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NEW PARAMETERIZED MOCK THETA FUNCTIONS AND HECKE-TYPE DOUBLE SUMS

Li-Jun Hao and Liang-Liang Xu

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Abstract. In this paper we express two new mock theta functions with one parameter as the Appell-Lerch sums using the Bailey mechanism. Meanwhile, we also obtain some Hecke-type double sums for some new q-series. In addition, we establish the relationships between the mock theta functions and the classical sixth and eighth order mock theta functions. Furthermore, we give the Hecke-type double sums for the second order mock theta function $D_5(q)$.

1. Introduction

Throughout this paper, let q denote a complex number with $|q| < 1$. Here and in what follows, we adopt the standard q-series notation [**6**]. For any positive integer n,

$$(a;q)_0 := 1, \qquad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k),$$
$$(a_1,a_2,a_3,\cdots,a_m;q)_n := (a_1;q)_n(a_2;q)_n(a_3;q)_n\cdots(a_m;q)_n,$$
$$(a_1,a_2,a_3,\cdots,a_m;q)_\infty := (a_1;q)_\infty(a_2;q)_\infty(a_3;q)_\infty\cdots(a_m;q)_\infty.$$

For convenience, we use $(a)_n$ to denote $(a;q)_n$.

The Jacobi's triple product identity can be stated as follows.

$$j(x;q) := (x)_\infty(q/x)_\infty(q)_\infty = \sum_{n=-\infty}^{\infty}(-1)^n q^{\binom{n}{2}}x^n. \tag{1.1}$$

From the definition of $j(x;q)$, we have

$$j(x;q) = j(q/x;q) \tag{1.2}$$

and

$$j(q^n x;q) = (-1)^n q^{-\binom{n}{2}} x^{-n} j(x;q), \quad n \in \mathbb{Z}. \tag{1.3}$$

Let m and a be integers with m positive. Define

$$J_{a,m} := j(q^a;q^m), \quad \overline{J}_{a,m} := j(-q^a;q^m),$$
$$J_m := \prod_{i\geq 1}(1-q^{mi}), \quad \overline{J}_m := \prod_{i\geq 1}(1+q^{mi}), \text{ and}$$
$$j(b_1,b_2,\cdots,b_m;q) := j(b_1;q)j(b_2;q)\cdots j(b_m;q).$$

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Mock theta functions have been studied deeply by many scholars. They were introduced by Ramanujan in his last letter to G.H. Hardy on January 12, 1920. In that letter, Ramanujan listed seventeen mock theta functions and divided them into four classes, one class of third order, two of fifth order, and one of seventh order. However, Ramanujan neither rigorously defined a mock theta function nor the order of a mock theta function.

Motivated by Ramanujan's work, mock theta functions have received a great deal of attention as in [**1, 17, 18**]. Until 2002, it was not known how these functions fit into the theory of modular forms. A new chapter in the study of mock theta functions was opened due to the work of Zwegers [**20**] and Bringmann and Ono [**4, 5**]. Hickerson and Mortenson [**9**] defined Appell-Lerch sums as follows.

Definition 1.1. Let x and z be generic complex numbers with $z \neq 0$, with neither z nor xz an integer power of q. Then

$$m(x, q, z) := \frac{-z}{j(z;q)} \sum_{r=-\infty}^{\infty} \frac{(-1)^r q^{\binom{r+1}{2}} z^r}{1 - q^r xz}.$$

Notice that $m(0, q, z) = 1$.

Following [**9**], the term "generic" means that the parameters do not cause poles in the Appell-Lerch sums or in the quotients of theta functions.

Specializations of the Appell-Lerch sums are perhaps the most important class of mock theta functions. In other words, for any function $f(z)$, if we can express $f(z)$ as Appell-Lerch sums up to the addition of a weakly holomorphic modular form, then the function $f(z)$ is a mock theta function. Hickerson and Mortenson [**9**] studied the properties of Appell-Lerch sums and established the representations of mock theta functions in terms of Appell-Lerch sums.

In q-series, the Hecke-type double sum has played an important role and received a lot of attention. It was defined by Hickerson and Mortenson [**9**].

Definition 1.2. Let x, $y \in \mathbb{C}^*$ and define $sg(r) := 1$ for $r \geq 0$ and $sg(r) := -1$ for $r < 0$. Then

$$f_{a,b,c}(x, y, q) := \sum_{sg(r)=sg(s)} sg(r)(-1)^{r+s} x^r y^s q^{a\binom{r}{2}+brs+c\binom{s}{2}}.$$

It is clear that $f_{a,b,a}(x, y, q) = f_{a,b,a}(y, x, q)$.

Recently, using Bailey's lemma and Bailey pairs, many scholars established mock theta functions and gave their different forms in the modern sense. We can refer to [**7, 8, 13, 14, 15, 16, 19**]. Inspired by those works, we present two new Eulerian series and determine their Hecke-type double sum forms and Appell-Lerch sums by applying Bailey pairs.

Define

$$R_1(x, q) := \sum_{n=0}^{\infty} \frac{q^n (-xq)_n (-x^{-1})_n}{(q)_{2n}}, \tag{1.4}$$

$$R_2(x, q) := \sum_{n=0}^{\infty} \frac{xq^{n+1} (-xq)_n (-x^{-1})_n}{(q^2;q^2)_n}. \tag{1.5}$$

Then by means of the Bailey pairs, we can express the above two Eulerian series as Appell-Lerch sums as follows. Based on the modern sense of mock theta functions, we find two new mock theta functions with one parameter.

Theorem 1.3. *We have*

$$\begin{aligned} R_1(x,q) &= \frac{1}{J_1^2} f_{1,2,1}(q,-xq^2,q) \\ &= \frac{j(-xq^2;q)}{J_1^2} m(x^{-2}q^{-1},q^3,-1) - \frac{qJ_3^3 j(xq;q) j(-xq^5;q^3)}{J_1^2 \bar{J}_{0,3} j(x^{-1}q,-x^2q^4;q^3)}, \end{aligned} \tag{1.6}$$

$$\begin{aligned} R_2(x,q) &= \frac{1}{J_1^2} f_{2,2,1}(q^2,-xq^2,q) \\ &= \frac{xqj(-xq^2;q)}{J_1 J_2} m(1/(x^2q),q^2,-1) - \frac{2J_2^2 j(xq,xq^2;q^2)}{J_1 \bar{J}_{0,1} \bar{J}_{0,2} j(-x^2q^3;q^2)}. \end{aligned} \tag{1.7}$$

The two sixth order mock theta functions appeared in [**3**] are as follows.

$$\phi(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2} (q;q^2)_n}{(-q)_{2n}},$$

$$\mu(q) := \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1} (1+q^n)(q;q^2)_n}{(-q)_{n+1}}.$$

In 2014, Hickerson and Mortenson [**9**] provided the Appell-Lerch sums for them.

$$\phi(q) = 2m(q,q^3,-1), \tag{1.8}$$

$$\mu(q) = 2m(q^2,q^6,-1) - \frac{J_{1,2}\overline{J}_{1,3}}{2\overline{J}_{1,4}}. \tag{1.9}$$

Then, we get the following relationships.

Theorem 1.4. *We have*

$$R_1(q^{-1},q) = \frac{\overline{J}_1^2}{J_1} \phi(q),$$

$$R_1(q^{-2},q^2) - \frac{\overline{J}_2^2}{J_2} \mu(q) = \frac{\overline{J}_2^2 J_{1,2} \overline{J}_{1,3}}{2J_2 \overline{J}_{1,4}}.$$

Applying the Bailey pairs, we obtain some Hecke-type double sums for the parameterized q-series.

Theorem 1.5. *We have*

$$\sum_{n=0}^{\infty} \frac{q^n (-xq)_n (-x^{-1})_n}{(q)_n} = \frac{1}{(q)_\infty^2} \left(f_{6,3,1}(-q^5,-xq^2,q) - qf_{6,3,1}(-q^7,-xq^3,q) \right), \tag{1.10}$$

$$\sum_{n=0}^{\infty} \frac{q^n (-xq)_n (-x^{-1})_n}{(q;q^2)_n} = \left(f_{2,2,1}(q,-xq^2,q) + qf_{2,2,1}(q^3,-xq^2,q) \right) \frac{1}{(q;q^2)_\infty (q)_\infty (1+q)}, \tag{1.11}$$

$$\sum_{n=0}^{\infty} q^n(-xq)_n(-x^{-1})_n = \frac{1}{(q)_\infty} f_{3,2,1}(q^3, -xq^2, q), \tag{1.12}$$

$$\sum_{n=0}^{\infty} \frac{q^n(-xq)_n(-x^{-1})_n}{(-q;q^2)_{n+1}} = \frac{1}{(-q;q^2)_\infty(q)_\infty} f_{4,2,1}(q^4, -xq^2, q). \tag{1.13}$$

In 2006, Hikami [**10**] introduced the second order mock theta functions and gave a transformation formula in [**11**] as follows.

$$D_5(q) = \sum_{n=0}^{\infty} \frac{(-q)_n q^n}{(q;q^2)_{n+1}} = \frac{1}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} (q;q^2)_n^2 q^{2n}. \tag{1.14}$$

Letting $q \to q^2$ and $x = -q^{-1}$ in (1.12), we establish the Hecke-type double sums for $D_5(q)$.

Theorem 1.6. *We have*

$$D_5(q) = \frac{1}{j(q;q^2)} f_{3,2,1}(q^6, q^3, q^2).$$

The paper is organized as follows. In Section 2, we first state some lemmas which are used to prove the main results. In Section 3, we prove the main theorems.

2. Preparations

In this section, we recall some definitions and known results to prove our theorems.

The pair of sequences (α_n, β_n) is called a Bailey pair relative to (a, q) if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}.$$

Lemma 2.1. [**2**, Lemma 3] *The (α_n, β_n) is a Bailey pair relative to $(1, q)$, where*

$$\alpha_n = \begin{cases} 1, & n = 0, \\ q^{\binom{n}{2}}(x^n q^n + x^{-n}), & n \geq 1. \end{cases}$$
$$\beta_n = \frac{(-xq)_n(-x^{-1})_n}{(q)_{2n}}. \tag{2.1}$$

Lemma 2.2. [**12**, Theorem 1.1 (parts 1,5,6,12), Corollary 1.3 (parts 1,2)]

1.If (α_n, β_n) is a Bailey pair relative to (a, q), then

$$\sum_{n\geq 0} q^n \beta_n = \frac{1}{(aq, q)_\infty} \sum_{r,n\geq 0} (-a)^n q^{\binom{n+1}{2}+(2n+1)r} \alpha_r. \tag{2.2}$$

2.If (α_n, β_n) is a Bailey pair relative to (a, q), then

$$\sum_{n\geq 0} (aq;q^2)_n q^n \beta_n = \frac{1}{(aq^2;q^2)_\infty (q)_\infty} \sum_{r,n\geq 0} (-a)^n q^{n^2+2rn+r+n} \alpha_r. \tag{2.3}$$

3.If (α_n, β_n) is a Bailey pair relative to (a^2, q), then

$$\sum_{n\geq 0} \frac{(a^2q)_{2n} q^n}{(aq)_n} \beta_n = \frac{1}{(aq, q)_\infty} \sum_{r,n\geq 0} a^{3n} q^{3n^2+2n+3rn+r}(1 - aq^{2n+r+1})\alpha_r. \tag{2.4}$$

4.If (α_n,β_n) is a Bailey pair relative to (a,q), then

$$\sum_{n\geq 0}(aq^2;q^2)_n q^n\beta_n=\frac{1}{(aq;q^2)_\infty(q)_\infty(1+q)}\sum_{r,n\geq 0}(-a)^n q^{n^2+2rn+r}(1+q^{2n+1})\alpha_r. \tag{2.5}$$

5.If (α_n,β_n) is a Bailey pair relative to (a,q), then

$$\sum_{n\geq 0}(aq)_{2n}q^n\beta_n=\frac{1}{(q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{3n(n+1)/2+(2n+1)r}\alpha_r. \tag{2.6}$$

6.If (α_n,β_n) is a Bailey pair relative to (a,q), then

$$\sum_{n\geq 0}\frac{(aq)_{2n}q^n}{(-aq;q^2)_{n+1}}\beta_n=\frac{1}{(-aq;q^2)_\infty(q)_\infty}\sum_{r,n\geq 0}(-a)^n q^{2n^2+2n+2rn+r}\alpha_r. \tag{2.7}$$

In fact, there exist rich relationships between the Appell-Lerch sums and the Hecke-type double sums. They were stated as follows.

Theorem 2.3. *Let a,b,c be positive integers with $ac<b^2$ and b divisible by a,c. Then*

$$f_{a,b,c}(x,y,q)=h_{a,b,c}(x,y,q,-1,-1)-\frac{1}{\overline{J}_{0,b^2/a-c}\overline{J}_{0,b^2/c-a}}\cdot\theta_{a,b,c}(x,y,q),$$

where

$$\begin{aligned}h_{a,b,c}(x,y,q,z_1,z_0):=&j(x;q^a)m(-q^{a\binom{b/a+1}{2}-c}(-y)(-x)^{-b/a},q^{b^2/a-c},z_1)\\&+j(y;q^c)m(-q^{c\binom{b/c+1}{2}-a}(-x)(-y)^{-b/c},q^{b^2/c-a},z_0)\end{aligned}$$

and

$$\begin{aligned}&\theta_{a,b,c}(x,y,q):=\\&\sum_{d=0}^{b/c-1}\sum_{e=0}^{b/a-1}\sum_{f=0}^{b/a-1}q^{(b^2/a-c)\binom{d+1}{2}+(b^2/c-a)\binom{e+f+1}{2}+a\binom{f}{2}}j(q^{(b^2/a-c)(d+1)+bf}y;q^{b^2/a})\times\\&(-x)^f j(q^{b(b^2/(ac)-1)(e+f+1)-(b^2/a-c)(d+1)+b^3(b-a)/(2a^2c)}(-x)^{b/a}y^{-1};q^{(b^2/a)(b^2/(ac)-1)})\times\\&\frac{J^3_{b(b^2/(ac)-1)}j(q^{(b^2/c-a)(e+1)+(b^2/a-c)(d+1)-c\binom{b/c}{2}-a\binom{b/a}{2}}(-x)^{1-b/a}(-y)^{1-b/c};q^{b(b^2/(ac)-1)})}{j(q^{(b^2/c-a)(e+1)-c\binom{b/c}{2}}(-x)(-y)^{-b/c},q^{(b^2/a-c)(d+1)-a\binom{b/a}{2}}(-x)^{-b/a}(-y);q^{b(b^2/(ac)-1)})}.\end{aligned}$$

Lemma 2.4. [**9**, (1.7)] *We have*

$$\begin{aligned}f_{1,2,1}(x,y,q)=&j(x;q)m(q^2y/x^2,q^3,-1)+j(y;q)m(q^2x/y^2,q^3,-1)\\&-\frac{yJ_3^3j(-x/y;q)j(q^2xy;q^3)}{\overline{J}_{0,3}j(-qy^2/x,-qx^2/y;q^3)}.\end{aligned} \tag{2.8}$$

3. Proofs of the Main Results

In this section, we mainly prove the results by the Bailey pairs, the properties for the Hecke-type double sums and Appell-Lerch sums.

Proof of Theorem 1.3. Substituting the Bailey pair (2.1) into (2.2) with $a=1$, we deduce

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^n(-xq)_n(-x^{-1})_n}{(q)_{2n}}\\
&\quad=\frac{1}{J_1^2}\left(\sum_{r\geq 1,n\geq 0}(-1)^nq^{\binom{n+1}{2}+(2n+1)r+\binom{r}{2}}(x^rq^r+x^{-r})+\sum_{n\geq 0}(-1)^nq^{\binom{n+1}{2}}\right)\\
&\quad=\frac{1}{J_1^2}\left(\sum_{r,n\geq 0}(-1)^nq^{\binom{n+1}{2}+(2n+2)r+\binom{r}{2}}x^r+\sum_{r\geq 1,n\geq 0}(-1)^nq^{\binom{n+1}{2}+(2n+1)r+\binom{r}{2}}x^{-r}\right)\\
&\quad=\frac{1}{J_1^2}\Bigg(\sum_{r,n\geq 0}(-1)^nq^{\binom{n}{2}+2nr+\binom{r}{2}+n+2r}x^r\\
&\qquad\qquad+\sum_{r,n\leq -1}(-1)^{-n-1}q^{\binom{-n}{2}+(-2n-1)(-r)+\binom{-r}{2}}x^r\Bigg)\\
&\quad=\frac{1}{J_1^2}\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^nq^{\binom{n}{2}+2nr+\binom{r}{2}+n+2r}x^r\\
&\quad=\frac{1}{J_1^2}\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{\binom{n}{2}+2nr+\binom{r}{2}}q^n(-xq^2)^r\\
&\quad=\frac{1}{J_1^2}f_{1,2,1}(q,-xq^2,q).
\end{aligned}$$

Applying (2.8), we have

$$f_{1,2,1}(q,-xq^2,q)=j(-xq^2;q)m(x^{-2}q^{-1},q^3,-1)-\frac{qJ_3^3j(xq;q)j(-xq^5;q^3)}{\bar{J}_{0,3}j(x^{-1}q,-x^2q^4;q^3)},$$

which implies (1.6).

For (1.7), substituting the Bailey pair (2.1) into (2.3) with $a=1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^n(-xq)_n(-x^{-1})_n}{(q^2;q^2)_n}\\
&\quad=\frac{1}{J_1J_2}\left(\sum_{r\geq 1,n\geq 0}(-1)^nq^{n^2+2rn+r+n+\binom{r}{2}}(x^rq^r+x^{-r})+\sum_{n\geq 0}(-1)^nq^{n^2+n}\right)\\
&\quad=\frac{1}{J_1J_2}\left(\sum_{r,n\geq 0}(-1)^nq^{n^2+2rn+2r+n+\binom{r}{2}}x^r+\sum_{r\geq 1,n\geq 0}(-1)^nq^{n^2+2rn+r+n+\binom{r}{2}}x^{-r}\right)\\
&\quad=\frac{1}{J_1J_2}\Bigg(\sum_{r,n\geq 0}(-1)^nq^{n^2+2rn+2r+n+\binom{r}{2}}x^r\\
&\qquad\qquad+\sum_{r,n\geq 1}(-1)^{n-1}q^{(n-1)^2+2r(n-1)+r+(n-1)+\binom{r}{2}}x^{-r}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{J_1J_2}\Bigg(\sum_{r,n\geq 0}(-1)^n q^{n^2+2rn+\binom{r}{2}+n+2r}x^r \\
&\qquad\qquad + \sum_{r,n\leq -1}(-1)^{-n-1}q^{n^2+2rn+\binom{r}{2}+n+2r}x^r\Bigg) \\
&= \frac{1}{J_1J_2}\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{2\binom{n}{2}+2rn+\binom{r}{2}}q^{2n}(-xq^2)^r \\
&= \frac{1}{J_1J_2}f_{2,2,1}(q^2,-xq^2,q).
\end{aligned}
$$

Based on Theorem 2.3, we have

$$
\begin{aligned}
f_{2,2,1}(q^2,-xq^2,q) =&h_{2,2,1}(q^2,-xq^2,q,-1,-1) - \frac{1}{\bar{J}_{0,1}\bar{J}_{0,2}}\theta_{2,2,1}(q^2,-xq^2,q) \\
=&j(-xq^2;q)m(1/(x^2q),q^2,-1) - \frac{J_2^3}{\bar{J}_{0,1}\bar{J}_{0,2}}\times \\
&\left(\frac{j(-xq^3,xq,x^{-1};q^2)}{j(-x^{-2}q^{-1},-xq;q^2)}+\frac{qj(-xq^4,x^{-1},x^{-1}q;q^2)}{j(-x^{-2}q^{-1},-xq^2;q^2)}\right) \\
=&j(-xq^2;q)m(1/(x^2q),q^2,-1) - \frac{J_2^3}{\bar{J}_{0,1}\bar{J}_{0,2}}\times \\
&\frac{j(x^{-1}q,x^{-1};q^2)\big(j(-xq^3,-xq^2;q^2)+qj(-xq^4,-xq;q^2)\big)}{j(-x^2q^3,-xq,-xq^2;q^2)}.
\end{aligned}
\tag{3.1}
$$

On the other hand, we have

$$
\begin{aligned}
j(-xq;q^2) &= xqj(-xq^3;q^2), \\
j(-xq^4;q^2) &= \frac{1}{xq^2}j(-xq^2;q^2).
\end{aligned}
$$

Combining (3.1) and the above identities, we arrive at

$$
\begin{aligned}
f_{2,2,1}(q^2,-xq^2,q) =&j(-xq^2;q)m(1/(x^2q),q^2,-1) - \frac{J_2^3}{\bar{J}_{0,1}\bar{J}_{0,2}}\times \\
&\frac{2j(x^{-1}q,x^{-1};q^2)j(-xq^3,-xq^2;q^2)}{j(-x^2q^3,-xq,-xq^2;q^2)} \\
=&j(-xq^2;q)m(1/(x^2q),q^2,-1) - \frac{J_2^3}{\bar{J}_{0,1}\bar{J}_{0,2}}\frac{2j(xq,xq^2;q^2)}{xqj(-x^2q^3;q^2)},
\end{aligned}
$$

which yields (1.7). ∎

Proof of Theorem 1.4. Based on (1.6), we have

$$
\begin{aligned}
R_1(q^{-1},q) &= 2\frac{\overline{J}_1^2}{J_1}m(q,q^3,-1), \\
R_1(q^{-2},q^2) &= 2\frac{\overline{J}_2^2}{J_2}m(q^2,q^6,-1).
\end{aligned}
$$

Combining (1.8) and (1.9) with the above identities, we arrive at Theorem 1.4 by simplifying. This completes the proof. □

Proof of Theorem 1.5. Substituting the Bailey pair (2.1) into (2.4) with $a = 1$, we have

$$\begin{aligned}
&\sum_{n\geq 0}\frac{q^n(-xq)_n(-x^{-1})_n}{(q)_n}\\
&\quad=\frac{1}{J_1^2}\Bigg(\sum_{r\geq 1,n\geq 0}q^{3n^2+2n+3rn+r}(1-q^{2n+r+1})q^{\binom{r}{2}}(x^rq^r+x^{-r})\\
&\qquad\qquad+\sum_{n\geq 0}q^{3n^2+2n}(1-q^{2n+1})\Bigg)\\
&\quad=\frac{1}{J_1^2}\Bigg(\sum_{r,n\geq 0}q^{3n^2+2n+3rn+2r+\binom{r}{2}}(1-q^{2n+r+1})x^r\\
&\qquad\qquad+\sum_{r,n\geq 1}q^{3n^2+3rn+\binom{r}{2}-4n-2r+1}(1-q^{2n+r-1})x^{-r}\Bigg)\\
&\quad=\frac{1}{J_1^2}\Big(\sum_{r,n\geq 0}q^{3n^2+3rn+\binom{r}{2}+2n+2r}x^r-\sum_{r,n\geq 0}q^{3n^2+3rn+\binom{r}{2}+4n+3r+1}x^r\\
&\qquad\qquad-\sum_{r,n\leq -1}q^{3n^2+3rn+\binom{r}{2}+2n+2r}x^r+\sum_{r,n\leq -1}q^{3n^2+3rn+\binom{r}{2}+4n+3r+1}x^r\Big)\\
&\quad=\frac{1}{J_1^2}\Bigg(\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{6\binom{n}{2}+3nr+\binom{r}{2}}(-q^5)^n(-xq^2)^r\\
&\qquad\qquad-q\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{6\binom{n}{2}+3nr+\binom{r}{2}}(-q^7)^n(-xq^3)^r\Bigg)\\
&\quad=\frac{1}{J_1^2}\Big(f_{6,3,1}(-q^5,-xq^2,q)-qf_{6,3,1}(-q^7,-xq^3,q)\Big),
\end{aligned}$$

which is (1.10).

Substituting the Bailey pair (2.1) into (2.5) with $a = 1$, we get

$$\begin{aligned}
&\sum_{n\geq 0}\frac{q^n(-xq)_n(-x^{-1})_n}{(q;q^2)_n}\\
&\quad=\frac{1}{(q;q^2)_\infty(q)_\infty(1+q)}\times\Bigg(\sum_{r\geq 1,n\geq 0}(-1)^nq^{n^2+2rn+r+\binom{r}{2}}(1+q^{2n+1})(x^rq^r+x^{-r})\\
&\qquad\qquad+\sum_{n\geq 0}(-1)^nq^{n^2}(1+q^{2n+1})\Bigg).
\end{aligned}$$

For the sums of the right-hand side of the above identity, we have

$$\sum_{r\geq 1,n\geq 0}(-1)^nq^{n^2+2rn+r+\binom{r}{2}}(1+q^{2n+1})(x^rq^r+x^{-r})+\sum_{n\geq 0}(-1)^nq^{n^2}(1+q^{2n+1})$$

$$
\begin{aligned}
&= \sum_{r,n\geq 0}(-1)^n q^{n^2+2rn+2r+\binom{r}{2}}(1+q^{2n+1})x^r \\
&\qquad + \sum_{r\geq 1, n\geq 0}(-1)^n q^{n^2+2rn+r+\binom{r}{2}}(1+q^{2n+1})x^{-r} \\
&= \sum_{r,n\geq 0}(-1)^n q^{n^2+2rn+\binom{r}{2}+2r}(1+q^{2n+1})x^r \\
&\qquad + \sum_{r.n\leq -1}(-1)^{-n-1} q^{n^2+2rn+\binom{r}{2}+2n+2r+1}(1+q^{-2n-1})x^r \\
&= \left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{2\binom{n}{2}+2rn+\binom{r}{2}}q^n(-xq^2)^r \\
&\qquad + q\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{2\binom{n}{2}+2rn+\binom{r}{2}}q^{3n}(-xq^2)^r \\
&= f_{2,2,1}(q,-xq^2,q)+qf_{2,2,1}(q^3,-xq^2,q),
\end{aligned}
$$

which is (1.11).

Substituting the Bailey pair (2.1) into (2.6) with $a=1$, we obtain

$$
\begin{aligned}
&\sum_{n=0}^{\infty} q^n(-xq)_n(-x^{-1})_n \\
&\quad = \frac{1}{J_1}\Bigg(\sum_{r\geq 1, n\geq 0}(-1)^n q^{3n(n+1)/2+(2n+1)r+\binom{r}{2}}(x^rq^r+x^{-r}) \\
&\qquad\qquad + \sum_{n\geq 0}(-1)^n q^{3n(n+1)/2}\Bigg) \\
&\quad = \frac{1}{J_1}\Bigg(\sum_{r,n\geq 0}(-1)^n q^{3n(n+1)/2+(2n+2)r+\binom{r}{2}}x^r \\
&\qquad\qquad + \sum_{r\geq 1, n\geq 0}(-1)^n q^{3n(n+1)/2+(2n+1)r+\binom{r}{2}}x^{-r}\Bigg) \\
&\quad = \frac{1}{J_1}\Bigg(\sum_{r,n\geq 0}(-1)^n q^{\frac{3n^2}{2}+\frac{3n}{2}+2nr+\binom{r}{2}+2r}x^r \\
&\qquad\qquad + \sum_{r,n\leq -1}(-1)^{-n-1} q^{\frac{3n^2}{2}+\frac{3n}{2}+2nr+\binom{r}{2}+2r}x^r\Bigg) \\
&\quad = \frac{1}{J_1}\left(\sum_{r,n\geq 0}-\sum_{r,n\leq -1}\right)(-1)^{n+r}q^{3\binom{n}{2}+2nr+\binom{r}{2}}q^{3n}(-xq^2)^r \\
&\quad = \frac{1}{J_1}f_{3,2,1}(q^3,-xq^2,q),
\end{aligned}
$$

which is (1.12).

Substituting the Bailey pair (2.1) into (2.7) with $a=1$, we deduce

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{q^n(-xq)_n(-x^{-1})_n}{(-q;q^2)_{n+1}}\\
&\quad=\frac{1}{(-q;q^2)_\infty(q)_\infty}\Bigg(\sum_{r\geq1,n\geq0}(-1)^nq^{2n^2+2n+2rn+r+\binom{r}{2}}(x^rq^r+x^{-r})\\
&\qquad\qquad+\sum_{n\geq0}(-1)^nq^{2n^2+2n}\Bigg)\\
&\quad=\frac{1}{(-q;q^2)_\infty(q)_\infty}\Bigg(\sum_{r,n\geq0}(-1)^nq^{2n^2+2nr+\binom{r}{2}+2n+2r}x^r\\
&\qquad\qquad+\sum_{r\geq1,n\geq0}(-1)^nq^{2n^2+2nr+\binom{r}{2}+2n+r}x^{-r}\Bigg)\\
&\quad=\frac{1}{(-q;q^2)_\infty(q)_\infty}\left(\sum_{r,n\geq0}-\sum_{r,n\leq-1}\right)(-1)^{n+r}q^{4\binom{n}{2}+2nr+\binom{r}{2}}q^{4n}(-xq^2)^r\\
&\quad=\frac{1}{(-q;q^2)_\infty(q)_\infty}f_{4,2,1}(q^4,-xq^2,q),
\end{aligned}$$

which is (1.13). This completes the proofs. □

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Li-Jun Hao
Zhejiang Sci-Tech University,
School of Science,
Hangzhou 310018,
P.R. China
haolijun152@163.com

Liang-Liang Xu
Zhejiang Sci-Tech University,
School of Science,
Hangzhou 310018,
P.R. China
liangliang_xu@163.com