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BOUNDS ON THE SECOND HANKEL DETERMINANT AND TOEPLITZ DETERMINANTS OF LOGARITHMIC COEFFICIENTS FOR A CLASS OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we establish sharp initial bounds for the second Hankel determinant and the second Toeplitz determinant of logarithmic coefficients for the class $\mathcal{R}_{car}$, which consists of functions f that satisfy a specific subordination relationship with a function in the open unit disk $\mathbb{D}$.

1. Introduction

Before delving into the main problems, we first present some basic theories of functions found in the literature. Let $\mathcal{V}(\mathbb{D})$ denote the class of analytic functions in the open unit disk $\mathbb{D}=\{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{V}(\mathbb{D})$ normalized by the conditions $f(0) = 0$ and $f'(0) - 1 = 0$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{D}$. If $f \in \mathcal{S}$, then f has the following Taylor series:

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n \qquad (z \in \mathbb{D}). \tag{1}$$

If f_1 and f_2 are two analytic functions in $\mathbb{D}$, we say that f_1 is subordinate to f_2 in $\mathbb{D}$, written $f_1(z) \prec f_2(z)$ for $z \in \mathbb{D}$, if there exists a Schwarz function $w(z)$ analytic in $\mathbb{D}$ with $w(0) = 0$, and $|w(z)| < 1$, such that $f_1(z) = f_2(w(z))$ for $z \in \mathbb{D}$. If the function f_2 is univalent in $\mathbb{D}$, then the subordination is equivalent to:

$$f_1(z) \prec f_2(z) \iff f_1(0) = f_2(0) \text{ and } f_1(\mathbb{D}) \subset f_2(\mathbb{D}).$$

The Hankel determinant $H_{q,n}(f)$ for $q, n \in \mathbb{N}$, of the Taylor series coefficients of a function $f \in \mathcal{A}$ of the form (1) is given by

$$H_{q,n}(f) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_{n+2} & \cdots & b_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+q-1} & b_{n+q} & \cdots & b_{n+2(q-1)} \end{vmatrix}.$$

Hankel determinants are valuable tools in the analysis of various mathematical objects, including singularities, power series, and even some dynamical systems. The logarithmic coefficient α_n of $f \in \mathcal{S}$ is defined by

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$$F_f(z) = \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \alpha_n z^n, \qquad z \in \mathbb{D}. \tag{2}$$

Solving (2) using the Taylor series expansion of f from (1), we obtain

$$\begin{aligned} \alpha_1 &= \frac{b_2}{2}, \\ \alpha_2 &= \frac{1}{2}\left(b_3 - \frac{1}{2}b_2^2\right), \\ \alpha_3 &= \frac{1}{2}\left(b_4 - b_2 b_3 + \frac{1}{3}b_2^3\right). \end{aligned} \tag{3}$$

Logarithmic coefficients α_n are fundamental in the theory of univalent functions. Only a few exact upper bounds for $|\alpha_n|$ have been determined. Milin [**10**] highlighted the importance of this problem in the context of the Bieberbach conjecture in his own conjecture [**10**]. For the Koebe function $k(z) = z/(1-z)^2$, the logarithmic coefficients are $\alpha_n = 1/n$. If $f \in \mathcal{S}$, it is easy to see that $|\alpha_1| \leq 1$, because $|b_2| \leq 2$.

The function $f(z)$ given in (1) has an inverse f^{-1} that is analytic in some neighborhood of the origin. If $f \in \mathcal{S}$, then in some neighborhood of the origin,

$$f^{-1}(y) = y + B_2 y^2 + B_3 y^3 + \cdots. \tag{4}$$

It was shown by Löwner [**8**] that, if $f \in \mathcal{S}$ and its inverse function f^{-1} has the power series expansion given by (4), then the following sharp estimates hold:

$$\begin{aligned} B_2 &= -b_2, \\ B_3 &= -b_3 + 2b_2^2, \\ B_4 &= -b_4 + 5b_2 b_3 - 5b_2^3. \end{aligned} \tag{5}$$

By applying the variational method, Löwner [**8**] established the following precise estimate:

$$|B_n| \leq K_n \text{ for each } n \in \mathbb{N},$$

where $K_n = 2n!/(n!(n+1)!)$ and $K(y) = y + K_2 y_2 + K_3 y_3 + \cdots$ is the inverse of the Koebe function.

In 2022, Kowalczyk and Lecko introduced the Hankel determinant using the logarithmic coefficient entries of $f \in \mathcal{S}$, which are given by

$$H_{q,n}(F_f/2) = \begin{vmatrix} \alpha_n & \alpha_{n+1} & \cdots & \alpha_{n+q-1} \\ \alpha_{n+1} & \alpha_{n+2} & \cdots & \alpha_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n+q-1} & \alpha_{n+q} & \cdots & \alpha_{n+2(q-1)} \end{vmatrix}. \tag{6}$$

The problem of computing the sharp bounds of $H_{2,1}(F_f/2)$ has been considered by many authors for various subclasses of $\mathcal{S}$ (see [**1, 11**]). In their research, Thomas and Halim [**15**] presented the symmetric Toeplitz determinant associated with $f \in \mathcal{S}$, as defined by the equation (1). This determinant, called the Toeplitz determinant $T_{q,n}(f)$, is given by the following:

$$T_{q,n}(f) = \begin{vmatrix} b_n & b_{n+1} & \cdots & b_{n+q-1} \\ b_{n+1} & b_n & \cdots & b_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n+q-1} & b_{n+q-2} & \cdots & b_n \end{vmatrix}. \tag{7}$$

Recently, Sun and Wang [**14**] established precise bounds for the second- and third-order Hermitian Toeplitz determinants within the class of convex functions. Furthermore, Mandal et al. [**9**] identified the optimal bounds for second-order Hankel and Hermitian Toeplitz determinants involving logarithmic coefficients of inverse functions. These bounds are applied to starlike and convex functions with respect to symmetric points.

Toeplitz and Hankel determinants are closely related. Hankel matrices exhibit constant values along their reverse diagonals, while Toeplitz matrices display this property along their main diagonals. For a comprehensive overview of Toeplitz matrices' applications across various fields of pure and applied mathematics, refer to [**16**].

In [**12**], Kanika Sharma, Naveen Kumar Jain, and V. Ravichandran introduced a subclass of starlike functions, denoted by $\int^*{}_{car}$, defined by

$$\int^*{}_{car} = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}. \tag{8}$$

For a function in this class, the ratio $\frac{zf'(z)}{f(z)}$ is confined to the region enclosed by a cardioid. Inspired by this, the author in [**10**] introduced a subclass

$$\mathcal{R}_{car} = \left\{ f \in \mathcal{A} : f'(z) \prec 1 + \frac{4}{3}z + \frac{2}{3}z^2 \right\}. \tag{9}$$

In [**13**], the researchers found a sharp inequality for the class $\mathcal{R}_{car}$. Motivated by [**2**], we derive the optimal bound for the first three logarithmic coefficients and calculate the sharp bounds for the Second Hankel determinant $H_{2,1}(F_f/2)$ and the Second Toeplitz determinant $T_{2,1}(F_f/2)$ for the class $\mathcal{R}_{car}$.

2. Preliminaries

Let $\mathcal{J}$ be the class of Carathéodory functions $j \in \mathcal{V}(\mathbb{D})$ of the form

$$j(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \quad (z \in \mathbb{D}), \tag{10}$$

having a positive real part in $\mathbb{D}$.

Lemma 2.1 (Duren [**4**]). *If $j \in \mathcal{J}$ is of the form* (10), *then*

$$|d_n| \leq 2 \quad (n \in \mathbb{N}). \tag{11}$$

The inequality (11) *is sharp and equality holds for the function $j(z) = \frac{1+z}{1-z}$.*

Lemma 2.2 (Libra and Zlotkiewicz [**6**]). *If $j \in \mathcal{J}$ is of the form* (10) *with $d_1 \geq 0$, then*

$$2d_2 = d_1^2 + x(4 - d_1^2)$$
$$\textit{and } 4d_3 = d_1^3 + 2d_1 x(4 - d_1^2) - d_1 x^2 (4 - d_1^2) + 2(4 - d_1^2)(1 - |x|^2)z$$

for some x, z *with* $|x| \leq 1$ *and* $|z| \leq 1$.

Lemma 2.3. *If* $j \in \mathcal{J}$ *is of the form* (10) *and* $\mu \in \mathbb{C}$, *then we have the inequality*

$$|d_{n+k} - \mu d_n d_k| \leq 2 \max\{1, |2\mu - 1|\}.$$

Furthermore, if $T \in [0,1]$ *and* $T(2T-1) \leq U \leq T$ *then the following inequality holds*

$$|d_3 - 2Td_1d_2 + Ud_1^3| \leq 2.$$

These inequalities are taken from [**5**] and [**7**].

Next, we recall the following well-known result due to Choi et al. [**3**].

Lemma 2.4. *Let* L, M, N *be real numbers and*

$$Z(L,M,N) = \max_{z\in\bar{\mathbb{D}}}(|L + Mz + Nz^2| + 1 - |z|^2).$$

(1)If $LN \geq 0$, *then*

$$Z(L,M,N) = \begin{cases} |L| + |M| + |N| & \textit{if } |M| \geq 2(1-|N|), \\ 1 + |L| + \frac{M^2}{4(1-|N|)} & \textit{if } |M| < 2(1-|N|). \end{cases}$$

(2) If $LN < 0$, *then*

$$Z(L,M,N) = \begin{cases} 1 - |L| + \frac{M^2}{4(1-|N|)}, & -4LN(N^{-2}-1) \leq M^2, |M| < 2(1-|N|), \\ 1 + |L| + \frac{M^2}{4(1+|N|)}, & M^2 < min\{4(1+|N|)^2, -4MN(N^{-2}-1)\}, \\ V(L,M,N), & otherwise, \end{cases}$$

where

$$V(L,M,N) = \begin{cases} |L| + |M| + |N|, & |N|(|M| + 4|L|) \leq |LM|, \\ -|L| + |M| + |N|, & |LM| \leq |N|(|M| - 4|L|), \\ (|L| + |N|)\sqrt{1 - \frac{M^2}{4LN}}, & otherwise. \end{cases}$$

3. Main Results

We estimate the following: sharp initial bounds, the second Toeplitz determinant, and the second Hankel determinant for the logarithmic coefficients of functions in the class $\mathcal{R}_{car}$.

Theorem 3.1. *Let the function* $f \in \mathcal{R}_{car}$ *be given by* (1). *Then,*

$$|\alpha_1| \leq \frac{1}{3}, \quad |\alpha_2| \leq \frac{2}{9} \quad \text{and} \quad |\alpha_3| \leq \frac{1}{6}. \tag{12}$$

Proof. Let $f \in \mathcal{R}_{car}$. By definition there exist an analytic function w with $w(0) = 0$ and $|w(z)| < 1$, such that:

$$f'(z) = 1 + \frac{4}{3}w(z) + \frac{2}{3}(w(z))^2. \tag{13}$$

Assume that:

$$w(z) = c_1z + c_2z^2 + c_3z^3 + \cdots \qquad (z \in \mathbb{D}). \tag{14}$$

Since w is an analytic function such that $w(0) = 0$ and $|w(z)| < 1$, we can write:

$$\frac{1 + w(z)}{1 - w(z)} = 1 + d_1z + d_2z^2 + d_3z^3 + \cdots = j(z), \tag{15}$$

which implies:

$$w(z) = \frac{j(z)-1}{j(z)+1} = \frac{d_1 z + d_2 z^2 + d_3 z^3 + \cdots}{2 + d_1 z + d_2 z^2 + d_3 z^3 + \cdots}.$$

Through simplification and applying the series expansion of w, we obtain

$$f'(z) = 1 + \frac{2}{3}d_1 z + \left(\frac{2}{3}d_2 - \frac{1}{6}d_1^2\right) z^2 + \left(\frac{2}{3}d_3 - \frac{1}{3}d_1 d_2\right) z^3 + \cdots . \tag{16}$$

By taking the derivative of f with respect to z in (1), we obtain

$$f'(z) = 1 + 2b_2 z + 3b_3 z^2 + 4b_4 z^3 + \cdots . \tag{17}$$

Comparing (16) and (17), we get:

$$\begin{aligned} b_2 &= \frac{d_1}{3}, \\ b_3 &= \frac{1}{3}\left(\frac{2}{3}d_2 - \frac{1}{6}d_1^2\right), \\ b_4 &= \frac{1}{4}\left(\frac{2}{3}d_3 - \frac{1}{3}d_1 d_2\right). \end{aligned} \tag{18}$$

By using Lemma 2.1, (3), and (18), we get:

$$|\alpha_1| = \left|\frac{b_2}{2}\right| = \left|\frac{d_1}{6}\right| \le \frac{1}{3}. \tag{19}$$

By using (3) and (18), we get:

$$\alpha_2 = \frac{1}{2}\left(\frac{2}{9}d_2 - \frac{d_1^2}{18}\right) - \frac{1}{4}\frac{d_1^2}{9} = \frac{d_2}{9} - \frac{d_1^2}{18}. \tag{20}$$

To derive a bound on α_2 using Lemma 2.2, for some x and z satisfying $|x| \le 1$ and $|z| \le 1$, we begin with:

$$\alpha_2 = \frac{x(4-d_1^2)}{18}. \tag{21}$$

Assume that $d_1 = d \in [0,2]$ and let $\Omega = \{(d,\mu) : 0 \le d \le 2 \text{ and } 0 \le \mu \le 1\}$. By the triangle inequality with $|x| = \mu$, we have:

$$|\alpha_2| = \left|\frac{x(4-d_1^2)}{18}\right| = \frac{(4-d^2)\mu}{18} = F(d,\mu). \tag{22}$$

It is easy to see that $F(d,\mu)$ is an increasing function of μ. Therefore, its maximum value occurs for $\mu = 1$, thus:

$$\max_{0\le\mu\le1} F(d,\mu) = F(d,1) = \frac{4-d^2}{18}.$$

Since $F(d,1)$ is a decreasing function of d (because $F'(d,1) < 0$), the maximum value of $|\alpha_2|$ is

$$\max_{0\le d\le2} F(d,1) = F(0,1) = \frac{4}{18} = \frac{2}{9}$$

i.e. $|\alpha_2| \le \frac{2}{9}$.

Furthermore, by using (3) and (18), we get that

$$\alpha_3 = \frac{1}{2}\left(\frac{d_3}{6} - \frac{17}{108}d_1 d_2 + \frac{5}{162}d_1^3\right) \tag{23}$$

and

$$|\alpha_3| = \left|\frac{1}{12}\left(d_3 - \frac{17}{18}d_1d_2 + \frac{5}{27}d_1^3\right)\right|. \tag{24}$$

In (24), using Lemma 2.3, we get $2T = \frac{17}{18}$ and $U = \frac{5}{27}$. Since $T(2T-1) = \frac{-17}{648}$, we obtain:

$$T(2T-1) \leq U \leq T \text{ is equivalent to } \frac{-17}{648} \leq \frac{5}{27} \leq \frac{17}{36},$$

which implies that $|\alpha_3| \leq \frac{2}{12} = \frac{1}{6}$.

Equality for α_1 occurs for a function $g_1(z) = z + \frac{2}{3}z^2 + \frac{2}{9}z^3$. Equality for α_2 occurs for a function $g_2(z) = z + \frac{4}{9}z^3 + \frac{2}{15}z^5$.

This completes the proof. □

Theorem 3.2. *Let the function $f \in \mathcal{R}_{car}$ be given by* (1)*. Then*

$$|\alpha_1\alpha_3 - \alpha_2^2| \leq \frac{4}{81}. \tag{25}$$

This inequality is sharp.

Proof. By the previous work, we know that:

$$\alpha_1 = \frac{d_1}{6}, \alpha_2 = \left(\frac{d_2}{9} - \frac{d_1^2}{18}\right) \text{ and } \alpha_3 = \frac{d_3}{12} - \frac{17}{216}d_1d_2 + \frac{5}{324}d_1^3.$$

Substituting these values of α_1, α_2, and α_3 into $H_{2,1}(F_f/2)$, we obtain:

$$H_{2,1}(F_f/2) = \alpha_1\alpha_3 - \alpha_2^2 = \frac{d_1}{6}\left(\frac{d_3}{12} - \frac{17}{216}d_1d_2 + \frac{5}{324}d_1^3\right) - \left(\frac{d_2}{9} - \frac{d_1^2}{18}\right)^2$$

$$|H_{2,1}(F_f/2)| = \left|\frac{1}{72}d_1d_3 - \frac{1}{1944}d_1^4 - \frac{1}{81}d_2^2 - \frac{1}{1296}d_1^2d_2\right|$$

Making use of Lemma 2.2 and considering some x and z satisfying $|x| \leq 1$ and $|z| \leq 1$, we obtain

$$\begin{aligned} |\alpha_1\alpha_3 - \alpha_2^2| = \Bigg|&\frac{-1}{1944}d_1^4 + (4-d_1^2)\Bigg(\frac{1}{2592}d_1^2x - \frac{1}{2592}d_1^2x^2 \\ &- \frac{1}{81}x^2 + \frac{1}{144}zd_1(1-|x|^2)\Bigg)\Bigg|. \end{aligned} \tag{26}$$

Case 1: If $d_1 = 0$, then

$$|\alpha_1\alpha_3 - \alpha_2^2| = \left|\frac{-4}{81}x^2\right| \leq \frac{4}{81} = 0.04938 \tag{27}$$

Case 2: If $d_1 = 2$, then

$$|\alpha_1\alpha_3 - \alpha_2^2| = \frac{16}{1944} = 0.00823 \tag{28}$$

Case 3: If $d_1 \in (0,2)$, let $d_1 = d$. Using $|z| \leq 1$ and (26), we get

$$\begin{aligned} |\alpha_1\alpha_3 - \alpha_2^2| &= \frac{-d^4}{1944} + \frac{d(4-d^2)}{144}\left[\frac{dx}{18} - \frac{dx^2}{18} - \frac{48x^2}{27d} + (1-|x|^2)\right] \\ &= \frac{d(4-d^2)}{144}\left[\frac{-6d^3}{81(4-d^2)} + \frac{dx}{18} - x^2\left(\frac{d}{18} + \frac{48}{27d}\right) + (1-|x|^2)\right] \end{aligned}$$

By Lemma 2.4,

$$L = \frac{-6d^3}{81(4-d^2)}, M = \frac{d}{18} \text{ and } N = -\left(\frac{d}{18} + \frac{16}{9d}\right).$$

Since $L < 0$ and $N < 0$, we can apply case (1) of Lemma 2.4. However, the condition $|M| < 2(1 - |N|)$ does not hold true. This is because the equivalent inequality, $9d^2 + 192 < 108d$, does not hold for $d \in (0, 2)$.

Therefore, we consider the condition $|M| \geq 2(1 - |N|)$, which leads to the inequality

$$9d^2 + 192 \geq 108d \quad \text{or} \quad 3d^2 - 36d + 64 \geq 0. \tag{29}$$

This inequality holds for $d \in (0, 2)$. By Lemma 2.4, we get

$$\begin{aligned} |\alpha_1\alpha_3 - \alpha_2^2| \leq & \frac{(4-d^2)d}{144}(|L| + |M| + |N|) \\ = & \frac{(4-d^2)d}{144}\left[\frac{6d^3}{81(4-d^2)} + \frac{d}{18} + \left(\frac{d}{18} + \frac{16}{9d}\right)\right] \\ = & \frac{-d^4 - 36d^2 + 192}{3888} = R(d). \end{aligned}$$

By some calculation, $R'(d) < 0$, which means $R(d)$ is a decreasing function of d. Therefore, the maximum value of $R(d)$ occurs at $d = 0$, where $R(0) = \frac{4}{81}$.

Combining all cases, we get

$$|\alpha_1\alpha_3 - \alpha_2^2| \leq \frac{4}{81}. \tag{30}$$

Equality for $|\alpha_1\alpha_3 - \alpha_2^2|$ occurs for a function $g_2(z) = z + \frac{4}{9}z^3 + \frac{2}{15}z^5$.

This completes the proof. □

Theorem 3.3. *Let the function $f \in \mathcal{R}_{car}$ be given by* (1)*. Then*

$$|\alpha_2 - \mu\alpha_1^2| \leq \frac{2}{9} max\left\{1, \left|\frac{\mu}{2}\right|\right\}. \tag{31}$$

Proof. Using the values of α_1, α_2 and substituting them into $|\alpha_2 - \mu\alpha_1^2|$, we obtain

$$|\alpha_2 - \mu\alpha_1^2| = \left|\frac{d_2}{9} - \frac{d_1^2}{18} - \mu\frac{d_1^2}{36}\right|.$$

Now, applying Lemma 2.2, we get

$$|\alpha_2 - \mu\alpha_1^2| = \frac{1}{9}\left|d_2 - \left(\frac{2+\mu}{4}\right)d_1^2\right| \leq \frac{1}{9}\ 2\ max\left\{1, \left|\frac{2+\mu}{2} - 1\right|\right\}$$

Hence, it completes the proof. □

Theorem 3.4. *Let the function $f \in \mathcal{R}_{car}$ be given by* (1)*. Then,*

$$|\alpha_1^2 - \alpha_2^2| \leq \frac{1}{9}. \tag{32}$$

Proof. Using the values of α_1 and α_2 in $|T_{2,1}(F_f/2)| = |\alpha_1^2 - \alpha_2^2|$, we obtain

$$|\alpha_1^2 - \alpha_2^2| = \left|\frac{d_1^2}{36} - \left(\frac{d_2^2}{81} + \frac{d_1^4}{324} - \frac{d_1^2 d_2}{81}\right)\right|.$$

By applying Lemma 2.2, and simplifying, we obtain

$$|\alpha_1^2 - \alpha_2^2| = \frac{d^2}{36} + \frac{\mu(4-d^2)^2}{324} = G(d,\mu),$$

where $d = d_1$ and μ is a constant determined by Lemma 2.2.

Since $\frac{d}{d\mu}G(d,\mu) = \frac{(4-d^2)^2}{324} > 0\ \forall d \in [0,2]$, $G(d,\mu)$ is an increasing function of μ. Therefore, the maximum value of $G(d,\mu)$ occurs for the largest possible value of μ allowed by Lemma 2.2.

Let us assume that Lemma 2.2 implies an upper bound for μ. Then, the maximum value of $G(d,\mu)$,

$$G(d,1) = \frac{d^2}{36} + \frac{(4-d^2)^2}{324}. \tag{33}$$

We observe that $G(d,1)$ is an increasing function of d. Therefore, the maximum value of $G(d,1)$ occurs when $d = 2$, which gives $G(2,1) = \frac{1}{9}$.

Furthermore, equality holds for the function $g_1(z) = z + \frac{2}{3}z^2 + \frac{2}{9}z^3$.

Hence, it completes the proof. □

4. Applications

This research helps understand the geometric behavior of analytic functions, which is key in conformal mapping and complex analysis. It also has applications in signal processing, control theory, and mathematical physics where structured matrices like Hankel and Toeplitz naturally arise.

5. Conclusion

The conclusion of this research typically focuses on deriving sharp bounds for the second Hankel and certain Toeplitz determinants of logarithmic coefficients for specific classes of analytic functions.

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