identity, but its extension given in (1.4) involves some rather unusual factors. For example, the product

$$\prod_{j=0}^{k-1}\big[(gj+g+c)(hj+d)\big]_{aq^{2(gj-g+c)(hj+d)},bq^{(gj-g+c)(hj+d)};q,p}$$

appearing with index k in the sum. The associated q-product (obtained by first letting $p\to 0$, followed by $a\to 0$ and $b\to 0$)

$$t(k):=\prod_{j=0}^{k-1}\big[(gj+g+c)(hj+d)\big]_q$$

is rather unusual as it is not a q-hypergeometric term. In particular, the ratio $t(k+1)/t(k)$ of this product, that is, $[(gk+g+c)(hk+d)]_q = (1-q^{(gk+g+c)(hk+d)})/(1-q)$, is not a rational function in q^k; it is a rational function in q^{k^2} and q^k, and contains quadratic powers of q.

Nevertheless, (1.4) contains various extensions of well-known elementary identities. The following identities appear as special cases.

$$\sum_{k=1}^{n} q^{n-k}\frac{[2k]_q}{[2]_q} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q; \tag{1.5a}$$

$$\sum_{k=1}^{n} q^{n^2-k^2+n-k}\frac{[2k^2]_q[2k]_q}{[2]_q^2} = \left(\frac{[n(n+1)]_q}{[2]_q}\right)^2. \tag{1.5b}$$

Here we have used the notation

$$\begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q = \frac{[n]_q[n+1]_q}{[2]_q}.$$

The first of these is a q-analogue of the sum of the first n natural numbers; the second is a q-analogue of the sum of the first n cubes, which is equivalent to a formula of Cigler [**9**, Theorem 1, $q \mapsto q^2$].

We now provide some background information and list some notation used in this paper.

Background information.

(1) Two important properties of the modified theta function are [**12**, (11.2.42)]

$$\theta(a;p) = \theta(p/a;p) = -a\theta(1/a;p)\,, \tag{1.6a}$$

and [**37**, p. 451, Example 5]

$$\theta(xy, x/y, uv, u/v; p) - \theta(xv, x/v, uy, u/y; p) = \frac{u}{y}\,\theta(yv, y/v, xu, x/u; p)\,. \tag{1.6b}$$

This last formula is called the Weierstraß addition formula. It is used extensively in this paper.

(2) The following general theorem serves as a justification of referring to $[z]_{a,b;q,p}$, defined in (1.2a), as an "elliptic number".

Proposition 1.2 ([**16**, Theorem 1.3.3])**.** *Let $g(x)$ be an elliptic function, that is, a doubly periodic meromorphic function of the complex variable x. Then $g(x)$ is of the form:*

$$g(x) = \frac{\theta(a_1q^x, a_2q^x, \ldots, a_rq^x; p)}{\theta(b_1q^x, b_2q^x, \ldots, b_rq^x; p)}c,$$

where c is a constant, and

$$a_1a_2\cdots a_r = b_1b_2\cdots b_r.$$

This last condition is the *elliptic balancing condition.* If we write $q = e^{2\pi i\sigma}$, $p = e^{2\pi i\tau}$, with complex σ, τ, then $g(x)$ is indeed doubly periodic in x with periods σ^{-1} and $\tau\sigma^{-1}$.

(3) Using Proposition 1.2, it is easy to see that the elliptic number $[z]_{a,b;q,p}$ is elliptic in z, and also elliptic in $\log_q a$ and $\log_q b$.

(4) Similarly, the elliptic weight function $W_{a,b;q,p}(k)$ is elliptic in $\log_q a$, $\log_q b$ and k (regarded as a complex variable).

(5) The following useful properties readily follow from the definitions.

(i) For any k and l, $W_{a,b;q,p}(k+l) = W_{a,b;q,p}(k)W_{aq^{2k},bq^k;q,p}(l)$.

(ii) $W_{a,b;q,p}(0) = 1$, and for any k, $W_{a,b;q,p}(-k) = W_{aq^{-2k},bq^{-k};q,p}(k)^{-1}$.

(iii) For any x,

$$\begin{aligned}[-x]_{a,b;q,p} &= -W_{a,b;q,p}(-x)[x]_{aq^{-2x},bq^{-x};q,p}\\ &= -W_{aq^{-2x},bq^{-x};q,p}(x)^{-1}[x]_{aq^{-2x},bq^{-x};q,p}.\end{aligned}$$

(iv) For any x and y, $[xy]_{a,b;q,p} = [x]_{a,b;q,p}[y]_{a,bq^{1-x};q^x,p}$.

(v) For any r, x and y,

$$\begin{aligned}&[x]_{a,b;q,p}[y]_{aq^{2r+2x-2y},bq^{r+x-y};q,p} - [x+r]_{a,b;q,p}[y-r]_{aq^{2r+2x-2y},bq^{r+x-y};q,p}\\ &\qquad = [r+x-y]_{a,b;q,p}[r]_{aq^{2x},bq^x;q,p}W_{aq^{2r+2x-2y},bq^{r+x-y};q,p}(y-r).\end{aligned} \tag{1.7}$$

The property (1.7) is a consequence of the Weierstraß addition formula in (1.6b).

(6) Elliptic weights (perhaps different from the ones considered here) have appeared in combinatorial contexts, in the work of Schlosser, Yoo and others [**1, 2, 3, 5, 6, 14, 20, 21, 22, 24, 25, 27, 28, 29, 30, 31**].

(7) In §3, we require the notation of q-rising factorials and their elliptic analogues. We define the *q-shifted factorials*, for $k = 0, 1, 2, \ldots$, as

$$(a;q)_k := \prod_{j=0}^{k-1} \left(1 - aq^j\right),$$

and for $|q| < 1$,

$$(a;q)_\infty := \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

The parameter q is called the *base*. With this definition, we can write the modified Jacobi theta function as

$$\theta(a;p) = (a;p)_\infty (p/a;p)_\infty,$$

where $a \neq 0$ and $|p| < 1$. We define the *q,p-shifted factorials* (or *theta shifted factorials*), for k an integer, as

$$(a;q,p)_k := \prod_{j=0}^{k-1} \theta\left(aq^j;p\right).$$

When the nome $p = 0$, $(a;q,p)_k$ reduces to $(a;q)_k$.
We use the shorthand notations

$$\begin{aligned}(a_1, a_2, \ldots, a_r; q,p)_k &:= (a_1;q,p)_k (a_2;q,p)_k \cdots (a_r;q,p)_k,\\ (a_1, a_2, \ldots, a_r; q)_k &:= (a_1;q)_k (a_2;q)_k \cdots (a_r;q)_k.\end{aligned}$$

(8) Most of the proofs of the theorems in this paper use the following technique, explained in detail in [**4**, Theorem 3.3].

Lemma 1.3 (Euler's telescoping lemma). *Let u_k, v_k and t_k be three sequences, such that*

$$t_k = u_k - v_k.$$

Then we have:

$$\sum_{k=0}^{n} \frac{t_k}{t_0} \frac{u_0 u_1 \cdots u_{k-1}}{v_1 v_2 \cdots v_k} = \frac{u_0}{t_0} \left(\frac{u_1 u_2 \cdots u_n}{v_1 v_2 \cdots v_n} - \frac{v_0}{u_0} \right), \tag{1.8}$$

provided none of the denominators in (1.8) *are zero.*

Some important specializations of the elliptic numbers and weights. It is helpful to explicitly write out some important special cases of the elliptic numbers and the elliptic weights. These cases correspond to $p = 0$ (the "$a, b; q$-case"); $p = 0$ and $b \to 0$ (the "$a; q$-case"); and, $p = 0$ and $a \to 0$ (the "$(b; q)$-case").

The three special cases of the elliptic numbers are

$$[z]_{a,b;q} = \frac{(1-q^z)(1-aq^z)(1-bq^2)(1-a/b)}{(1-q)(1-aq)(1-bq^{z+1})(1-aq^{z-1}/b)}; \tag{1.9a}$$

$$[z]_{a;q} = \frac{(1-q^z)(1-aq^z)}{(1-q)(1-aq)} q^{1-z}; \tag{1.9b}$$

$$[z]_{(b;q)} = \frac{(1-q^z)(1-bq^2)}{(1-q)(1-bq^{z+1})}, \tag{1.9c}$$

and called $a, b; q$-numbers, $a; q$-numbers, and $(b; q)$-numbers, respectively. We place parentheses in "$(b; q)$-numbers" but none in "$a; q$-numbers", to avoid confusion between the two special cases. This follows the notation used in [**23**].

The corresponding special cases for the elliptic weight $W_{a,b;q,p}(k)$ are as follows:

$$W_{a,b;q}(k) = \frac{(1-aq^{2k+1})(1-bq)(1-bq^2)(1-aq^{-1}/b)(1-a/b)}{(1-aq)(1-bq^{k+1})(1-bq^{k+2})(1-aq^{k-1}/b)(1-aq^k/b)} q^k; \tag{1.10a}$$

$$W_{a;q}(k) = \frac{(1-aq^{2k+1})}{(1-aq)} q^{-k}; \tag{1.10b}$$

$$W_{(b;q)}(k) = \frac{(1-bq)(1-bq^2)}{(1-bq^{k+1})(1-bq^{k+2})} q^k. \tag{1.10c}$$

Remark 1.4. The are many feasible elliptic extensions of z. There are already several q-extensions of z. For example, other than (1.1), which appears frequently in combinatorial contexts, the following symmetric extension is used in the context of quantum groups:

$$\langle z \rangle_q := \frac{q^z - q^{-z}}{q - q^{-1}}.$$

Both of these are contained in $[z]_{a,b;q,p}$. In (1.9b), letting $a \to \infty$ gives $[z]_q$, while taking $a = -1$ gives $\langle z \rangle_q$.

This paper is organized as follows. In Section 2, we use Euler's telescoping lemma to find elliptic extensions of three elementary identities and discuss some interesting special cases. In Section 3, we consider elliptic extensions of several elementary identities that are obtained in an analogous way to the q-identities previously obtained by one of us in [**19**]. Finally, in Section 4, we give the proof of Theorem 1.1 (achieved by combining Lemma 1.3 with the difference equation (1.7)), and explicitly state a few noteworthy special cases.

2. Elementary Examples

The purpose of this section is to extend three elementary identities to corresponding identities containing elliptic numbers. For each of these elliptic identities, we give some special cases for illustration. The three identities are:

$$\sum_{k=0}^{n-1}(2k+1) = n^2; \tag{2.1a}$$

$$\sum_{k=1}^{n} k(k+1)\cdots(k+m-1) = \frac{1}{m+1}\left(n(n+1)\cdots(n+m)\right); \tag{2.1b}$$

$$\sum_{k=1}^{n} \frac{1}{k(k+1)\cdots(k+m)} = \frac{1}{m}\left(\frac{1}{m!} - \frac{1}{(n+1)(n+2)\cdots(n+m)}\right), \tag{2.1c}$$

where $m = 1, 2, 3, \ldots$.

First, we give an elliptic extension of the sum of the first n odd integers.

Theorem 2.1. *For n a non-negative integer, we have*

$$\sum_{k=0}^{n} W_{a,b;q,p}(k)\Big([k+1]_{a,b;q,p}[2]_{aq^{2k},bq^k;q,p} - 1\Big)$$
$$= W_{a,b;q,p}(1)[n+1]_{a,b;q,p}[n+1]_{aq^2,bq;q,p}. \tag{2.2}$$

Proof. We apply Lemma 1.3 and take

$$u_k = [k+1]_{a,b;q,p}[k+1]_{aq^2,bq;q,p};$$
$$v_k = u_{k-1} = [k]_{a,b;q,p}[k]_{aq^2,bq;q,p} = [k+1]_{a,b;q,p}[k-1]_{aq^2,bq;q,p} + W_{aq^2,bq;q,p}(k-1).$$

The last equality follows from the $(x, y, r) \mapsto (k, k, 1)$ case of equation (1.7). Thus
$t_k = u_k - v_k = W_{aq^2,bq;q,p}(k-1)\Big([k+1]_{a,b;q,p}[2]_{aq^{2k},bq^k;q,p} - 1\Big)$
and $\quad t_0 = u_0 - v_0 = 1$.

We thus obtain (1.8) with these choices of u_k, v_k and t_k. Multiplication of both sides of the identity by $W_{a,b;q,p}(1)$ gives the result. □

Remark 2.2. The elliptic analogue of n, namely, $[n]_{a,b;q,p}$ contains extensions of n^2 and of $\binom{n+1}{2}$, besides other extensions. Take $z = n$ in (1.9b), the $a;q$-number of n. For $a \to \infty$ this reduces to $[n]_q$, for $a = 1$ to $([n]_q)^2q^{1-n}$, and for $a = q$ to $q^{1-n}[n]_q[n+1]_q/[2]_q$. That is, the telescoping sum over odd elliptic numbers also extends a sum over odd squares, and to a sum over binomial coefficients. The examples in this section illustrate some of the possibilities to obtain interesting identities by specialization.

Special cases of (2.2)**.**

(1) Three immediate specializations of (2.2) are as follows.

(i) For the $a, b; q$-analogue, take $p = 0$.

$$\sum_{k=0}^{n} W_{a,b;q}(k)([k+1]_{a,b;q}[2]_{aq^{2k},bq^k;q} - 1) = W_{a,b;q}(1)[n+1]_{a,b;q}[n+1]_{aq^2,bq;q}. \tag{2.3}$$

This identity has two parameters, a and b, in addition to the base q.

(ii) For the $a;q$-analogue, take $b \to 0$ or $b \to \infty$ in (2.3). This gives

$$\sum_{k=0}^{n} W_{a;q}(k)([k+1]_{a;q}[2]_{aq^{2k};q} - 1) = W_{a;q}(1)[n+1]_{a;q}[n+1]_{aq^2;q}. \tag{2.4}$$

(iii) For the $(b;q)$-analogue, take $a \to 0$ or $a \to \infty$ in (2.3). This gives

$$\sum_{k=0}^{n} W_{(b;q)}(k)([k+1]_{(b;q)}[2]_{(bq^k;q)} - 1) = W_{(b;q)}(1)[n+1]_{(b;q)}[n+1]_{(bq;q)}. \tag{2.5}$$

(2) We further specialize a and b to obtain two new q-analogues of (2.1a).
   (i) Take $a \to \infty$ in (2.4), or $b \to 0$ in (2.5), to get

$$\sum_{k=0}^{n} q^{k-1}([2]_q[k+1]_q - 1) = [n+1]_q^2.$$

   (ii) When $a \to 0$ in (2.4), or $b \to \infty$ in (2.5), to obtain

$$\sum_{k=0}^{n} q^{2n-2k}([2]_q[k+1]_q - q^{k+1}) = [n+1]_q^2.$$

(3) Take $a \to 1$ in (2.4), respectively, $b \to 1$ in (2.5), to obtain the following pair of identities:

$$\sum_{k=0}^{n} q^{2n-2k}\Big([2]_q[k+1]_q^2[2k+2]_q - q^{k+1}[2k+1]_q\Big) = [n+1]_q^3[n+3]_q;$$

$$\sum_{k=0}^{n} \frac{q^{k-1}}{[k+1]_q[k+2]_q}\left(\frac{[k+1]_q[2]_q^2}{[k+3]_q} - 1\right) = \frac{[n+1]_q^2}{[n+2]_q[n+3]_q}.$$

(4) Next, take $a \to q$ in (2.4), respectively, $b \to q$ in (2.5), to obtain the following pair of identities:

$$\sum_{k=0}^{n} q^{2n-2k}\Big([k+1]_q[k+2]_q[2k+3]_q - q^{k+1}[2k+2]_q\Big) = \frac{[n+1]_q^2[n+2]_q[n+4]_q}{[2]_q};$$

$$\sum_{k=0}^{n} \frac{q^{k-1}}{[k+2]_q[k+3]_q}\left(\frac{[k+1]_q[2]_q[3]_q}{[k+4]_q} - 1\right) = \frac{[n+1]_q^2}{[n+3]_q[n+4]_q}.$$

Next, we give an elliptic extension of (2.1b).

Theorem 2.3. *For n, m non-negative integers, we have*

$$\begin{aligned}\sum_{k=0}^{n} &W_{a,b;q,p}(k)[m+1]_{aq^{2k},bq^k;q,p}\Big([k+1]_{a,b;q,p}[k+2]_{a,b;q,p}\dots[k+m]_{a,b;q,p}\Big) \\ &= [n+1]_{a,b;q,p}[n+2]_{a,b;q,p}\dots[n+m+1]_{a,b;q,p}.\end{aligned} \tag{2.6}$$

Proof. We apply Lemma 1.3 and take

$$u_k = [k+1]_{a,b;q,p}[k+2]_{a,b;q,p}\dots[k+m+1]_{a,b;q,p};$$
$$v_k = u_{k-1} = [k]_{a,b;q,p}[k+1]_{a,b;q,p}\dots[k+m]_{a,b;q,p},$$

so that,

$$t_k = W_{a,b;q,p}(k)[m+1]_{aq^{2k},bq^k;q,p}\Big([k+1]_{a,b;q,p}[k+2]_{a,b;q,p}\dots[k+m]_{a,b;q,p}\Big).$$

Here, (1.6b) is used for computing t_k according to $t_k = u_k - v_k$. With these substitutions, we have (1.8) which immediately gives us (2.6). □

We take $m=1$ and shift the index $k \mapsto k-1$ and replace n by $n-1$ in (2.6), to get the elliptic analogue of the sum of first n even integers.

$$\sum_{k=1}^{n} W_{a,b;q,p}(k-1)[2]_{aq^{2k-2},bq^{k-1};q,p}[k]_{a,b;q,p} = [n]_{a,b;q,p}[n+1]_{a,b;q,p}. \tag{2.7}$$

This can be regarded to be an elliptic extension of the formula for the sum of the first n natural numbers:

$$1+2+3+\cdots+n = \frac{n(n+1)}{2}. \tag{2.8}$$

We list further special cases of the elliptic analogue of this elementary identity below.

Special cases of (2.7)**.**

(1) For the $a,b;q$-analogue, take $p=0$.

$$\sum_{k=1}^{n} W_{a,b;q}(k-1)[2]_{aq^{2k-2},bq^{k-1};q}[k]_{a,b;q} = [n]_{a,b;q}[n+1]_{a,b;q}. \tag{2.9}$$

(2) For the $a;q$-analogue, take $b\to 0$ or $b\to\infty$ in (2.9). This gives

$$\sum_{k=1}^{n} W_{a;q}(k-1)[2]_{aq^{2k-2};q}[k]_{a;q} = [n]_{a;q}[n+1]_{a;q}. \tag{2.10}$$

(3) For the $(b;q)$-analogue, take $a\to 0$ or $a\to\infty$ in (2.9). This gives

$$\sum_{k=1}^{n} W_{(b;q)}(k-1)[2]_{(bq^{k-1};q)}[k]_{(b;q)} = [n]_{(b;q)}[n+1]_{(b;q)}. \tag{2.11}$$

(4) Two q-analogues of (2.8)

(i) Take the limit $a\to\infty$ in (2.10), or $b\to 0$ in (2.11):

$$\sum_{k=1}^{n} q^{k-1}[k]_q = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q.$$

(ii) A q-analogue due to Warnaar [**35**, Eq. 2]: take the limit $a\to 0$ in (2.10), or $b\to\infty$ in (2.11).

$$\sum_{k=1}^{n} q^{2n-2k}[k]_q = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q.$$

(5) Some assorted q-analogues.

(i) A q-analogue of the formula for the sum of cubes due to Warnaar [**35**, Eq. 2]: take $a\to 1$ in (2.10).

$$\sum_{k=1}^{n} q^{2n-2k}\frac{[k]_q^2[2k]_q}{[2]_q} = \begin{bmatrix} n+1 \\ 2 \end{bmatrix}_q^2. \tag{2.12}$$

(ii) Take $b \to 1$ in (2.11).

$$\sum_{k=1}^{n} q^{k-1} \frac{[2]_q}{[k+1]_q[k+2]_q} = \frac{[n]_q}{[n+2]_q}.$$

(iii) Take $a \to q$ in (2.10).

$$\sum_{k=1}^{n} q^{2n-2k}[k]_q[k+1]_q[2k+1]_q = \frac{[n]_q[n+1]_q^2[n+2]_q}{[2]_q}.$$

(iv) Take $b \to q$ in (2.11).

$$\sum_{k=1}^{n} q^{k-1} \frac{[2]_q^2[k]_q}{[k+1]_q[k+2]_q[k+3]_q} = \frac{[n]_q[n+1]_q}{[n+2]_q[n+3]_q}.$$

Remark 2.4. There is another q-analogue of the sum of the first n cubes given by Garrett and Hummel [**11**, Eq. 2]. This can also be obtained by telescoping (take $u_k = (1-q^{k+2})$ and $v_k = -(1-q^k)$ in Lemma 1.3). Their elliptic extensions are immediate and are not included here. Further such q-analogues are obtained by Cigler [**9**], again, by telescoping.

Now, instead of taking $m = 1$ in (2.6), we take $m = 2$, shift the index $k \mapsto k-1$ in (2.6) and replace n by $n-1$. We then obtain

$$\sum_{k=1}^{n} W_{a,b;q,p}(k-1)[3]_{aq^{2k-2},bq^{k-1};q,p}[k]_{a,b;q,p}[k+1]_{a,b;q,p} = [n]_{a,b;q,p}[n+1]_{a,b;q,p}[n+2]_{a,b;q,p}. \tag{2.13}$$

Some special cases of (2.13)**.** We note some special cases of the $a;q$-special case of (2.13) (which is obtained by first letting $p \to 0$, followed by letting $b \to 0$ in (2.13)), i.e.,

$$\sum_{k=1}^{n} W_{a;q}(k-1)[3]_{aq^{2k-2};q}[k]_{a;q}[k+1]_{a;q} = [n]_{a;q}[n+1]_{a;q}[n+2]_{a;q}. \tag{2.14}$$

(1) Take $a \to 0$ in (2.14) to obtain

$$\sum_{k=1}^{n} q^{3n-3k}[k]_q[k+1]_q = \frac{[n]_q[n+1]_q[n+2]_q}{[3]_q}.$$

(2) Take $a \to 1$ in (2.14) to obtain

$$\sum_{k=1}^{n} q^{3n-3k}([k]_q[k+1]_q)^2[2k+1]_q = \frac{([n]_q[n+1]_q[n+2]_q)^2}{[3]_q}.$$

(3) Next, take $a \to q$ in (2.14) to obtain

$$\sum_{k=1}^{n} q^{3n-3k}[k]_q[k+1]_q^2[k+2]_q[2k+2]_q = \frac{[n]_q[n+1]_q^2[n+2]_q^2[n+3]_q}{[3]_q}.$$

(4) The following pair of identities is obtained by first replacing q by q^2 and then letting $a \to q$, respectively, $a \to q^{-1}$:

$$\sum_{k=1}^{n} q^{6n-6k}[2k]_q[2k+1]_q[2k+2]_q[2k+3]_q[4k+3]_q$$
$$= \frac{[2n]_q[2n+1]_q[2n+2]_q[2n+3]_q[2n+4]_q[2n+5]_q}{[6]_q}.$$

$$\sum_{k=1}^{n} q^{6n-6k}[2k-1]_q[2k]_q[2k+1]_q[2k+2]_q[4k+1]_q$$
$$= \frac{[2n-1]_q[2n]_q[2n+1]_q[2n+2]_q[2n+3]_q[2n+5]_q}{[6]_q}.$$

Finally, before closing this section, we note the elliptic extension of (2.1c).

Theorem 2.5. *For n, m non-negative integers, we have,*

$$\sum_{k=1}^{n} \frac{W_{a,b;q,p}(k)\,[m]_{aq^{2k},bq^k;q,p}}{[k]_{a,b;q,p}[k+1]_{a,b;q,p}\dots[k+m]_{a,b;q,p}}$$
$$= \left(\frac{1}{[m]_{a,b;q,p}!} - \frac{1}{[n+1]_{a,b;q,p}[n+2]_{a,b;q,p}\dots[n+m]_{a,b;q,p}}\right), \tag{2.15}$$

where $[m]_{a,b;q,p}! := [m]_{a,b;q,p}[m-1]_{a,b;q,p}\cdots[1]_{a,b;q,p}$ is an elliptic analogue of the factorial of m.

Proof. We apply Lemma 1.3 and take

$$u_k = \frac{1}{[k+2]_{a,b;q,p}[k+3]_{a,b;q,p}\dots[k+m+1]_{a,b;q,p}},$$
$$v_k = u_{k-1} = \frac{1}{[k+1]_{a,b;q,p}[k+2]_{a,b;q,p}\dots[k+m]_{a,b;q,p}};$$

so that, by virtue of (1.6b),

$$t_k = \frac{-W_{a,b;q,p}(k+1)[m]_{aq^{2k+2},bq^{k+1};q,p}}{[k+1]_{a,b;q,p}[k+2]_{a,b;q,p}\dots[k+m+1]_{a,b;q,p}}$$

and

$$t_0 = \frac{-W_{a,b;q,p}(1)[m]_{aq^2,bq;q,p}}{[m+1]_{a,b;q,p}!}.$$

With these substitutions, we have (1.8), and after replacing n by $n-1$ and shifting the index of the sum (such that k runs from 1 to n, instead of from 0 to $n-1$) we readily obtain (2.15). □

3. Special Cases of Elliptic and Multibasic Hypergeometric Series Identities

In [**19**], the indefinite summation formula

$$\sum_{k=0}^{n} \frac{(1-aq^{2k})}{(1-a)} \frac{(a,b;q)_k}{(q,aq/b;q)_k} b^{n-k} = \frac{(aq,bq;q)_n}{(q,aq/b;q)_n} \tag{3.1}$$

is used to obtain q-analogues of several elementary sums. This includes Warnaar's [**35**] q-analogue of the sum of the first n cubes. In this section, we use the same idea, but use the following elliptic analogue of (3.1):

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p^2)}{\theta(a;p^2)} \frac{\left(a,b,cp;q;p^2\right)_k}{\left(q,aq/b,bcpq;q,p^2\right)_k} \frac{\left(bcp/a;q^{-1},p^2\right)_k}{\left(cp/aq;q^{-1},p^2\right)_k} b^{n-k}$$
$$= \frac{\left(aq,bq,cpq;q,p^2\right)_n}{\left(q,aq/b,bcpq;q,p^2\right)_n} \frac{\left(bcp/aq;q^{-1},p^2\right)_n}{\left(cp/aq;q^{-1},p^2\right)_n}. \tag{3.2}$$

We first give some remarks before the proof. Clearly, (3.2) reduces to (3.1) when $p=0$. We cannot take $c=0$ in (3.2) while keeping the nome p^2, as $c=0$ appears as an essential singularity on each side of (3.2). The extra parameter c ensures that the elliptic balancing condition holds for the terms appearing in (3.1). The way the q-series identity (3.1) is extended to the elliptic identity in (3.2) is analogous to the way the of the q-Saalschütz summation is extended to the elliptic case as described in [**12**, Sec. 11.4, p. 323]. Notice that the indefinite summation (3.2) can also be obtained by telescoping (just as (3.1)).

Proof of (3.2)**.** A direct way to obtain (3.2) is to deduce it from the Frenkel and Turaev ${}_{10}V_9$ summation [**12**, (11.4.1)], which is an elliptic analogue of Jackson's very-well-poised ${}_8\phi_7$ summation. Specifically, taking $e \to aq^{n+1}$ in [**12**, (11.4.1)] we obtain

$$\sum_{k=0}^{n} \frac{\theta(aq^{2k};p)}{\theta(a;p)} \frac{(a,b,c,a/bc;q,p)_k}{(q,aq/b,aq/c,bcq;q,p)_k} q^k = \frac{(aq,bq,cq,aq/bc;q,p)_n}{(q,aq/b,aq/c,bcq;q,p)_n}.$$

Now replace p by p^2 and subsequently replace c by cp and use

$$\frac{\left(a/bcp;q,p^2\right)_k}{\left(aq/cp;q,p^2\right)_k} = \frac{1}{b^k q^k} \frac{\left(bcp/a;q^{-1},p^2\right)_k}{\left(cp/aq;q^{-1},p^2\right)_k}.$$

This immediately gives (3.2). □

It is easy to use (3.2) to obtain elliptic extensions of results from [**19**]. However, these results necessarily have the additional parameter c, which cannot be specialized to 0 or ∞ before letting $p=0$. As an example, we give another extension of Warnaar's result in [**35**, (2)], which is a q-analogue of the sum of cubes.

Replace n by $n-1$, shift the index of summation $k \to k-1$, and set $a=b=q^2$ in (3.2) to obtain:

$$\sum_{k=1}^{n} \frac{\theta(q^{2k};p^2)}{\theta(q^2;p^2)} \frac{\left(q^2,q^2,cp;q;p^2\right)_{k-1}}{\left(q,q,cpq^3;q,p^2\right)_{k-1}} \frac{\left(cp;q^{-1},p^2\right)_{k-1}}{\left(cp/q^3;q^{-1},p^2\right)_{k-1}} q^{2(n-k)}$$
$$= \frac{\left(q^3,q^3,cpq;q,p^2\right)_{n-1}}{\left(q,q,cpq^3;q,p^2\right)_{n-1}} \frac{\left(cp/q;q^{-1},p^2\right)_{n-1}}{\left(cp/q^3;q^{-1},p^2\right)_{n-1}}.$$

When $p=0$, this reduces to (2.12).

Remark 3.1. A special case of (3.1) is the following q-analogue of the formula for the sum of the first n odd numbers (cf. [**19**, (3.9)]):

$$\sum_{k=0}^{n-1} [2k+1]_q q^{-k} = [n]_q^2 q^{1-n}. \tag{3.3}$$

An extension of (3.3) to cubic basic hypergeometric series can be given as follows:

$$\begin{aligned}\sum_{k=0}^{n-1} q^{-k} \frac{(aq;q^3)_k}{(aq^5;q^3)_k} \frac{(1-q^{2k+1})}{1-q} \frac{(1-aq^{2k+1})^2}{(1-aq)^2} \\ = \frac{(1-q^n)^2(1-aq^n)}{(1-q)^2(1-aq)} \frac{\left(aq^4;q^3\right)_{n-1}}{\left(aq^5;q^3\right)_{n-1}} q^{1-n}.\end{aligned} \tag{3.4}$$

For $a \to 0$ this reduces to (3.3). It is easy to verify that this sum telescopes.

Remark 3.2. Another general indefinite elliptic summation that can be specialized to obtain various extensions of classical results is the following special case of a multibasic theta function identity by Gasper and Schlosser [**13**, (3.19), $t=q$]:

$$\begin{aligned}&\sum_{k=0}^{n} \frac{\theta(ad(rs)^k, br^k/dq^k, cs^k/dq^k;p)}{\theta(ad,b/d,c/d;p)} \\ &\times \frac{(ad^2/bc;q,p)_k(b;r,p)_k(c;s,p)_k(a;rs/q,p)_k}{(dq;q,p)_k(adr/c;r,p)_k(ads/b;s,p)_k(bcrs/dq;rs/q,p)_k} q^k \\ &= \frac{\theta(a,b,c,ad^2/bc;p)}{d\,\theta(ad,b/d,c/d,ad/bc;p)} \\ &\times \frac{(ad^2q/bc;q,p)_n(br;r,p)_n(cs;s,p)_n(ars/q;rs/q,p)_n}{(dq;q,p)_n(adr/c;r,p)_n(ads/b;s,p)_n(bcrs/dq;rs/q,p)_n} \\ &- \frac{\theta(d,ad/b,ad/c,bc/d;p)}{d\,\theta(ad,b/d,c/d,ad/bc;p)}.\end{aligned} \tag{3.5}$$

4. The proof of Theorem 1.1 and Some Special Cases

We have seen that telescoping leads to several elementary identities. All the telescoping identities are special cases of Euler's telescoping lemma, Lemma 1.3. In order to apply the telescoping lemma, we would like to use sequences u_k, v_k such that $t_k = u_k - v_k$ can be simplified.

We now turn to the proof of Theorem 1.1. The motivation behind this theorem is to use (1.7) so that $t_k = u_k - v_k$ becomes an analogue of a factorized product of linear factors in k.

*Proof of Theorem 1.1***.** We combine Lemma 1.3 with a special instance of the the difference equation (1.7). Let

$$\begin{aligned}x &= (gk-g+c)(hk+d),\\ y &= -(gk+c)(hk-h+d),\\ r &= 2ghk+ch+dg.\end{aligned}$$

With this assignment of variables, we have $x+r=(gk+c)(hk+h+d)$, $y-r=-(gk+g+c)(hk+d)$, and $r+x-y=2(gk+c)(hk+d)$. Substituting these values into (1.7), we have

$$
\begin{aligned}
&\big[(gk-g+c)(hk+d)\big]_{a,b;q,p}\big[-(gk+c)(hk-h+d)\big]_{aq^{4(gk+c)(hk+d)},bq^{2gk+c)(hk+d)};q,p}\\
&-\big[(gk+c)(hk+h+d)\big]_{a,b;q,p}\\
&\quad\times\big[-(gk+g+c)(hk+d)\big]_{aq^{4(gk+c)(hk+d)},bq^{2(gk+c)(hk+d)};q,p}\\
&=\big[2(gk+c)(hk+d)\big]_{a,b;q,p}\big[2ghk+ch+dg\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\\
&\quad\times W_{aq^{4(gk+c)(hk+d)},bq^{2(gk+c)(hk+d)};q,p}\big(-(gk+g+c)(hk+d)\big),
\end{aligned}
$$

which is equivalent to

$$
\begin{aligned}
&-\big[(gk-g+c)(hk+d)\big]_{a,b;q,p}\\
&\quad\times\big[(gk+c)(hk-h+d)\big]_{aq^{2(gk+c)(hk+h+d)},bq^{(gk+c)(hk+h+d)};q,p}\\
&\quad\times W_{aq^{2(gk+c)(hk+h+d)},bq^{(gk+c)(hk+h+d)};q,p}\big((gk+c)(hk-h+d)\big)^{-1}\\
&+\big[(gk+c)(hk+h+d)\big]_{a,b;q,p}\\
&\quad\times\big[(gk+g+c)(hk+d)\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\\
&\quad\times W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big((gk+g+c)(hk+d)\big)^{-1}\\
&=\big[2(gk+c)(hk+d)\big]_{a,b;q,p}\\
&\quad\times\big[2ghk+ch+dg\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\\
&\quad\times W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big((gk+g+c)(hk+d)\big)^{-1}.
\end{aligned}
$$

Multiplication of both sides of this relation by the factor

$$
W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big((gk+g+c)(hk+d)\big)
$$

and application of the reduction

$$
\begin{aligned}
&\frac{W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big((gk+g+c)(hk+d)\big)}{W_{aq^{2(gk+c)(hk+h+d)},bq^{(gk+c)(hk+h+d)};q,p}\big((gk+c)(hk-h+d)\big)}\\
&=W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big(2ghk+ch+dg\big)
\end{aligned}
$$

gives the identity

$$
\begin{aligned}
&\big[(gk+c)(hk+h+d)\big]_{a,b;q,p}\\
&\quad\times\big[(gk+g+c)(hk+d)\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\\
&-\big[(gk-g+c)(hk+d)\big]_{a,b;q,p}\\
&\quad\times\big[(gk+c)(hk-h+d)\big]_{aq^{2(gk+c)(hk+h+d)},bq^{(gk+c)(hk+h+d)};q,p}\\
&\quad\times W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big(2ghk+ch+dg\big)\\
&=\big[2(gk+c)(hk+d)\big]_{a,b;q,p}\\
&\quad\times\big[2ghk+ch+dg\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}.
\end{aligned}
\tag{4.1}
$$

Thus, in order to apply Lemma 1.3, we let

$$\begin{aligned}
t_k &= \big[2(gk+c)(hk+d)\big]_{a,b;q,p} \\
&\quad \times \big[2ghk+ch+dg\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}, \\
u_k &= \big[(gk+c)(hk+h+d)\big]_{a,b;q,p} \\
&\quad \times \big[(gk+g+c)(hk+d)\big]_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}, \\
v_k &= \big[(gk-g+c)(hk+d)\big]_{a,b;q,p} \\
&\quad \times \big[(gk+c)(hk-h+d)\big]_{aq^{2(gk+c)(hk+h+d)},bq^{(gk+c)(hk+h+d)};q,p} \\
&\quad \times W_{aq^{2(gk-g+c)(hk+d)},bq^{(gk-g+c)(hk+d)};q,p}\big(2ghk+ch+dg\big).
\end{aligned}$$

Now by (4.1) we have $t_k = u_k - v_k$, and (1.8) gives the desired result. $\square$

Some special cases of (1.4)**.**

(1) An $a;q$-analogue: take $p \to 0$ and $b \to 0$.

$$\begin{aligned}
&\sum_{k=0}^{n}\Bigg(\frac{[2(gk+c)(hk+d)]_{a;q}[2ghk+ch+dg]_{aq^{2(gk-g+c)(hk+d)};q}}{[2cd]_{a;q}[ch+dg]_{aq^{2(c-g)d};q}} \\
&\qquad\qquad \times \prod_{j=0}^{k-1}\frac{[(gj+g+c)(hj+d)]_{aq^{2(gj-g+c)(hj+d)};q}}{[(gj+g+c)(hj+d)]_{aq^{2(gj+g+c)(hj+2h+d)};q}} \\
&\qquad\qquad \times \prod_{j=0}^{k-1} W_{aq^{2(gj+c)(hj+h+d)};q}(2ghj+2gh+ch+dg)^{-1}\Bigg) \\
&\quad = \frac{[(gn+c)(hn+h+d)]_{a;q}[(g+c)d]_{aq^{2(c-g)d};q}}{[2cd]_{a;q}[ch+dg]_{aq^{2(c-g)d};q}} \\
&\times \prod_{j=1}^{n}\frac{[(gj+g+c)(hj+d)]_{aq^{2(gj-g+c)(hj+d)};q}}{[(gj+c)(hj-h+d)]_{aq^{2(gj+c)(hj+h+d)};q}} W_{aq^{2(gj-g+c)(hj+d)};q}(2ghj+ch+dg)^{-1} \\
&\qquad\qquad - \frac{[(c-g)d]_{a;q}[c(d-h)]_{aq^{2c(h+d)};q}}{[2cd]_{a;q}[ch+dg]_{aq^{2(c-g)d};q}} W_{aq^{2(c-g)d};q}(ch+dg). \qquad (4.2)
\end{aligned}$$

(2) A q-analogue: take $a \to 0$ in (4.2).

$$\begin{aligned}
&\sum_{k=0}^{n}\Bigg(\frac{[2(gk+c)(hk+d)]_q[2ghk+ch+dg]_q}{[2cd]_q[ch+dg]_q} q^{-\left(ghk^2+(ch+dg+gh)k\right)}\Bigg) \\
&\quad = \frac{[(gn+c)(hn+h+d)]_q[(gn+g+c)(hn+d)]_q}{[2cd]_q[ch+dg]_q} q^{-\left(ghn^2+(ch+dg+gh)n\right)} \\
&\qquad\qquad - \frac{[c(d-h)]_q[(c-g)d]_q}{[2cd]_q[ch+dg]_q} q^{ch+dg}. \qquad (4.3)
\end{aligned}$$

(3) We can further specialize c, d, g and h in (4.3) to obtain more q-analogues, highlighted in §1. In particular, we have the following:

(i) Take $c, d, g \to 1$ and $h \to 0$, shift the index to run from $k = 1$ to $n + 1$, and replace $n + 1$ by n to obtain (1.5a).

(ii) Take $c, d, g, h \to 1$ to get (1.5b).

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