GENERALIZATION OF A REAL-ANALYSIS RESULT TO
A CLASS OF TOPOLOGICAL VECTOR SPACES

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Abstract. In this paper, we generalize an elementary real-analysis result to a class of topological vector spaces. We also give an example of a topological vector space to which the result cannot be generalized.

1. Introduction

This paper draws its inspiration from the following result, which appears to be a popular real-analysis exam problem (see [3], for example):

Let \((x_n)_{n \in \mathbb{N}}\) be a sequence in \(\mathbb{R}\). If \(\lim_{n \to \infty} (2x_{n+1} - x_n) = x\) for some \(x \in \mathbb{R}\), then \(\lim_{n \to \infty} x_n = x\).

An expedient proof can be given using the Stolz-Cesàro Theorem as follows:

**Proof.** Define sequences \((a_n)_{n \in \mathbb{N}}\) and \((b_n)_{n \in \mathbb{N}}\) in \(\mathbb{R}\) by

\[
\forall n \in \mathbb{N} : \quad a_n \overset{df}{=} 2^n x_n \quad \text{and} \quad b_n \overset{df}{=} 2^n.
\]

Then \((b_n)_{n \in \mathbb{N}}\) is strictly increasing and diverges to \(\infty\), and as

\[
\lim_{n \to \infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n \to \infty} \frac{2^{n+1} x_{n+1} - 2^n x_n}{2^{n+1} - 2^n} = \lim_{n \to \infty} \frac{2^{n+1} x_{n+1} - 2^n x_n}{2^n} = \lim_{n \to \infty} (2x_{n+1} - x_n) = x.
\]

the Stolz-Cesàro Theorem immediately tells us that \(\lim_{n \to \infty} x_n = \lim_{n \to \infty} \frac{a_n}{b_n} = x\). \(\square\)

A natural question to ask is: Is this result still valid if \(\mathbb{R}\) is replaced by another topological vector space? The answer happens to be affirmative for a wide class of topological vector spaces that includes all the locally convex ones.

We will also exhibit a topological vector space for which the result is not valid, which indicates that it is rather badly behaved.

In this paper, we adopt the following conventions:

- \(\mathbb{N}\) denotes the set of all positive integers, and for each \(n \in \mathbb{N}\), let \([n] \overset{df}{=} \mathbb{N}_{\leq n}\).
- All vector spaces are over the field \(\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}\).

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2. Good Topological Vector Spaces

Recall that a topological vector space is an ordered pair \((V, \tau)\), where:

- \(V\) is a vector space, and
- \(\tau\) is a topology on \(V\), under which vector addition and scalar multiplication are continuous operations.

**Definition 2.1.** Let \((V, \tau)\) be a topological vector space, and \((x_\lambda)_{\lambda \in \Lambda}\) a net in \(V\). Then \(x \in V\) is called a \(\tau\)-limit for \((x_\lambda)_{\lambda \in \Lambda}\) — which we write as \((x_\lambda)_{\lambda \in \Lambda} \xrightarrow{\tau} x\) — if and only if for each \(\tau\)-neighborhood \(U\) of \(x\), there is a \(\lambda_0 \in \Lambda\) such that \(x_\lambda \in U\) for all \(\lambda \in \Lambda \geq \lambda_0\).

**Remark 2.2.** We do not assume that \(\tau\) is a Hausdorff topology on \(V\).

**Definition 2.3.** A topological vector space \((V, \tau)\) is said to be good if and only if any sequence \((x_n)_{n \in \mathbb{N}}\) in \(V\) has a \(\tau\)-limit whenever \((2x_{n+1} - x_n)_{n \in \mathbb{N}}\) has a \(\tau\)-limit.

A topological vector space that is not good is said to be bad.

**Proposition 2.4.** Let \((V, \tau)\) be a topological vector space, and \((x_n)_{n \in \mathbb{N}}\) a sequence in \(V\) such that \((2x_{n+1} - x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} x\) for some \(x \in V\). Then either

- \((x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} x\) also, or
- \((x_n)_{n \in \mathbb{N}}\) has no \(\tau\)-limit.

**Proof.** If \((x_n)_{n \in \mathbb{N}}\) has no \(\tau\)-limit, then we are done.

Next, suppose that \((x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} y\) for some \(y \in V\). Then

\[(2x_{n+1} - x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} 2y - y = y,
\]
so \(y\) is a \(\tau\)-limit for \((2x_{n+1} - x_n)_{n \in \mathbb{N}}\) in addition to \(x\). It follows that

\[(O_V)_{n \in \mathbb{N}} = ((2x_{n+1} - x_n) - (2x_{n+1} - x_n))_{n \in \mathbb{N}} \xrightarrow{\tau} x - y,
\]
which yields

\[(y)_{n \in \mathbb{N}} = (O_V + y)_{n \in \mathbb{N}} \xrightarrow{\tau} (x - y) + y = x.
\]

Therefore, any \(\tau\)-neighborhood of \(x\) also contains \(y\), giving us \((x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} x\). \(\square\)

**Proposition 2.4** tells us: To prove that a topological vector space \((V, \tau)\) is good, it suffices to prove that for each sequence \((x_n)_{n \in \mathbb{N}}\) in \(V\), if \((2x_{n+1} - x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} x\) for some \(x \in V\), then \((x_n)_{n \in \mathbb{N}} \xrightarrow{\tau} x\) also.

**Definition 2.5.** Let \(p \in (0, 1]\). A \(p\text{-homogeneous seminorm}\) on a vector space \(V\) is then a function \(\sigma : V \to \mathbb{R}_{\geq 0}\) with the following properties:

1. **The Triangle Inequality:** \(\sigma(x + y) \leq \sigma(x) + \sigma(y)\) for all \(x, y \in V\).
2. **\(p\text{-Homogeneity:}\)** \(\sigma(kx) = |k|^p \sigma(x)\) for all \(k \in \mathbb{k}\) and \(x \in V\).

**Remark 2.6.**

- **By letting \(k = 0\) and \(x = 0_V\) in (2), we find that \(\sigma(0_V) = 0\).**
- **A 1-homogeneous seminorm is the same as a seminorm in the ordinary sense.**
- **No extra generality is gained by postulating that \(\sigma(kx) \leq |k|^p \sigma(x)\) for all \(k \in \mathbb{k}\) and \(x \in V\). If \(k \in \mathbb{k} \setminus \{0\}\), then replacing \(k\) by \(1/k\) gives us the reverse inequality, which leads to equality; if \(k = 0\), then equality automatically holds.**
We do not consider $p \in (2, \infty)$ because
\[ \forall x \in V : \quad 2^p \sigma(x) = \sigma(2x) \quad (\text{By } p\text{-homogeneity.}) \]
\[ = \sigma(x + x) \leq 2 \sigma(x), \quad (\text{By the Triangle Inequality.}) \]
so if $\sigma$ is non-trivial, then $2^p \leq 2$, which implies that $p \in (0, 1]$ if $p \in \mathbb{R}_{>0}$.

Let $V$ be a vector space, and $S$ a collection of $p$-homogeneous seminorms on $V$ where $p \in (0, 1]$ may not be fixed. Define a function $U : V \times S \times \mathbb{R}_{>0} \to P(V)$ by
\[ \forall x \in V, \forall \sigma \in S, \forall \epsilon \in \mathbb{R}_{>0} : \quad \mathcal{U}_{x, \sigma, \epsilon} \overset{df}{=} \{ y \in V \mid \sigma(y - x) < \epsilon \}. \]

Then let $\tau_S$ denote the topology on $V$ that is generated by the sub-base
\[ \{ \mathcal{U}_{x, \sigma, \epsilon} \in P(V) \mid (x, \sigma, \epsilon) \in V \times S \times \mathbb{R}_{>0} \}. \]

**Proposition 2.7.** The following statements about $\tau_S$ hold:

1. $\tau_S$ is a vector-space topology on $V$.
2. Let $(x_\lambda)_{\lambda \in \Lambda}$ be a net in $V$. Then for each $x \in V$, we have
\[ (x_\lambda)_{\lambda \in \Lambda} \overset{\tau_S}{\rightarrow} x \iff \lim_{\lambda \in \Lambda} \sigma(x_\lambda - x) = 0 \text{ for all } \sigma \in S. \]

**Proof.** One only has to imitate the proof in the case of locally convex topological vector spaces that the initial topology generated by a collection of seminorms is a vector-space topology. We refer the reader to Chapter 1 of [2] for details. \qed

**Proposition 2.8.** $(V, \tau_S)$ is a good topological vector space.

**Proof.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $V$. Suppose that $(2x_{n+1} - x_n)_{n \in \mathbb{N}} \overset{\tau_S}{\rightarrow} x$ for some $x \in V$. Then without loss of generality, we may assume that $x = 0_V$. To see why, define a new sequence $(y_n)_{n \in \mathbb{N}}$ in $V$ by $y_n \overset{df}{=} x_n - x$ for all $n \in \mathbb{N}$, so that
\[ \forall n \in \mathbb{N} : \quad 2y_{n+1} - y_n = 2(x_{n+1} - x) - (x_n - x) = 2x_{n+1} - 2x - x_n + x = (2x_{n+1} - x_n) - x. \]

Hence,
\[ (2y_{n+1} - y_n)_{n \in \mathbb{N}} = ((2x_{n+1} - x_n) - x)_{n \in \mathbb{N}} \overset{\tau_S}{\rightarrow} x - x = 0_V, \]
so if we can prove that $(y_n)_{n \in \mathbb{N}} \overset{\tau_S}{\rightarrow} 0_V$, then $(x_n)_{n \in \mathbb{N}} \overset{\tau_S}{\rightarrow} x$ as desired.

Let $\sigma \in S$ and $\epsilon > 0$, and suppose that $\sigma$ is $p$-homogeneous for some $p \in (0, 1]$. Then by (2) of [2] there is an $N \in \mathbb{N}$ such that
\[ \forall n \in \mathbb{N} : \quad \sigma(2x_{n+1} - x_n) = \sigma((2x_{n+1} - x_n) - 0_V) < (2^p - 1) \epsilon. \]

By $p$-homogeneity, we thus have
\[ \forall k \in \mathbb{N} : \quad \sigma(2^k x_{N+k} - 2^{k-1} x_{N+k-1}) = \sigma(2^{k-1} (2x_{N+k} - x_{N+k-1})) = 2^{(k-1)p} \sigma(2x_{N+k} - x_{N+k-1}) < 2^{(k-1)p} (2^p - 1) \epsilon. \]

Next, a telescoping sum in conjunction with the Triangle Inequality yields
\[ \forall m \in \mathbb{N} : \quad \sigma(2^m x_{N+m} - x_N) = \sigma \left( \sum_{k=1}^{m} (2^k x_{N+k} - 2^{k-1} x_{N+k-1}) \right) \]
\[
\leq \sum_{k=1}^{m} \sigma \left( 2^k x_{N+k} - 2^{k-1} x_{N+k-1} \right)
\]
\[
< \sum_{k=1}^{m} 2^{(k-1)p}(2^p - 1) \epsilon
\]
\[
= (2^{mp} - 1) \epsilon.
\]

Then by \(p\)-homogeneity again,

\[
\forall m \in \mathbb{N} : \quad \sigma \left( x_{N+m} - \frac{1}{2m} x_N \right) = \sigma \left( \frac{1}{2m} (2^m x_{N+m} - x_N) \right)
\]
\[
= \frac{1}{2mp} \sigma \left( 2^m x_{N+m} - x_N \right)
\]
\[
< \left( 1 - \frac{1}{2mp} \right) \epsilon.
\]

Applying the Triangle Inequality and \(p\)-homogeneity once more, we get

\[
\forall m \in \mathbb{N} : \quad \sigma(x_{N+m}) < \sigma \left( \frac{1}{2^m} x_N \right) + \left( 1 - \frac{1}{2mp} \right) \epsilon
\]
\[
= \frac{1}{2mp} \sigma(x_N) + \left( 1 - \frac{1}{2mp} \right) \epsilon.
\]

Consequently,

\[
\limsup_{n \to \infty} \sigma(x_n) = \limsup_{m \to \infty} \sigma(x_{N+m}) \leq \limsup_{m \to \infty} \left[ \frac{1}{2mp} \sigma(x_N) + \left( 1 - \frac{1}{2mp} \right) \epsilon \right] = \epsilon.
\]

As \(\epsilon > 0\) is arbitrary, we obtain

\[
\lim_{n \to \infty} \sigma(x_n - 0_V) = \lim_{n \to \infty} \sigma(x_n) = 0.
\]

Finally, as \(\sigma \in \mathcal{S}\) is arbitrary, (2) of Proposition 2.7 says that \((x_n)_{n \in \mathbb{N}} \xrightarrow{\mathcal{S}} 0_V. \]

By Proposition 2.8, the class of good topological vector spaces includes:

- All locally convex topological vector spaces.
- All \(L^p\)-spaces for \(p \in (0, 1)\), which are generally not locally convex.

In the next section, we will give an example of a bad topological vector space.

3. A Bad Topological Vector Space from Probability Theory

Before we present the example, let us first fix some probabilistic terminology.

**Definition 3.1.** Let \((\Omega, \Sigma, P)\) be a probability space.

- A measurable function from \((\Omega, \Sigma)\) to \((\mathbb{R}, \mathcal{B}(\mathbb{R}))\) is called a random variable.
- The \(\mathbb{R}\)-vector space of random variables on \((\Omega, \Sigma)\) is denoted by \(\text{RV}(\Omega, \Sigma)\).
- Let \((X_\lambda)_{\lambda \in \Lambda}\) be a net in \(\text{RV}(\Omega, \Sigma)\), and let \(X \in \text{RV}(\Omega, \Sigma)\). Then \((X_\lambda)_{\lambda \in \Lambda}\) is said to converge in probability to \(X\) (for \(P\)) if and only if for each \(\epsilon > 0\), we have

\[
\lim_{\lambda \in \Lambda} P\{ \omega \in \Omega \mid |X_\lambda(\omega) - X(\omega)| > \epsilon \} = 0,
\]

in which case, we write \((X_\lambda)_{\lambda \in \Lambda} \xrightarrow{p} X\).

The following theorem says that convergence in probability is convergence with respect to a vector-space topology on the vector space of random variables.

\[1\] \(\mathcal{B}(\mathbb{R})\) denotes the Borel \(\sigma\)-algebra generated by the standard topology on \(\mathbb{R}\).
Theorem 3.2. Let \((\Omega, \Sigma, \mathbb{P})\) be a probability space, and define a pseudo-metric \(\rho_\mathbb{P}\) on \(RV(\Omega, \Sigma)\) by
\[
\forall X, Y \in RV(\Omega, \Sigma) : \quad \rho_\mathbb{P}(X, Y) \equiv \int_\Omega \frac{|X - Y|}{1 + |X - Y|} \, d\mathbb{P}.
\]
Then the topology \(\tau_\mathbb{P}\) on \(RV(\Omega, \Sigma)\) generated by \(\rho_\mathbb{P}\) has the following properties:

• \(\tau_\mathbb{P}\) is a vector-space topology.

• Let \((X_\lambda)_{\lambda \in \Lambda}\) be a net in \(RV(\Omega, \Sigma)\). Then for each \(X \in RV(\Omega, \Sigma)\), we have
\[
(X_\lambda)_{\lambda \in \Lambda} \overset{\mathbb{P}}{\to} X \iff (X_\lambda)_{\lambda \in \Lambda} \overset{\tau_\mathbb{P}}{\to} X.
\]

Proof. Please refer to Problems 6, 10 and 14 in Section 5.2 of [1]. □

Now, for each \(k \in \mathbb{N}\), define a probability measure \(c_k\) on \((\{k\}, \mathbb{P}(\{k\}))\) by
\[
\forall A \subseteq \{k\} : \quad c_k(A) \equiv \frac{\text{Card}(A)}{k},
\]
and let \((\Omega, \Sigma, \mathbb{P})\) denote the product probability space \(\prod_{k=1}^{\infty} (\{k\}, \mathbb{P}(\{k\}), c_k)\). Define a sequence \((S_n)_{n \in \mathbb{N}}\) in \(\Sigma\) by
\[
\forall n \in \mathbb{N} : \quad S_n \equiv \left\{ \mathbf{v} \in \prod_{k=1}^{\infty} \{k\} \middle| \mathbf{v}(n) = 1 \right\}.
\]
Then \(\mathbb{P}(S_n) = \frac{1}{n}\) for all \(n \in \mathbb{N}\), and the \(S_n\)'s form mutually-independent events.

Next, define a sequence \((Y_n)_{n \in \mathbb{N}}\) in \(RV(\Omega, \Sigma)\) by
\[
\forall n \in \mathbb{N} : \quad Y_n \equiv 2^n \chi_{S_n},
\]
where \(\chi_{S_n}\) denotes the indicator function of \(S_n\). Then we get for each \(\epsilon > 0\) that
\[
\lim_{n \to \infty} \mathbb{P}(\{\omega \in \Omega \mid |Y_n(\omega)| > \epsilon\}) = \lim_{n \to \infty} \mathbb{P}(S_n) = \lim_{n \to \infty} \frac{1}{n} = 0.
\]
The first equality is obtained because, for each \(\epsilon > 0\), we have \(2^n > \epsilon\) for all \(n \in \mathbb{N}\) large enough. Consequently, \((Y_n)_{n \in \mathbb{N}} \overset{\mathbb{P}}{\to} 0_{\Omega \to \mathbb{R}}\).

Define a new sequence \((X_n)_{n \in \mathbb{N}}\) in \(RV(\Omega, \Sigma)\) by
\[
\forall n \in \mathbb{N} : \quad X_n \equiv \begin{cases} 0_{\Omega \to \mathbb{R}} & \text{if } n = 1; \\ \frac{1}{2^{n-k}} Y_k & \text{if } n \geq 2. \end{cases}
\]
Then \(2X_2 - X_1 = 2X_2 = Y_1\), and
\[
\forall n \in \mathbb{N} \geq 2 : \quad 2X_{n+1} - X_n = \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} Y_k - \frac{1}{2^{n-k}} Y_k = \sum_{k=1}^{n-1} \frac{1}{2^{n-k}} Y_k = X_n.
\]
It follows that \((2X_{n+1} - X_n)_{n \in \mathbb{N}} = (Y_n)_{n \in \mathbb{N}} \overset{\mathbb{P}}{\to} 0_{\Omega \to \mathbb{R}}\).
Gathering what we have thus far, observe that

\[ \forall n \in \mathbb{N} : \quad X_{2n+1} = \sum_{k=1}^{2n} \frac{1}{2^{2n+1-k}} Y_k \]

\[ = \sum_{k=1}^{2n} \frac{1}{2^{2n+1-k}} (2^k \chi_{S_k}) \]

\[ = \sum_{k=1}^{2n} 2^{2k-2n-1} \chi_{S_k} \]

\[ \geq \sum_{k=n+1}^{2n} 2^{2k-2n-1} \chi_{S_k} \]

\[ \geq \sum_{k=n+1}^{2n} \chi_{S_k} \]

\[ \geq \chi_{\bigcup_{k=n+1}^{2n} S_k}. \]

As the $S_k$’s are mutually independent, their complements are as well, so

\[ \forall n \in \mathbb{N} : \quad P\left( \left\{ \omega \in \Omega \mid |X_{2n+1}(\omega)| > \frac{1}{2} \right\} \right) \geq P\left( \bigcup_{k=n+1}^{2n} S_k \right) \]

\[ = 1 - P\left( \Omega \setminus \bigcup_{k=n+1}^{2n} S_k \right) \]

\[ = 1 - P\left( \bigcap_{k=n+1}^{2n} \Omega \setminus S_k \right) \]

\[ = 1 - \prod_{k=n+1}^{2n} P(\Omega \setminus S_k) \]

\[ = 1 - \prod_{k=n+1}^{2n} \left( 1 - \frac{1}{k} \right) \]

\[ = 1 - \prod_{k=n+1}^{2n} \frac{k-1}{k} \]

\[ = 1 - \frac{n}{2^n} \]

\[ = 1 - \frac{1}{2} \]

\[ = \frac{1}{2}. \]

Hence, $(X_n)_{n \in \mathbb{N}}$ does not converge to $0_{\Omega \to \mathbb{R}}$ in probability. By Theorem 3.2

**Proposition 3.3.** $(RV(\Omega, \Sigma), \tau_p)$ is therefore a bad topological vector space.

By Proposition 2.4 $(X_n)_{n \in \mathbb{N}}$ does not, in fact, converge in probability at all.
4. Acknowledgments

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