NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 50 (2020), 29–48

# BEREZIN KARPELEVICH FORMULA FOR $\chi$-SPHERICAL FUNCTIONS ON COMPLEX GRASSMANNIANS

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(Received 30 July, 2019)

Abstract. In [**5**], Berezin and Karpelevich gave, without a proof, an explicit formula for spherical functions on complex Grassmannian manifolds. A first attempt to give a proof of Berezin-Karpelevich formula was taken, in [**16**], by Takahashi. His proof contained a gap, which was fixed later, in [**10**], by Hoogenboom. The aim of this paper is to generalize Berezin-Karpelevich formula to the case of $\chi$-spherical functions on complex Grassmannian manifolds $SU(p+q)/S\left(U(p)\times U(q)\right)$.

Contents



## 1. Introduction

Spherical functions lie at the heart of harmonic analysis on Gelfand pairs. Recall that for $G$ a locally compact group, $K$ a compact subgroup, the pair $(G,\,K)$ is called a Gelfand pair if $L^1\left(K\diagdown G\diagup K\right)$ is commutative under convolution. In the case where $G$ is a connected Lie group and $K$ a compact subroup, it is known from [**17**] that $(G,\,K)$ is a Gelfand pair if and only if the algebra $\mathcal{D}\left(G/K\right)$ of $G$-invariant differential operators on $G/K$ is commutative.

In this context, spherical functions are normalized joint eigenfunctions of the commutative algebras $\mathcal{D}\left(G/K\right)$ and they are in one to one correspondence with irreducible unitary representations of $G$ with a $K$-invariant vector.

Spherical functions appear also in the Peter-Weyl Theorem on $L^2\left(K\diagdown G\diagup K\right)$ and in the Placherel Theorem for spherical transform, which is a generalization of the Fourier transform to Gelfand pairs.

Given a Gelfand pair $(G,\,K)$, to find explicitly the corresponding spherical functions is a very difficult problem, which is widely open. In some special cases such as

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2010 *Mathematics Subject Classification* 43A85, 28C10, 43A77, 43A90, 53C35.
*Key words and phrases:* Spherical Functions, Hermitian Symmetric Spaces, Jacobi Function, Laplace-Beltrami Operator.

complex Grassmannians and their noncompact duals, an explicit formula is given (see [**5**], [**16**] and [**10**]).

In the case where $G/K$ is a symmetric space of noncompact type, Harish-Chandra gave an integral representation for spherical functions on the pair $(G, K)$.

Our main motivation for studying spherical functions came from the study of convolution of orbital measures on symmetric spaces. In a series of papers, [**3**], [**4**], [**1**], and [**2**], an explicit formula for spherical functions, or some estimates of those functions, was crucial in the study of the regularity of the Radon-Nikodym derivative of convolutions of orbital measures on symmetric spaces.

For a character $\chi : K \to \mathbb{C}$, a notion of $\chi$-spherical functions on $(G, K)$ is defined, and reduces to the usual notion of spherical functions in the case of a trivial character. For more details see Section 2 of this paper, [**15**], or [**7**].

The aim of this paper is to extend Hoogenboom's work, [**10**], to the case of $\chi$-spherical functions. To state our main result, let us fix some notations. Consider the non-compact symmetric space $G/K$, where $G = SU(p, q)$, $K = S\left(U(p) \times U(q)\right)$, $p, q$ are integers such that, $1 \leq q \leq p$, and let $\mathfrak{g}$ (resp. $\mathfrak{k}$) be the Lie algebra of the Lie group $G$ (resp. K), $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ a Cartan decomposition of $\mathfrak{g}$, $\mathfrak{u} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ be a compact real form of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$, $\mathfrak{a}$ a maximal abelian subspace of $\mathfrak{p}$, and $\mathfrak{a}^*$ its dual. For an integer $l$, let $\chi_l : K \longrightarrow \mathbb{C}$ be a character of $K$ defined as in section 2. Let $U$ be the group with lie algebra $\mathfrak{u}$, then $U = SU(p+q)$. Let $\Lambda_l^+$ be the set of highest restricted weights of $\chi_l$-spherical representations of $U$. By fixing a basis of $\mathfrak{a}$ we can identify $\mathfrak{a}$ with $\mathbb{R}^q$ and for an element $H_T = (t_1, \ldots, t_q) \in \mathfrak{a}$, we put $a_T = \exp(H_T)$. Our goal in this paper is to prove the following.

**Theorem 1.1.** *Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, where $\mathfrak{a}_{\mathbb{C}}^*$ is the complexification of $\mathfrak{a}^*$. The $\chi_l$-spherical function $\varphi_{\lambda,l}$ on $G = SU(p, q)$ is given by*

$$\begin{aligned}
&\varphi_{\lambda,l}(a_T) \\
&\quad = C \frac{\det\left[\left({}_2F_1\left(\frac{1}{2}\left(k+|l|+1+\lambda_i\right), \frac{1}{2}\left(k+|l|+1-\lambda_i\right), k+1; -\sinh^2(t_j)\right)\right)_{i,\,j}\right]}{2^{\frac{1}{2}q(q-1)} \prod_{i<j}(\lambda_i^2 - \lambda_j^2) \prod_{i<j}(\cosh(2t_i) - \cosh(2t_j))} \\
&\qquad \cdot \prod_{i=1}^{q} \cosh^{|l|}(t_i),
\end{aligned}$$

*where for $T = (t_1, \ldots, t_q) \in \mathbb{R}^q$, and*

$$C = 2^{2q(q-1)} \prod_{i=1}^{q-1} \left[(k+j)^{q-j} j!\right]$$

As a consequence of Theorem 1.1 we deduce the following.

**Theorem 1.2.** *Let $\lambda \in \Lambda_l^+$ such that*

$$\lambda = (2m_1 + |l|, 2m_2 + |l|, \ldots, 2m_q + |l|), m_i \in \mathbb{Z}, m_1 \geq m_2 \geq \cdots \geq m_q \geq 0.$$

*Then the $\chi_l$-spherical function $\psi_\lambda$ on $U = SU(p+q)$ is given by*

$$\psi_{\lambda,l}\left(\exp\left(\sqrt{-1}H_T\right)\right)$$
$$= \frac{C\det\left[\left({}_2F_1\left(p+|l|-i+1+m_i, i-m_i-q, k+1; \sin^2(t_j)\right)\right)_{i,j}\right]}{\prod_{i<j}(c(m_i)-c(m_j))\prod_{i<j}(\cos(2t_i)-\cos(2t_j))}$$
$$\cdot\prod_{i=1}^{q}\cos^{|l|}(t_i)$$

*where*

$$C = 2^{\frac{1}{2}q(q-1)}\prod_{i=1}^{q-1}\left[(k+j)^{q-j}j!\right],$$

*and*

$$c(m_i) = (m_i+q-i)(m_i+q-i+|l|+k+1).$$

The paper is organized as follows. In Section 2 we introduce some notations and structural results on $\chi$-spherical functions on Hermitian symmetric spaces and we review basic facts about complex Grassmannians. In Section 3, we give an explicit formula for the radial part of the Laplace-Beltrami operator on $SU(p,q)/S(U(p)\times U(q))$. In Section 4 we give a proof of the Theorem 1.1. In Section 5 we review some facts about $\chi$-spherical representations on Hermitian symmetric spaces and we state Theorem 1.2.

## 2. Preliminary results

Let $G$ be a locally compact Hausdorff group and $K \subseteq G$ a compact subgroup. Let $\chi : K \longrightarrow \mathbb{C}$ be a character of $K$. A continuous function $\varphi : G \longrightarrow \mathbb{C}$ is called $\chi$-spherical function or elementary $\chi$-spherical function if $\varphi$ is not identically zero and for any $h, g \in H$,

$$\int_K \varphi(gkh)\chi(k)d\mu_L(k) = \varphi(g)\varphi(h),$$

where $\mu_K$ is the normalized haar measure on $K$. For $G$ a non-compact, connected, simply connected, and semisimple Lie group, let $\mathfrak{g}$ be its Lie algebra and

$$\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p},$$

be a Cartan decomposition of $\mathfrak{g}$. Then

$$\mathfrak{u} = \mathfrak{k}\oplus\sqrt{-1}\mathfrak{p},$$

is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, the complexification of $\mathfrak{g}$. Let $U$, $K$ be Lie groups with Lie algebras $\mathfrak{u}, \mathfrak{k}$, respectively. Denote by $\mathfrak{z}(\mathfrak{k})$ and $Z(K)$ the centers of $\mathfrak{k}$ and $K$ respectevely. Let $[K,K]$ be the commutator subgroup of $K$, i.e., the subgroup generated by the set

$$\{aba^{-1}b^{-1} \mid a, b \in K\}.$$

Then $[K,K]$ is the Lie subgroup $K$ with Lie algebra $[\mathfrak{k},\mathfrak{k}]$. Moreover, we have

$$[\mathfrak{k},\mathfrak{k}]\oplus\mathfrak{z}(\mathfrak{k}) = \mathfrak{k} \qquad \text{and} \qquad [K,K]Z(K) = K.$$

Let $H \in \mathfrak{z}(\mathfrak{k})$ be a non-zero element such that $\exp(tH) \in Z(K)$ and $\exp(tH) \in [K,K]$ if and only if $t \in 2\pi\mathbb{Z}$. For an integer $l$, let $\chi_l : K \longrightarrow \mathbb{C}$ be a character of K such that

$$\chi_l(x) = \begin{cases} 1 & \text{if } x \in [K,\, K]; \\ \\ \exp(itl) & \text{if } x = \exp(tH). \end{cases}$$

It is known that every character of $K$ is of the form $\chi_l$ for some integer $l$ (see [**12**], Proposition 3.4). In all what follows, we will assume that $\chi_l$ is not trivial i.e., $\mathfrak{z}(\mathfrak{k}) \neq \{0\}$. For $g \in G$, let $\kappa(g) \in K$ and $H(g) \in \mathfrak{a}$ be elements uniquely determined by the Iwasawa decomposition $G = KAN$, i.e., $g \in \kappa(g) \exp H(g) N$. For $l \in \mathbb{Z}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ let

$$\varphi_{\lambda,l}(g) = \int_K e^{-(\lambda+\rho)(H(g^{-1}k))} \chi_l(k^{-1}\kappa(g^{-1}k))\, d\mu_K(k).$$

The function $\varphi_{\lambda,l}$ is a $\chi_l$-spherical function and any $\chi_l$-spherical function on $G/K$ is of this form for some $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ (see [**14**], Proposition 6.1).

In what follows, we consider the non-compact Lie group

$$G = SU(p,q) = \{g \in SL(p+q, \mathbb{C}) \mid g^* I_{p,q} g = I_{p,q}\},$$

where $I_k$ is the $k \times k$-identity matrix and

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

Let $\mathfrak{g} = \mathfrak{su}(p,q)$ be the Lie algebra of $SU(p,q)$, then for $n = p+q$,

$$\mathfrak{su}(p,q) = \{A \in M_n(\mathbb{C}) \mid A^* I_{p,q} + I_{p,q} A = 0, \mathrm{Tr}(A) = 0\}.$$

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where

$$\mathfrak{k} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathfrak{u}(p),\ B \in \mathfrak{u}(q) \ \text{ and } \ \mathrm{Tr}(A) + \mathrm{Tr}(B) = 0 \right\},$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Z \\ \overline{Z}^T & 0 \end{pmatrix} \mid Z \in M_{p,q}(\mathbb{C}) \right\}.$$

Let $\mathfrak{u} = \mathfrak{k} + \sqrt{-1}\mathfrak{p}$ be a compact real form of $\mathfrak{sl}(n,\mathbb{C})$, the complexification of $\mathfrak{su}(p,q)$. Thus $U = SU(n)$ is the lie group with lie algebra $\mathfrak{u}$. Let $T = (t_1, t_2, \dots, t_q) \in \mathbb{R}^q$, and let

$$H_T = \left( \begin{array}{c:c:c} 0 & 0 & \begin{matrix} & & t_1 \\ & \iddots & \\ t_q & & \end{matrix} \\ \hdashline 0 & 0 & 0 \\ \hdashline \begin{matrix} & & t_q \\ & \iddots & \\ t_1 & & \end{matrix} & 0 & 0 \end{array} \right).$$

It can be shown that $\mathfrak{a} = \{H_T \mid T = (t_1, t_2, \dots, t_q) \in \mathbb{R}^q\}$ is a maximal abelian subspace of $\mathfrak{p}$. Let $\mathfrak{a}^*$ denote the dual space of $\mathfrak{a}$ and let $\alpha_i \in \mathfrak{a}^*$ be such that

$$\alpha_i \left(H_{(t_1,\dots,t_q)}\right) = t_i.$$

Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a})$, be the set of restricted roots which consist of

$$\pm\alpha_i, \pm 2\alpha_i, (1 \le i \le q), \text{ and } \pm(\alpha_i \pm \alpha_j), (1 \le i < j \le q),$$

with multiplicities

$$m_{\alpha_i} = 2k, \;\; m_{2\alpha_i} = 1, \text{ and } m_{\alpha_i \pm \alpha_j} = 2,$$

where $k = p - q$. Let $\mathfrak{a}^+$ be a Weyl chamber in $\mathfrak{a}$ defined by

$$\mathfrak{a}^+ = \left\{H_{(t_1,\dots,t_q)} \in \mathfrak{a} \mid t_1 > t_2 > \dots > t_q > 0\right\}.$$

The corresponding system of positive restricted roots $\Sigma^+$ consists of

$$\alpha_i, 2\alpha_i, (1 \le i \le q), \text{ and } (\alpha_i \pm \alpha_j), (1 \le i < j \le q).$$

With respect to $\mathfrak{a}^+$ the simple roots $\Lambda^+$ consist of the following

$$\alpha_1 - \alpha_2, \alpha_2 - \alpha_3, \dots, \alpha_{q-1} - \alpha_q, \;\beta_q,$$

where

$$\beta_q = \begin{cases} 2\alpha_q & \text{if } p = q, \\ \alpha_q & \text{if } p > q. \end{cases}$$

For $T = (t_1, \dots, t_q)$, $a_T = \exp H_T$ and $l \in \mathbb{Z}$, put

$$\mathcal{O}_s^+ = \{\alpha_i\}_{i=1}^q, \qquad \mathcal{O}_{s'}^+ = \{2\alpha_i\}_{i=1}^q, \qquad \mathcal{O}_m^+ = \{\alpha_i \pm \alpha_j\}_{1 \le i < j \le q},$$
$$m_{\alpha_i}(l) = m_{\alpha_i} - 2|l|, \quad m_{2\alpha_i}(l) = m_{2\alpha_i} + 2|l|, \quad m_{\alpha_i \pm \alpha_j}(l) = m_{\alpha_i \pm \alpha_j},$$

and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha, \qquad \rho(l) = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha(l) \alpha.$$

Then

$$\rho = \sum_{i=1}^q (k + 1 + 2(q - i)) \alpha_i,$$

and

$$\rho(l) = \rho + |l| \sum_{\alpha \in \mathcal{O}_s^+} \alpha = \sum_{i=1}^q (k + |l| + 1 + 2(q - i)) \alpha_i.$$

## 3. The radial part of the Laplace-Beltrami operator

For $X = G/K$ a non-compact symmetric space, let $\Delta_l(\mathcal{L}_X)$ be the $\chi_l$-radial part of the Laplace-Beltrami Operator on $X$. Then

$$\Delta_l(\mathcal{L}_X) = \mathcal{L}_A + \sum_{\alpha\in\Sigma^+} m_\alpha(l)\coth(\alpha)A_\alpha,$$

where $\mathcal{L}_A$ is the Laplacian of $A.o$, where $o = eK$, where $e$ is the identity of $G$, and $A_\alpha \in \mathfrak{a}$ is determined by

$$\langle A_\alpha, H\rangle = \alpha(H), \text{ for all } H \in \mathfrak{a}.$$

The goal of this section is to prove the following

**Proposition 3.1.** *For $X = SU(p,q)/S(u(p)\times U(q))$, we have*

$$\Delta_l(\mathcal{L}_X) = \frac{1}{4n}\left[\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\circ\omega - c_l\right],$$

*where*

$$\omega(a_T) = 2^{-\frac{1}{2}q(q-1)}\prod_{i<j}\left(\cosh(2t_i)-\cosh(2t_j)\right),$$

$$\mathcal{L}_{i,l} = \frac{\partial^2}{\partial t_i^2} + \left(2k\coth(t_i) + 2\coth(2t_i) + 2\,|l|\tanh(t_i)\right)\frac{\partial}{\partial t_i},$$

*and*

$$c_l = q(q-1)\left(\frac{4}{3}(q+1) + 2(k+|l|)\right).$$

To prove Proposition 3.1 we need the following

**Lemma 3.2.** *Let $a_1,\dots a_d \in \mathbb{R}$ be such that $a_i \neq a_j$ for all $i \neq j$. Thus for $d \geq 3$, we have*

$$\sum_{i=1}^{d}\left[[a_i + a_i^2]\sum_{\substack{j=1\\ j\neq i}}^{d-1}\frac{1}{a_i - a_j}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{d}\frac{1}{a_i - a_s}\right)\right] = \frac{1}{6}d(d-1)(d-2). \tag{3.1}$$

*Proof.* From the identity

$$\frac{a_i + a_i^2}{(a_i - a_j)(a_i - a_s)} + \frac{a_j + a_j^2}{(a_j - a_i)(a_j - a_s)} + \frac{a_s + a_s^2}{(a_s - a_i)(a_s - a_j)} = 1$$

where $a_i \neq a_j$ for all $i \neq j$, we deduce that for $d = 3$ we have

$$\begin{aligned}
&\sum_{i=1}^{3}\left[[a_i + a_i^2]\sum_{\substack{j=1\\ j\neq i}}^{2}\frac{1}{a_i - a_j}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{3}\frac{1}{a_i - a_s}\right)\right]\\
&\qquad = \frac{a_1 + a_1^2}{(a_1 - a_2)(a_1 - a_3)} + \frac{a_2 + a_2^2}{(a_2 - a_1)(a_2 - a_3)} + \frac{a_3 + a_3^2}{(a_3 - a_1)(a_3 - a_2)}\\
&\qquad = 1.
\end{aligned}$$

Assume that formula (3.1) is true for some $d \geq 3$, then

$$\begin{aligned}
&\sum_{i=1}^{d+1}\left[[a_i+a_i^2]\sum_{\substack{j=1\\ j\neq i}}^{d}\frac{1}{a_i-a_j}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{d+1}\frac{1}{a_i-a_s}\right)\right]\\
&=\sum_{i=1}^{d}\left[[a_i+a_i^2]\sum_{\substack{j=1\\ j\neq i}}^{d-1}\frac{1}{a_i-a_j}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{d}\frac{1}{a_i-a_s}\right)\right]\\
&\qquad+\sum_{i=1}^{d}\left[[a_i+a_i^2]\sum_{\substack{j=1\\ j\neq i}}^{d}\frac{1}{a_i-a_j}\left(\frac{1}{a_i-a_{d+1}}\right)\right]\\
&\qquad+(a_{d+1}+a_{d+1}^2)\sum_{j=1}^{d}\frac{1}{a_{d+1}-a_j}\left(\sum_{s=j+1}^{d}\frac{1}{a_{d+1}-a_s}\right)\\
&=\frac{1}{6}d(d-1)(d-2)+\sum_{i<j}\frac{a_i+a_i^2}{(a_i-a_j)(a_i-a_{d+1})}\\
&\qquad+\sum_{i<j}\frac{a_j+a_j^2}{(a_j-a_i)(a_j-a_{d+1})}+\sum_{i<j}\frac{a_{d+1}+a_{d+1}^2}{(a_{d+1}-a_i)(a_{d+1}-a_j)}\\
&=\frac{1}{6}d(d-1)(d-2)+\sum_{i<j}\left(\frac{a_i+a_i^2}{(a_i-a_j)(a_i-a_{d+1})}\right.\\
&\qquad\left.+\frac{a_j+a_j^2}{(a_j-a_i)(a_j-a_{d+1})}+\frac{a_{d+1}+a_{d+1}^2}{(a_{d+1}-a_i)(a_{d+1}-a_j)}\right)\\
&=\frac{1}{6}d(d-1)(d-2)+\frac{d(d-1)}{2}\\
&=\frac{1}{6}d(d+1)(d-1).
\end{aligned}$$

□

**Lemma 3.3.**

$$\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\omega = q(q-1)\left(\frac{4}{3}(q+1)+2(k+|l|)\right).$$

*Proof.* Note that

$$\omega^{-1}\frac{\partial\omega}{\partial t_i}=\sum_{\substack{j=1\\ j\neq i}}^{q}\frac{2\sinh(2t_i)}{\cosh(2t_i)-\cosh(2t_j)},$$

and

$$\omega^{-1}\frac{\partial^2\omega}{\partial t_i^2}=\sum_{\substack{j=1\\ j\neq i}}^{q}\frac{4\cosh(2t_i)}{\cosh(2t_i)-\cosh(2t_j)}$$
$$+8\sinh^2(2t_i)\sum_{\substack{j=1\\ j\neq i}}^{q-1}\frac{1}{\cosh(2t_i)-\cosh(2t_j)}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{q}\frac{1}{\cosh(2t_i)-\cosh(2t_s)}\right).$$

Then

$$\begin{aligned}&\sum_{i=1}^{q}\left[\sum_{\substack{j=1\\ j\neq i}}^{q}\frac{4\cosh(2t_i)}{\cosh(2t_i)-\cosh(2t_j)}+(2k\coth(t_i)+2\coth(2t_i)+2\,|l|\tanh(t_i))\,\omega^{-1}\frac{\partial\omega}{\partial t_i}\right]\\&=4\sum_{i=1}^{q}\sum_{\substack{j=1\\ j\neq i}}^{q}\frac{(2+2\,|l|)\cosh(2t_i)+2(k-|l|)\cosh^2(t_i)}{\cosh(2t_i)-\cosh(2t_j)}\\&=2q(q-1)(k+|l|+2).\end{aligned}\tag{3.2}$$

By Lemma 3.2, we get

$$\begin{aligned}&\sum_{i=1}^{q}\left[\sinh^2(2t_i)\sum_{\substack{j=1\\ j\neq i}}^{q-1}\frac{1}{\cosh(2t_i)-\cosh(2t_j)}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{q}\frac{1}{\cosh(2t_i)-\cosh(2t_s)}\right)\right]\\&=\sum_{i=1}^{q}\left[(\sinh^2(t_i)+\sinh^4(t_i))\sum_{\substack{j=1\\ j\neq i}}^{q-1}\frac{1}{\sinh^2(t_i)-\sinh^2(t_j)}\left(\sum_{\substack{s=j+1\\ s\neq i}}^{q}\frac{1}{\sinh^2(t_i)-\sinh(t_s)}\right)\right]\\&=\frac{1}{6}q(q-1)(q-2).\end{aligned}\tag{3.3}$$

The lemma follows from (3.2) and (3.3). □

**Proof of Proposition 3.1.** Let $\{H_i\}_{i=1}^{q}$ be a basis of $\mathfrak{a}$, $H_j=H_{(0,\ldots,0,1,0,\ldots,0)}$, where 1 is on the $j$-th position. Note that

$$A_{\alpha_i}=\frac{1}{4n}H_i,\qquad A_{2\alpha_i}=\frac{1}{2n}H_i,\qquad\text{and}\qquad A_{(\alpha_i\pm\alpha_j)}=\frac{1}{4n}(H_i\pm H_j).$$

So

$$\begin{aligned}
\sum_{i=1}^{q}\omega^{-1}\frac{\partial\omega}{\partial t_i}\frac{\partial}{\partial t_i} &= \sum_{i=1}^{q}\left[2\sinh(2t_i)\sum_{\substack{1\le j\le q\\ j\neq i}}\frac{1}{\cosh(2t_i)-\cosh(2t_j)}\right]\frac{\partial}{\partial t_i}\\
&= \sum_{i<j}\left(2\frac{\sinh(2t_i)}{\cosh(2t_i)-\cosh(2t_j)}\frac{\partial}{\partial t_i}-2\frac{\sinh(2t_j)}{\cosh(2t_i)-\cosh(2t_j)}\frac{\partial}{\partial t_j}\right)\\
&= \sum_{i<j}\left(\frac{\sinh(2t_i)}{\sinh(t_i-t_j)\sinh(t_i+t_j)}\frac{\partial}{\partial t_i}-\frac{\sinh(2t_j)}{\sinh(t_i-t_j)\sinh(t_i+t_j)}\frac{\partial}{\partial t_j}\right)\\
&= \sum_{i<j}\coth(t_i-t_j)\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_j}\right)+\sum_{i<j}\coth(t_i+t_j)\left(\frac{\partial}{\partial t_i}+\frac{\partial}{\partial t_j}\right).
\end{aligned}$$

Thus by definition of $\Delta_l$ we have

$$\begin{aligned}
4n\Delta_l(\mathcal{L}_X) &= 4n\mathcal{L}_A+4n\sum_{\alpha\in\Sigma^+}m_\alpha(l)\coth(\alpha)A_\alpha\\
&= \sum_{j=1}^{q}\frac{\partial^2}{\partial t_j^2}+\sum_{j=1}^{q}(2(k-|l|)\coth(t_j)+2(1+2\,|l|)\coth(2t_j))\frac{\partial}{\partial t_j}\\
&\quad+\sum_{i<j}2\coth(t_i-t_j)\left(\frac{\partial}{\partial t_i}-\frac{\partial}{\partial t_j}\right)+\sum_{i<j}2\coth(t_i+t_j)\left(\frac{\partial}{\partial t_i}+\frac{\partial}{\partial t_j}\right)\\
&= \sum_{i=1}^{q}\left[\frac{\partial^2}{\partial t_i^2}+2\left((k-|l|)\coth(t_i)+(1+2\,|l|)\coth(2t_i)+\omega^{-1}\frac{\partial\omega}{\partial t_i}\right)\frac{\partial}{\partial t_i}\right]\\
&= \sum_{i=1}^{q}\left[\frac{\partial^2}{\partial t_i^2}+2\left(k\coth(t_i)+\coth(2t_i)+|l|\tanh(t_i))+\omega^{-1}\frac{\partial\omega}{\partial t_i}\right)\frac{\partial}{\partial t_i}\right]\\
&= \sum_{i=1}^{q}\left[\mathcal{L}_{i,l}+2\omega^{-1}\frac{\partial\omega}{\partial t_i}\frac{\partial}{\partial t_i}\right].
\end{aligned}$$

For $f\in C^2(\mathbb{R})$, we have

$$\begin{aligned}
\mathcal{L}_{i,l}\circ\omega f &= \left(\frac{\partial^2}{\partial t_i^2}+2((k-|l|)\coth(t_i)+(1+2\,|l|)\coth(2t_i))\frac{\partial}{\partial t_i}\right)(\omega f)\\
&= \left(2\frac{\partial\omega}{\partial t_i}\frac{\partial}{\partial t_i}+\omega\mathcal{L}_{i,l}+\mathcal{L}_{i,l}\omega\right)f.
\end{aligned}$$

Hence

$$\omega^{-1}\left(\mathcal{L}_{i,l}\circ\omega\right)-\omega^{-1}\mathcal{L}_{i,l}\omega=\mathcal{L}_{i,l}+2\omega^{-1}\frac{\partial\omega}{\partial t_i}\frac{\partial}{\partial t_i}. \tag{3.4}$$

Therefore by using (3.4)) we have

$$\begin{aligned}4n\Delta_l(\mathcal{L}_X) &= \sum_{i=1}^{q}\left[\mathcal{L}_{i,l} + 2\omega^{-1}\frac{\partial\omega}{\partial t_i}\frac{\partial}{\partial t_i}\right]\\ &= \sum_{i=1}^{q}\omega^{-1}\mathcal{L}_{i,l}\circ\omega - \sum_{i=1}^{q}\omega^{-1}\mathcal{L}_{i,l}\omega\end{aligned}$$

The Proposition follows from Lemma 3.3. □

## 4. Proof of Theorem 1.1

The aim of this section is to give a proof of Theorem 1.1 and to prove it the following is needed.

**Lemma 4.1.** *Let $\alpha, \beta, \lambda \in \mathbb{C}$, $\lambda \neq 1, 2, \dots$, $t > 0$. Then*

$$\begin{aligned}\Phi_\lambda^{\alpha,\beta}(t) &= \left(e^t - e^{-t}\right)^{\lambda-(\alpha+\beta+1)}\\ &\cdot {}_2F_1\left(\frac{1}{2}(-\alpha+\beta+1-\lambda), \frac{1}{2}(\alpha+\beta+1-\lambda), 1-\lambda; -\sinh^{-2}(t)\right).\end{aligned}$$

*is a solution of*

$$\delta_{\alpha,\beta}^{-1}(t)\frac{d}{dt}\left\{\delta_{\alpha,\beta}(t)\frac{du(t)}{dt}\right\} = (\lambda^2 - (\alpha+\beta+1)^2)u(t), \tag{4.1}$$

*where ${}_2F_1\,(.\,,\,.,.;.)$ is the Gauss hypergeometric function and*

$$\begin{aligned}\delta_{\alpha,\beta}(t) &= \left(e^t - e^{-t}\right)^{2\alpha+1}\left(e^t + e^{-t}\right)^{2\beta+1}\\ &= 2^{2(\alpha+\beta+1)}\sinh^{2\alpha+1}(t)\cosh^{2\beta+1}(t).\end{aligned}$$

*Proof.* The differential equation (4.1) can be written as follows

$$\begin{aligned}\frac{d^2u(t)}{dt^2} &+ \left[(2\alpha+1)\cosh(t)\sinh^{-1}(t) + (2\beta+1)\cosh^{-1}(t)\sinh(t)\right]\frac{du(t)}{dt}\\ &= (\lambda^2 - (\alpha+\beta+1)^2)u(t).\end{aligned}$$

Let $a = \frac{1}{2}(\alpha+\beta+1-\lambda)$, $b = \frac{1}{2}(\alpha+\beta+1+\lambda)$, $c = \alpha+1$, and $z = -\sinh^2(t)$. Then

$$\begin{aligned}(ab)u &= -\frac{1}{4}\left((\lambda^2 - (\alpha+\beta+1)^2)u\right)\\ &= -\frac{1}{4}\left(\frac{d^2u}{dt^2} + \left[(2\alpha+1)\cosh(t)\sinh^{-1}(t) + (2\beta+1)\cosh^{-1}(t)\sinh(t)\right]\frac{du}{dt}\right)\\ &= z(1-z)\frac{d^2u}{dz^2} + \frac{1}{2}(1-2z)\frac{du}{dz} + \frac{1}{2}\left[(2\alpha+1)\cosh^2(t) + (2\beta+1)\sinh^2(t)\right]\frac{du}{dz}\\ &= z(1-z)\frac{d^2u}{dz^2} + (c - (a+b+1)z)\frac{du}{dz}.\end{aligned}$$

By [**6**] Equation 2.9(9) the result follows. □

Since

$$\begin{aligned}\Phi_{\lambda_i,l} &= \Phi^{k,|l|}_{\lambda_i}\\ &= (e^t - e^{-t})^{\lambda-(k+|l|+1)}\\ &\quad \cdot {}_2F_1(\frac{1}{2}(-k+|l|+1-\lambda), \frac{1}{2}(k+|l|+1-\lambda), 1-\lambda; -\sinh^{-2}(t)),\end{aligned}$$

is a solution of

$$\delta^{-1}_{k,|l|}(t)\frac{\partial}{\partial t_i}\left\{\delta_{k,|l|}(t_i)\frac{\partial u(t)}{\partial t}\right\} = (\lambda_i^2 - (k+|l|+1)^2)u(t),$$

and since

$$\begin{aligned}\delta^{-1}_{k,|l|}(t)\frac{\partial}{\partial t}\left\{\delta_{k,|l|}(t)\frac{\partial u}{\partial t}\right\} &= \frac{\partial^2 u}{\partial t^2} + ((2\,|l|+1)\tanh(t) + (2k+1)\coth(t))\frac{\partial u}{\partial t}\\ &= \frac{\partial^2 u}{\partial t^2} + 2(k\coth(t) + \coth(2t) + |l|\tanh(t))\frac{\partial u}{\partial t}\\ &= \mathcal{L}_{i,l}u,\end{aligned}$$

we deduce that $\Phi_{\lambda_i,l}$ is an eigenvector of the operator $\mathcal{L}_{i,l}$ and the corresponding eigenvalue is $(\lambda_i^2 - (k+|l|+1)^2)$, i.e.,

$$\mathcal{L}_{i,l}\Phi_{\lambda_i,l} = (\lambda_i^2 - (k+|l|+1)^2)\Phi_{\lambda_i,l}.$$

For $H_T \in \mathfrak{a}^+$, $\lambda = (\lambda_1,\ldots,\lambda_q) \in \mathfrak{a}^*_{\mathbb{C}}$, $l \in \mathbb{Z}$ and $\lambda_i \neq 1,2,3,\ldots$, put

$$\Phi_{\lambda,l}(a_T) = \frac{\prod_{i=1}^q \Phi_{\lambda_i,l}(t_i)}{\omega(a_T)}, \tag{4.2}$$

and

$$\Lambda = \{\sum_{j=1}^s n_j\alpha_j \mid n_j \in \mathbb{Z}^+, \alpha_j \in \Sigma\}.$$

**Theorem 4.2.** *The function $\Phi_{\lambda,l}$, defined in* (4.2) *has the following properties*

(1) *$\Phi_{\lambda,l}$ satisfies*

$$\Delta_l(\mathcal{L}_X)\Phi_{\lambda,l} = (\langle\lambda,\lambda\rangle - \langle\rho(l),\rho(l)\rangle)\Phi_{\lambda,l}.$$

(2) *$\Phi_{\lambda,l}$ has the series expansion*

$$\Phi_{\lambda,l}(a_T) = e^{(\lambda-\rho(l))(T)}\sum_{\mu\in\Lambda}\Gamma_{\mu,l}(\lambda)e^{-\mu(T)},$$

*where $H_T \in \mathfrak{a}^+$, and $(\Gamma_{\mu,l})_{\mu\in\Lambda}$ is defined by the recurrence relation*

$$\begin{cases}\Gamma_{0,l} = 1,\\ \\ \langle\mu,\mu-2\lambda\rangle\Gamma_{\mu,l} = 2\sum_{\alpha\in\Sigma^+}(m(l))_\alpha\sum_{k\in\mathbb{N}}\Gamma_{\mu-2k\alpha,l}(\lambda)\langle\mu+\rho(l)-2k\alpha-\lambda,\alpha\rangle.\end{cases}$$

Let

$$\Omega^l_{\lambda_i,s} = \begin{cases}0, & \text{if } s \text{ is odd}\\ \\ \sum_{j=0}^{s/2}\gamma_{\lambda_i,j}\binom{-2j+\lambda_i-(k+|l|+1)}{s/2-j}, & \text{if } s \text{ is even}\end{cases} \tag{4.3}$$

where

$$\gamma_{\lambda_i,j} = \frac{2^{2j}\left(\left(\frac{1}{2}(k+|l|+1-\lambda_i)\right)_j\right)\left(\left(\frac{1}{2}(k-|l|+1-\lambda_i)\right)_j\right)}{(1-\lambda_i)_j\, j!}.$$

To prove Theorem 4.2 we need the following.

**Lemma 4.3.** *For $t>0$, $\Phi_{\lambda_i,l}$ has a convergent series expansion*

$$\Phi_{\lambda_i,l}(t) = e^{(\lambda_i-(k+|l|+1))t}\sum_{s=0}^{\infty}\Omega^l_{\lambda_i,s}e^{-st},$$

*where $\Omega^l_{\lambda_i,s}$ is as in* (4.3).

*Proof.* Using the identity

$${}_2F_1(a,b,c;z) = (1-z)^{-b}{}_2F_1\left(b, c-a, c; \frac{z}{z-1}\right),$$

we deduce that

$$\begin{aligned}
&\Phi_{\lambda_i,l}(t)\\
&= \left(e^t - e^{-t}\right)^{\lambda_i-(k+|l|+1)}{}_2F_1\left(\frac{1}{2}\left(-k+|l|+1-\lambda_i\right), \frac{1}{2}\left(k+|l|+1-\lambda_i\right), 1-\lambda_i; -\sinh^{-2}(t)\right)\\
&= \left[\left(e^t-e^{-t}\right)^2\left(1+\sinh^{-2}(t)\right)\right]^{\frac{1}{2}(\lambda_i-(k+|l|+1))}\\
&\qquad\cdot {}_2F_1\left(\frac{1}{2}\left(k+|l|+1-\lambda_i\right), \frac{1}{2}\left(k-|l|+1-\lambda_i\right), 1-\lambda_i; \frac{\sinh^{-2}(t)}{\sinh^{-2}(t)+1}\right)\\
&= \left(e^t+e^{-t}\right)^{\lambda_i-(k+|l|+1)}{}_2F_1\left(\frac{1}{2}\left(k+|l|+1-\lambda_i\right), \frac{1}{2}\left(k-|l|+1-\lambda_i\right), 1-\lambda_i; \cosh^{-2}(t)\right)\\
&= e^{(\lambda_i-(k+|l|+1))t}\left(1+e^{-2t}\right)^{\lambda_i-(k+|l|+1)}\\
&\qquad\cdot\sum_{j=0}^{\infty}\frac{\left(\left(\frac{1}{2}(k+|l|+1-\lambda_i)\right)_j\right)\left(\left(\frac{1}{2}(k-|l|+1-\lambda_i)\right)_j\right)}{(1-\lambda_i)_j\, j!}(\cosh^{-2}(t))^j
\end{aligned} \tag{4.4}$$

Since for $t\neq 0$, $0<\cosh^{-2}t<1$, and since $\lambda_i\neq 1,2,...$, the series (4.4) is absolutely convergent. Therefore, we have

$$\begin{aligned}
\Phi_{\lambda_i,l}(t) &= e^{(\lambda_i-(k+|l|+1))t}\left(1+e^{-2t}\right)^{\lambda_i-(k+|l|+1)}\sum_{j=0}^{\infty}\gamma_{\lambda_i,j}(e^t+e^{-t})^{-2j}\\
&= e^{(\lambda_i-(k+|l|+1))t}\sum_{j=0}^{\infty}\gamma_{\lambda_i,j}e^{-2jt}\left(1+e^{-2t}\right)^{-2j+\lambda_i-(k+|l|+1)}\\
&= e^{(\lambda_i-(k+|l|+1))t}\sum_{j=0}^{\infty}\gamma_{\lambda_i,j}e^{-2jt}\sum_{s=0}^{\infty}\binom{-2j+\lambda_i-(k+|l|+1)}{s}e^{-2st}\\
&= \sum_{s=0}^{\infty}\left(\sum_{j=0}^{s}\gamma_{\lambda_i,j}\binom{-2j+\lambda_i-(k+|l|+1)}{s-j}\right)e^{-2st}.
\end{aligned}$$

□

**Proof of Theorem 4.2.** (1) Denote by $\langle .,. \rangle$, the inner product induced by the Killing form of $\mathfrak{su}(p,q)$ on $\mathfrak{a}_{\mathbb{C}}^*$. Then for $\lambda = (\lambda_1, \dots, \lambda_q)$, $\mu = (\mu_1, \dots, \mu_q) \in \mathfrak{a}_{\mathbb{C}}^*$

$$\langle \lambda, \mu \rangle = 4n \sum_{i=1}^{q} \lambda_i \mu_i .$$

So we have

$$\begin{aligned}
&\left[\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\circ\omega\right]\Phi_{\lambda,l}(a_T)\\
&\quad=\left[\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\circ\omega\right]\left(\omega^{-1}\prod_{i=1}^{q}\Phi_{\lambda_i,l}(t_i)\right)\\
&\quad=\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\left(\prod_{i=1}^{q}\Phi_{\lambda_i,l}(t_i)\right)\\
&\quad=\omega^{-1}\left(\sum_{j=1}^{q}\mathcal{L}_{j,l}(\Phi_{\lambda_j,l}(t_j))\left(\prod_{\substack{i=1\\ i\neq j}}^{q}\Phi_{\lambda_i,l}(t_i)\right)\right)\\
&\quad=\omega^{-1}\left(\sum_{j=1}^{q}(\lambda_j^2-(k+|l|+1)^2)\Phi_{\lambda_i,l}(t_i)\left(\prod_{\substack{i=1\\ i\neq j}}^{q}\Phi_{\lambda_i,l}(t_i)\right)\right)\\
&\quad=\left(\sum_{j=1}^{q}(\lambda_j^2-(k+|l|+1)^2)\right)\omega^{-1}\prod_{i=1}^{q}\Phi_{\lambda_i,l}(t_i)\\
&\quad=\big(4n\langle\lambda,\lambda\rangle-q(k+|l|+1)^2\big)\Phi_{\lambda,l}(a_T).
\end{aligned}$$

Since

$$\begin{aligned}
4n\langle\rho(l),\rho(l)\rangle &= \sum_{i=1}^{q}(k+|l|+1+2(q-i))^2\\
&=\sum_{i=1}^{q}\big((k+|l|+1)^2+4(q-i)^2+4(k+|l|+1)(q-i)\big)\\
&=q(k+|l|+1)^2+\sum_{i=1}^{q-1}4i(i+k+|l|+1)\\
&\qquad+4\sum_{i=1}^{q-1}\Big[(q^2-2qi)+((k+|l|)q-2(k+|l|)i)+(q-2i)\Big]\\
&=q(k+|l|+1)^2+c_l,
\end{aligned}$$

we get

$$\begin{aligned}
4n\Delta(\mathcal{L}_X)\Phi_{\lambda,l} &= \left(\omega^{-1}\left(\sum_{i=1}^{q}\mathcal{L}_{i,l}\right)\circ\omega-c_l\right)\Phi_{\lambda,l}\\
&=4n\Big(\langle\lambda,\lambda\rangle-\langle\rho(l),\rho(l)\rangle\Big)\Phi_{\lambda,l}.
\end{aligned}$$

(2) Since $a_T \in A^+$ and $t_i - t_j > 0$ for all $i < j$ with

$$\frac{1}{1 - e^{-2(t_i - t_j)}} = \sum_{r=0}^{\infty} e^{-2r(t_i - t_j)},$$

so

$$\begin{aligned}
\Phi_{\lambda,l}(a_T) &= \frac{\prod_{i=1}^{q} \Phi_{\lambda_i,l}(t_i)}{\omega(a_T)} \\
&= \frac{e^{\sum_{i=1}^{q}(\lambda_i - (k+|l|+1))t_i} \prod_{i=1}^{q} \left[\sum_{s=1}^{\infty} \Omega_{\lambda_i,s}^{l} e^{-st_i}\right]}{e^{2(n-1)t_1 + 2(n-2)t_2 + \cdots + 2t_{n-1}} \prod_{i<j} \left(1 - e^{-2(t_i - t_j)}\right)\left(1 - e^{-2(t_i + 2t_j)}\right)} \\
&= e^{\sum_{i=1}^{\infty}(\lambda_i - (k+|l|+1) - 2(q-i))t_i} \prod_{i=1}^{q} \left[\sum_{s_i=0}^{\infty} \Omega_{\lambda_i,s}^{l} e^{-2s_i t_i}\right] \\
&\quad \cdot \prod_{i<j} \left[\sum_{r=0}^{\infty} e^{-2r(t_i - t_j)} \sum_{s=0}^{\infty} e^{-2s(t_i + t_j)}\right] \\
&= e^{(\lambda - \rho(l))(T)} \prod_{i=1}^{q} \left[\sum_{s_i=0}^{\infty} \Omega_{\lambda_i,s}^{l} e^{-2s_i t_i}\right] \prod_{i<j} \left[\sum_{r=0}^{\infty} e^{-2r(t_i - t_j)} \sum_{s=0}^{\infty} e^{-2s(t_i + t_j)}\right].
\end{aligned}$$

The recurrence relation follows from part (1) and the expansion in part (2). □

Let $\lambda = (\lambda_1, \ldots, \lambda_q) \in \mathfrak{a}_{\mathbb{C}}^*$ and suppose that $\lambda_i \notin \mathbb{Z}$, $\lambda_i \neq \lambda_j$ for $i \neq j$, and put

$$C(\lambda, l) = \frac{c_l(\lambda_1) \ldots c_l(\lambda_q)}{(-1)^{\frac{1}{2}q(q-1)} \det\left[\left(\lambda_i^{2(j-1)}\right)_{i\,j}\right]}, \tag{4.5}$$

where

$$c_l(\lambda_i) = \frac{2^{k+|l|+1-\lambda_i}\Gamma(1+k)\Gamma(\lambda_i)}{\Gamma(\frac{1}{2}\left(k+|l|+1+\lambda_i\right))\Gamma(\frac{1}{2}\left(k-|l|+1+\lambda_i\right))}. \tag{4.6}$$

**Lemma 4.4.** *Let* $C(\lambda, l)$ *be as in* (4.5), *then*

$$\begin{aligned}
C(\lambda, l) = c_0 \prod_{\alpha \in \mathcal{O}_m^+} &\frac{\Gamma(\langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle)}{\Gamma\left(\frac{1}{2}m_\alpha + \langle \lambda, \alpha \rangle / \langle \alpha, \alpha \rangle\right)} \\
&\cdot \prod_{i=1}^{q} \frac{2^{-\lambda_i}\Gamma(\lambda_i)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_{\alpha_i} + 1 + \lambda_i + |l|\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_{\alpha_i} + 1 + \lambda_i - |l|\right)\right)}.
\end{aligned}$$

*Proof.* Since

$$\begin{aligned}\prod_{\alpha\in\mathcal{O}_m^+}\frac{\Gamma(\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle)}{\Gamma\left(\frac{1}{2}m_\alpha+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\right)} &= \prod_{\alpha\in\mathcal{O}_m^+}\frac{\Gamma(\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle}{\Gamma\left(1+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\right)}\\ &= \prod_{\alpha\in\mathcal{O}_m^+}\frac{1}{\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle}\\ &= \prod_{i<j}\frac{4}{(\lambda_i^2-\lambda_j^2)}\\ &= \frac{(2)^{q(q-1)}(-1)^{\frac{1}{2}q(q-1)}}{\det\left[\left(\lambda_i^{2(j-1)}\right)_{i,\,j}\right]},\end{aligned}$$

we have

$$C(\lambda,l)=c_0\frac{2^{q(q-1)}(-1)^{\frac{1}{2}q(q-1)}}{\det\left[\left(\lambda_i^{2(j-1)}\right)_{i\,j}\right]}\cdot\prod_{i=1}^{q}\frac{2^{-\lambda_i}\Gamma(\lambda_i)}{\Gamma(\frac{1}{2}\left(k+|l|+1+\lambda_i\right))\Gamma(\frac{1}{2}\left(k-|l|+1+\lambda_i\right))}$$

where

$$c_0=2^{q(k+|l|-q+2)}\Gamma(1+k)^q.$$

□

**Proof of Theorem 1.1.** Assume that $\lambda_i\notin\mathbb{Z}$ and $\lambda_i\neq\pm\lambda_j$, for $i\neq j$. The idea of the proof is to make use of Harish-Chandra expansion (Theorem 3.6, [**15**] ), where we show that $\varphi_{\lambda,l}$ can be written as

$$\varphi_{\lambda,l}=\eta_l\sum_{w\in W}c(w\lambda,l)\Phi_{w\lambda,l},$$

where $W$ is the Weyl group of $G$, and $c(\lambda,l)$ is given by

$$\begin{aligned}c(\lambda,l)=c_0&\prod_{\alpha\in\mathcal{O}_m^+}\frac{\Gamma(\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle)}{\Gamma\left(\frac{1}{2}m_\alpha+\langle\lambda,\alpha\rangle/\langle\alpha,\alpha\rangle\right)}\\ &\cdot\prod_{i=1}^{q}\frac{2^{-\lambda_i}\Gamma(\lambda_i)}{\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_{\alpha_i}+1+\lambda_i+l\right)\right)\Gamma\left(\frac{1}{2}\left(\frac{1}{2}m_{\alpha_i}+1+\lambda_i-l\right)\right)}.\end{aligned}$$

Using the fact that $\varphi_{\lambda,l}(a_{(0,\ldots,0)})=1$ we deduce the value of the constant $C$.

Let

$$\tilde{\varphi}_{\lambda_i,l}(t_j)={}_2F_1\left(\frac{1}{2}\left(k+|l|+1+\lambda_i\right),\frac{1}{2}\left(k+|l|+1-\lambda_i\right),k+1;-\sinh^2(t_j)\right).$$

Since

$$\begin{aligned}{}_2F_1(a,b,c;z)=&B_1(-z)^{-a}{}_2F_1\left(a,1-c+a,1-b+a;z^{-1}\right)\\ &+B_2(-z)^{-b}{}_2F_1\left(b,1-c+b,1-a+b;z^{-1}\right),\end{aligned}$$

where

$$B_1 = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}, \qquad \text{and} \qquad B_2 = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)},$$

$$\begin{aligned}
&\tilde{\varphi}_{\lambda_i,l}(t_j) \\
&= {}_2F_1\left(\frac{1}{2}(k+|l|+1+\lambda_i), \frac{1}{2}(k+|l|+1-\lambda_i), k+1; -\sinh^2(t_j)\right) \\
&= B_1 \sinh^{-(k+|l|+1)-\lambda_i}(t_j) \\
&\qquad \cdot {}_2F_1\left(\frac{1}{2}(k+|l|+1+\lambda_i), \frac{1}{2}(-k+|l|+1+\lambda_i), 1+\lambda_i; -\sinh^{-2}(t_j)\right) \\
&\quad + B_2 \sinh^{-(k+|l|+1)+\lambda_i}(t_j) \\
&\qquad \cdot {}_2F_1\left(\frac{1}{2}(k+|l|+1-\lambda_i), \frac{1}{2}(-k+|l|+1-\lambda_i), 1-\lambda_i; -\sinh^{-2}(t_j)\right) \\
&= c_l(-\lambda_i)\Phi_{-\lambda_i,l}(t_j) + c_l(\lambda_i)\Phi_{\lambda_i,l}(t_j),
\end{aligned}$$

where $c_l(\lambda_i)$ as in (4.6). So we have

$$\begin{aligned}
&\frac{C^{-1}\omega(a_T)\varphi_{\lambda,l}(a_T)}{\eta_l} \\
&= \frac{\det\left[(\tilde{\varphi}_{\lambda_i,l}(t_j))_{i,j}\right]}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
&= \frac{\sum_{\sigma\in S_q}\left(\operatorname{sgn}(\sigma)\prod_{s=1}^q \tilde{\varphi}_{\lambda_{\sigma(s)}}(t_i)\right)}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
&= \frac{\sum_{\sigma\in S_q}\left(\operatorname{sgn}(\sigma)\prod_{s=1}^q (c_l(\lambda_{\sigma(s)})\Phi_{\lambda_{\sigma(s)},l}(t_i) + c_l(-\lambda_{\sigma(s)})\Phi_{-\lambda_{\sigma(s)},l}(t_i))\right)}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
&= \frac{\sum_{\sigma\in S_q}\left(\operatorname{sgn}(\sigma)\sum_{\substack{\varepsilon_i=\pm1\\ i=1,\dots,q}} c_l(\varepsilon_1\lambda_{\sigma(1)})\Phi_{\varepsilon_1\lambda_{\sigma(1)},l}(t_1)\dots c_l(\varepsilon_q\lambda_{\sigma(q)})\Phi_{\varepsilon_q\lambda_{\sigma(q)},l}(t_q)\right)}{\prod_{i<j}(\lambda_i^2-\lambda_j^2)} \\
&= \sum_{\substack{\sigma\in S_q\\ \varepsilon_i=\pm1}} \frac{\operatorname{sgn}(\sigma)c_l(\varepsilon_1\lambda_{\sigma(1)})\dots c_l(\varepsilon_q\lambda_{\sigma(q)})}{\prod_{\sigma(i)<\sigma(j)}((\varepsilon_i\lambda_{\sigma(i)})^2-(\varepsilon_j\lambda_{\sigma(j)})^2)} \prod_{s=1}^q \Phi_{\varepsilon_q\lambda_{\sigma(q)},l}(t_q).
\end{aligned}$$

Thus we have

$$\varphi_{\lambda,l}(a_T) = \eta_l \sum_{s\in W} C(s\lambda,l)\Phi_{s\lambda,l}(a_T)$$

where $C(\lambda,l)$ as in (4.5) and

$$W = \{s : s(t_1,\dots,t_q) = (\varepsilon_1 t_{\sigma(1)},\dots,\varepsilon_q t_{\sigma(q)}),\ \varepsilon_i=\pm1,\ \sigma\in S_q\}.$$

To complete the proof we need to show that

$$C = 2^{2q(q-1)}\prod_{i=1}^{q-1}\left[(k+j)^{q-j}j!\right].$$

Note that

$$\frac{d^m}{dz^m}\Big|_{z=0}\, {}_2F_1\left(\frac{1}{2}\left(k+|l|+1+\lambda_i\right), \frac{1}{2}\left(k+|l|+1-\lambda_i\right), k+1; -z\right) = \left(\frac{-1}{4}\right)^m \frac{\left[(k+|l|+1)^2-\lambda_i^2\right]\dots\left[(k+|l|+2m-1)^2-\lambda_i^2\right]}{(k+1)(k+2)\dots(k+m)}.$$

Put

$$\mathcal{I}_{i,\,j} = \left((k+|l|+1)^2-\lambda_j^2\right)\left((k+|l|+3)^2-\lambda_j^2\right)\cdots\left((k+|l|+2i-1))^2-\lambda_j^2\right),$$

by [**10**] Lemma 4.1, we have

$$\begin{aligned}
C^{-1} &= \varphi_{\lambda,l}(a_0) \\
&= C^{-1}\lim_{T\to 0}\varphi_{\lambda,l}(a_T) \\
&= \lim_{T\to 0}\frac{\det\left[\left({}_2F_1\left(\frac{1}{2}\left(k+|l|+1+\lambda_i\right), \frac{1}{2}\left(k+|l|+1-\lambda_i\right), k+1; -\sinh^2(t_j)\right)\right)_{i,\,j}\right]}{2^{\frac{1}{2}q(q-1)}\prod_{i<j}(\lambda_i^2-\lambda_j^2)\prod_{i<j}(\cosh(2t_i)-\cosh(2t_j))} \\
&= \lim_{T\to 0}\frac{\det\left[\left({}_2F_1\left(\frac{1}{2}\left(k+|l|+1+\lambda_i\right), \frac{1}{2}\left(k+|l|+1-\lambda_i\right), k+1; -\sinh^2(t_j)\right)\right)_{i,\,j}\right]}{2^{q(q-1)}\prod_{i<j}(\lambda_i^2-\lambda_j^2)\prod_{i<j}(\sinh^2(t_i)-\sinh^2(t_j))} \\
&= \frac{2^{-q(q-1)}(-1)^{\frac{1}{2}q(q-1)}}{\prod_{i=1}^{q-1}i!\prod_{i<j}(\lambda_i^2-\lambda_j^2)}\cdot\begin{vmatrix} 1 & \dots & 1 \\ \frac{(-1)\mathcal{I}_{1,\,1}}{2^2(k+1)} & \dots & \frac{(-1)\mathcal{I}_{1,\,q}}{2^2(k+1)} \\ \vdots & \vdots & \vdots \\ \frac{(-1)^{q-1}\mathcal{I}_{q-1,\,1}}{2^{2(q-1)}(k+1)(k+2)\cdots(k+q-1)} & \dots & \frac{(-1)^{q-1}\mathcal{I}_{q-1,\,q}}{2^{2(q-1)}(k+1)(k+2)\cdots(k+q-1)} \end{vmatrix} \\
&= \frac{(-1)^{q(q-1)}2^{-2q(q-1)}}{\prod_{i=1}^{q-1}i!(k+i)^{q-i}\prod_{i<j}(\lambda_i^2-\lambda_j^2)}\cdot\begin{vmatrix} 1 & \dots & 1 \\ \mathcal{I}_{1,\,1} & \dots & \mathcal{I}_{1,\,q} \\ \vdots & \vdots & \vdots \\ \mathcal{I}_{q-1,\,1} & \dots & \mathcal{I}_{q-1,\,q} \end{vmatrix} \\
&= \frac{2^{-2q(q-1)}}{\prod_{i=1}^{q-1}i!(k+i)^{q-i}\prod_{i<j}(\lambda_i^2-\lambda_j^2)}\cdot\begin{vmatrix} 1 & \dots & 1 \\ \left((k+|l|+1)^2-\lambda_1^2\right) & \dots & \left((k+|l|+1)^2-\lambda_q^2\right) \\ \vdots & \vdots & \vdots \\ \left((k+|l|+1)^2-\lambda_1^2\right)^{q-1} & \dots & \left((k+|l|+1)^2-\lambda_q^2\right)^{q-1} \end{vmatrix} \\
&= \frac{2^{-2q(q-1)}}{\prod_{i=1}^{q-1}i!(k+i)^{q-i}}.
\end{aligned}$$

The result for arbitrary $\lambda_j$ follows from analytic continuation (see [**10**], Lemma 4.1). □

## 5. $\chi$-spherical functions on complex Grassmannians

By the notation of Section 2, let $\pi : U \longrightarrow GL(E_\pi)$ be a representation of $U$, $\chi_l$ be a character of $K$ and let

$$E_\pi^l = \{X \in E_\pi \mid \pi(k)(X) = \chi_l(k) X \text{ for all } k \in K\}.$$

An irreducible unitary representation $(\pi, E_\pi)$ of $U$ is said to be $\chi_l$-spherical if $E_\pi^l \neq 0$. Lets denote by $\widehat{U}_l$ the set of $\chi_l$-spherical representations of $U$ and $\Lambda_l^+$ the set highest restricted weights of $\chi_l$-spherical representations of $U$.

**Theorem 5.1.** *Let $\{\alpha_1, \dots \alpha_r\}$ be an orthonormal basis of $\mathfrak{a}^*$, the dual space of $\mathfrak{a}$. Then the positive restricted roots $\Sigma^+$ is one of the following two sets:*

$$\textit{Case I}: \quad \Sigma^+ = \{2\alpha_i, (\alpha_j \pm \alpha_k) \mid 1 \le i \le r, \ 1 \le k < j \le r\}$$
$$\textit{Case II}: \quad \Sigma^+ = \{\alpha_i, 2\alpha_i, (\alpha_j \pm \alpha_k) \mid 1 \le i \le r, \ 1 \le k < j \le r\}.$$

Put

$$m_j = \begin{cases} \frac{\langle \lambda, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} & j = 1, \dots, r, \\ \lambda(iX) & j = 0. \end{cases}$$

then by [**12**], Proposition 7.1 and Theorem 7.2, we have the following

$$\Lambda_l^+ = \left\{ \lambda \in \mathfrak{a}^* \mid m_j - m_i \in 2\mathbb{Z}^+ (1 \le i < j \le n), \right. \tag{5.1}$$
$$\left. m_1 \in |l| + 2\mathbb{Z}^+, \ m_0 = \begin{cases} 0 & \text{(Case I)} \\ l & \text{(Case II)} \end{cases} \right\}.$$

For $\lambda \in \Lambda_l^+$, lets denote by $(\pi_\lambda, E_\lambda)$ a $\chi_l$-spherical representation of $U$ with highest weight $\lambda$ and put $E_\lambda^l = E_{\pi_\lambda}^l$. Let $e_\lambda \in E_\lambda^l$ be such that $\|e_\lambda\| = 1$. Let

$$\psi_{\lambda,l}(u) := (e_\lambda, \pi_\lambda(u) e_\lambda)_{E_\pi}, \qquad \text{for all } u \in U.$$

The function $\psi_{\lambda,l}$ is a $\chi$-spherical function on $U$ and any $\chi_l$-spherical function on $U$ is of this form (see [**9**], Lemma 4.3).

Let $U = SU(p+q)$, $K = S(U(p) \cdot U(q))$, $\lambda = (\lambda_1, \dots, \lambda_q) \in \Lambda_l^+$. Then by (5.1) we have

$$\frac{\langle \lambda, \alpha_1 \rangle}{\langle \alpha_1, \alpha_1 \rangle} = \lambda_1 \in |l| + 2\mathbb{Z}^+,$$

so

$$\lambda_1 = |l| + 2m_1, \text{ for some } m_1 \in \mathbb{Z}^+.$$

Also for $2 \le i \le q$, we have

$$\lambda_i - \lambda_1 \in 2\mathbb{Z}^+,$$

that is

$$\lambda_i = |l| + 2m_i, \text{ for some } m_i \in \mathbb{Z}^+.$$

Hence $\Lambda_l^+$ consist of all $\lambda \in \Lambda_l^+$ such that

$$\lambda = (2m_1 + |l|, 2m_2 + |l|, \dots, 2m_q + |l|), m_i \in \mathbb{Z}, m_1 \ge m_2 \ge \dots \ge m_q \ge 0$$

By Theorem 1.1 and [**9**] Lemma 4.6 we get Theorem 1.2.

To adopt Berezin-Karpelevich notation, let

$$P_n^{(\alpha,\beta)}(x) = \frac{(\alpha+1)_n}{n!} {}_2F_1(-n, n+\alpha+\beta+1, \alpha+1; \frac{1-x}{2})$$

be the Jacobi polynomial of degree $n$ and let

$$\tilde{P}_{n,l}(\cos(2x)) = \frac{P_n^{(k,|l|)}(\cos(2x))}{P_n^{(k,|l|)}(1)} = {}_2F_1(n+k+|l|+1, -n, k+1; \sin^2(x)).$$

Therefore we have

**Theorem 5.2.** *Let* $\lambda \in \Lambda_l^+$,

$$\lambda = (2m_1+|l|, 2m_2+|l|, \ldots, 2m_q+|l|), m_i \in \mathbb{Z}, m_1 \geq m_2 \geq \cdots \geq m_q \geq 0.$$

*Then the* $\chi_l$*-spherical function* $\psi_\lambda$ *on* $U = SU(p+q)$ *is given by*

$$\psi_{\lambda,l}(\sqrt{-1}H_{(t_1,\ldots,t_q)}) = \frac{C \det\left[\left(\tilde{P}_{n_i,l}(cos(2t_j)\right)_{i,\,j}\right] \prod_{i=1}^q \cos^{|l|}(t_i)}{\prod_{i<j}(c(n_i)-c(n_j)) \prod_{i<j}(\cos(2t_i)-\cos(2t_j))}$$

*where* $n_i = m_i + q - j$, $c(n_i) = n_i(n_i+|l|+k+1)$, *and*

$$C = 2^{\frac{1}{2}q(q-1)} \prod_{i=1}^{q-1} \left[(k+j)^{q-j} j!\right].$$

**Acknowledgment.** It is a great pleasure for me to thank Dr. Boudjemaa Anchouche for helpful suggestions, guidance, encouragement and advice provided to me during the preparation of this paper. I want also to take this opportunity to thank Kuwait University for hospitality, where part of this work was carried out. This work is part of my Ph.D. thesis.

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