NEW ZEALAND JOURNAL OF MATHEMATICS
Volume 50 (2020), 49-60

IDEALIZATION PROPERTIES OF COMULTIPLICATION MODULES

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(Received 9 July, 2019)

Abstract. In our previous work we gave a treatment of certain aspects of multiplication modules, projective modules, flat modules and like-cancellation modules via idealization. The purpose of this work is to continue our study and develop the tool of idealization in the context of comultiplication modules.

1. Introduction

Let R be a commutative ring and M an R-module. M is a multiplication module if every submodule N of M has the form IM for some ideal I of R. Or equivalently $N = [N : M]M$ by [**12**] and [**16**]. If M is a multiplication module then M is locally cyclic and the converse is true if M is finitely generated by [**16**] Proposition 1. The author and Smith introduced and investigated the concept of idempotent submodules in [**11**]. A submodule N of M is idempotent if $N = [N : M]N$. If M is multiplication then $N = [N : M]^2M$. Several properties of such submodules are given in [**9**] and [**11**]. A submodule N of M is called pure in M if $IN = N \cap IM$ for some ideal I of R ([**18**]).

Ansari and Farshadifar introduced and investigated the dual notions of such modules. Let N be a proper submodule of N. Then N is a comultiplication in M if $N = [0 :_M \operatorname{ann} N]$, N is coidempotent in M if $N = [0 :_M (\operatorname{ann} N)^2]$ and N is copure in M if $[N :_M I] = N + [0 :_M I]$ for some ideal of I of R, see [**14**], [**15**] and [**1**]. Several properties of these modules are also considered in [**10**]. It is shown for example that for any two submodules K and L of an R-module M, if K is comultiplication (resp. coidempotent) in M such that $K \subseteq L$ and $\operatorname{ann} K \subseteq \operatorname{ann} L$, then L is comultiplication (resp. coidempotent) in M ([**10**] Proposition 2.2). It is also proved that if K and L are comultiplication (resp. coidempotent, copure) in M such that $[K : L] + [L : K] = R$, then each of $K + L$ and $K \cap L$ is comultiplication (resp. coidempotent, copure) in M ([**10**] Proposition 3.2). We call a module M a comultiplication module if every submodule N of M is comultiplication in M, M is coidempotent if every submodule N of M is coidempotent in M and M is copure if every submodule N of M is copure M. Consequently, R is a comultplication if every ideal I of R is comultiplication, that is $I = \operatorname{ann}(\operatorname{ann} I)$, R is coidempotent if every ideal I of R is coidempotent, that is $I = \operatorname{ann}(\operatorname{ann} I)^2$ and R is copure if every ideal I is copure, that is $[I : J] = I + \operatorname{ann} J$ for every ideal J of R.

2010 *Mathematics Subject Classification* 13C13, 13C10, 13A15.
Key words and phrases: multiplication module, comultiplication submodule, coidempotent submodule, copure submodule.
Dedicated to my wife Zeinab Al-Khshali on the occasion of her 53rd birthday.

Let R be a commutatitve ring with identity and M and R-module. The R-module $R(M) = R(+)M$ becomes a commutative ring with identity under the product $(r,m)(r',m') = \left(rr^{'}, rm' + r'm\right)$, called the idealization of M. The idealization of a module is a well-established method to facilitate interaction between a ring on the one hand and a module over a ring on the other. The basic construction is to embed the module M as an ideal in the ring $R(M)$ which contains R as a subring. This technique was used with great success by Nagata in [**19**]. For a comprehensive survey on idealization, [**13**] and [**17**] can be consulted. $0(+)M$ is an ideal of $R(M)$ satisfying $(0(+)M)^2 = 0$ and the structure of $0(+)M$ as an ideal of $R(M)$ is essentially the same as the R-module structure of M. Every ideal which is contained in $0(+)M$ has the form $0(+)N$ for some submodule N of M and every ideal that contains $0(+)M$ has the form $I(+)M$ for some ideal I of R. Prime (maximal) ideals of $R(M)$ have the form $P(+)M$ where P is prime (maximal) ideal of R. Homogeneous ideals of $R(M)$ have the form $I\,(+)\,N$, where I is an ideal of R and N a submodule of M such that $IM \subseteq N$, that is $[N :_M I] = M$. Ideals of $R\,(M)$ need not be homogeneous. $R\,(M)$ is called a homogeneous ring if every ideal of $R\,(M)$ is homogeneous. Let $I\,(+)\,N$ and $J\,(+)\,K$ be ideals of $R\,(M)$, then the ideal

$$\left[I\,(+)\,N :_{R(M)} J\,(+)\,K\right] = [I : J] \cap [N : K]\,(+)\,[N :_M J]$$

is homogeneous by [**4**] Lemma 1 and [**3**] Lemma 1. Consequently,

$$\operatorname{ann}(I\,(+)\,N) = (\operatorname{ann} I \cap \operatorname{ann} N)\,(+)\,[0 :_M I]\,.$$

If M is faithful, $\operatorname{ann}(I\,(+)\,N) = \operatorname{ann} N\,(+)\,[0 :_M I]$. Let M be faithful multiplication or projective then $[0 :_M I] = (\operatorname{ann} I)\,M$ by [**3**] Lemma 5 and therefore $\operatorname{ann}(I\,(+)\,N) = \operatorname{ann} N\,(+)\,(\operatorname{ann} I)\,M$. In particular,

$$\operatorname{ann}(I\,(+)\,IM) = \operatorname{ann} I\,(+)\,(\operatorname{ann} I)\,M.$$

For any submodule N of M $\operatorname{ann}(0\,(+)\,N) = (\operatorname{ann} N)\,(+)\,M$. If M is a faithful R-module then $\operatorname{ann}(0\,(+)\,M) = 0\,(+)\,M$ and hence $0\,(+)\,M = \operatorname{ann}(\operatorname{ann}(0\,(+)\,M))$ is a comultiplication ideal of $R\,(M)$. This shows that $0\,(+)\,\mathbb{Z}$ is a comultiplication ideal of the ring $\mathbb{Z}\,(+)\,\mathbb{Z}$ but the $\mathbb{Z}-$module $\mathbb{Z}$ is not comultiplication because $2\mathbb{Z}$ is not a comultiplication submodule in $\mathbb{Z}$ ([**14**]). Also $0\,(+)\,\mathbb{Z}$ is a multiplication ideal of $\mathbb{Z}\,(+)\,\mathbb{Z}$. For the Prufer p-group $\mathbb{Z}p^\infty$, $0\,(+)\,\mathbb{Z}p^\infty$ is not a mulplication ideal of $\mathbb{Z}\,(+)\,\mathbb{Z}p^\infty$ but it is comultiplication. So neither multiplication modules are comultiplication nor comultiplication modules are multiplication.

In a series of works, the author developed more fully the tool of idealization of a module, particularly in the context of multiplication modules, projective modules, flat modules, cancellation like modules, generalizing Anderson's theorems and discussing the behavior under idealization of some ideals and some submodules assdociated with a module, see [**2**]-[**9**]. It is proved, for example, that if $I\,(+)\,N$ is a multiplication ideal of $R\,(M)$, then I is multiplication. Assuming further that M is multiplication then N is multiplication. Conversely, if I is multiplication and N multiplication such $\operatorname{ann} I + [IM : N] = R$, then $I\,(+)\,N$ is multiplication. See, for example, [**4**] Propositions 5 and 7, [**3**] Theorem 9 and [**2**] Theorem 9. If $I\,(+)\,N$ is idempotent in M, then I is idempotent in R and N is idempotent in M by [**9**] Theorem 17. It is also shown that if $I\,(+)\,N$ is a pure ideal of $R\,(M)$, then I is pure in R and N pure in M by [**5**] Theorem 3 or [**6**] Proposition 8.

In this work, we develop the tool of idealization in the context of comultiplication (resp. coidempotent, copure) submodules. We show in Theorem 2.4 that if $I\,(+)\,N$ is a comultiplication ideal of $R\,(M)$, then N is a comultiplication in M. Assuming further M is faithful multiplication then I is a comultiplication. The same statement holds for coidempotent submodules. If $I\,(+)\,N$ is copure in $R\,(M)$ then I is copure in R and N copure in M. Proposition 2.5 shows that if I is a comultiplication (resp. coidempotent) such that $\operatorname{ann} N \subseteq \operatorname{ann} I$ then $I\,(+)\,N$ is a comultiplication (resp. coidempotent) ideal of $R\,(M)$. Finally, Theorem 2.9 gives necessary and sufficient conditions for the ring $R\,(M)$ to be a comultiplication ring.

All rings are commutative with 1 and all modules are unital. For the basic concepts used, we refer the reader to [**17**] and [**18**].

2. Idealization of a Module

We start our work by a result showing how properties of a submodule N of M are related to those of the ideal $0\,(+)\,N$ of $R\,(M)$.

Proposition 2.1. *Let R be a ring, M an R-module and N a proper submodule of M. Then the following holds.*

(1) *$0\,(+)\,N$ is a comultiplication in $R\,(M)$ if and only if N is comultiplication in M and* $\operatorname{ann}(\operatorname{ann} N) \cap \operatorname{ann} M = 0$.

(2) *Let M be faithful. Then $0\,(+)\,N$ is coidempotent in $R\,(M)$ if and only if N is coidempotent in M and annN is faithful.*

(3) *If $0\,(+)\,N$ is copure in $R\,(M)$ then N is copure in M.*

Proof. (1) Suppose $0\,(+)\,N$ is a comultiplication. Then

$$\begin{aligned} 0\,(+)\,N &= \operatorname{ann}(\operatorname{ann}(0\,(+)\,N)) = \operatorname{ann}(\operatorname{ann} N\,(+)\,M) \\ &= \operatorname{ann}(\operatorname{ann} N) \cap \operatorname{ann} M + [0 :_M \operatorname{ann} N]. \end{aligned}$$

It follows that $N = [0 :_M \operatorname{ann} N]$ and hence N is comultiplication in M. Moreover, $\operatorname{ann}(\operatorname{ann} N) \cap \operatorname{ann} M = 0$. The statement is reversible.

(2) Assume M is faithful and $0\,(+)\,N$ is coidempotent. Then

$$\begin{aligned} 0\,(+)\,N &= \operatorname{ann}(\operatorname{ann}(0\,(+)\,N))^2 = \operatorname{ann}(\operatorname{ann} N\,(+)\,M)^2 \\ &= \operatorname{ann}\left((\operatorname{ann} N)^2\,(+)\,(\operatorname{ann} N)\,M\right) \\ &= \operatorname{ann}(\operatorname{ann} N)^2 \cap \operatorname{ann}((\operatorname{ann} N)\,M)\,(+)\left[0 :_M (\operatorname{ann} N)^2\right]. \end{aligned}$$

Since M is faithful, $\operatorname{ann}((\operatorname{ann} N)\,M) = \operatorname{ann}(\operatorname{ann} N)$. Also $\operatorname{ann}(\operatorname{ann} N) \subseteq \operatorname{ann}\left((\operatorname{ann} N)^2\right)$. It follows that $0\,(+)\,N = \operatorname{ann}(\operatorname{ann} N)\,(+)\left[0 :_M (\operatorname{ann} N)^2\right]$. This shows that $N = \left[0 :_M (\operatorname{ann} N)^2\right]$ and N is coidempotent in M, and $\operatorname{ann}(\operatorname{ann} N) = 0$. The statement is reversible.

(3) Let $0\,(+)\,N$ be copure in $R\,(M)$. Let J be an ideal of R. Then

$$\left[0\,(+)\,N :_{R(M)} J\,(+)\,JM\right] = \operatorname{ann} J \cap [N : JM]\,(+)\,[N :_M J].$$

But $\operatorname{ann} J \subseteq [N : JM]$. Thus

$$\left[0\,(+)\,N :_{R(M)} J\,(+)\,JM\right] = \operatorname{ann} J\,(+)\,[N :_M J].$$

On the other hand

$$\begin{aligned}\left[0\,(+)\,N :_{R(M)} J\,(+)\,JM\right] &= 0\,(+)\,N + \operatorname{ann}(J\,(+)\,JM)\\ &= 0\,(+)\,N + \operatorname{ann} J \cap \operatorname{ann}(JM)\,(+)\,[0 :_M J]\\ &= \operatorname{ann} J\,(+)\,N + [0 :_M J]\,.\end{aligned}$$

This gives that $[N :_M J] = N + [0 :_M J]$ and N is copure in M.

□

We have two remarks on the first part of Proposition 2.1. First, it is obvious now that for any proper submodule N of M, if $0\,(+)\,N$ is a comultiplication then N is a comultiplication in M and the converse is true if M is faithful. Second, let N be a proper submodule of a finitely generated projective R-module M. Then

$$\operatorname{ann}(\operatorname{ann} N) \cap \operatorname{ann} M \subseteq \operatorname{ann}(\operatorname{ann} M) \cap \operatorname{ann} M = \operatorname{Tr}(M) \cap \operatorname{ann}(\operatorname{Tr}(M)) = 0,$$

see [**12**] Corollary 3.2. This shows that if N is a comultiplication submodule of a finitely generated projective R-module M, then $0\,(+)\,N$ is a comultiplication. Recall that if M is projective then $M = \operatorname{Tr}(M)\,M$, $\operatorname{ann} M = \operatorname{ann}\operatorname{Tr}(M)$ and $\operatorname{Tr}(M)$ is a pure ideal of R. Note that

$$\operatorname{Tr}(M) = \sum_{f \in Hom(M,R)} f(M)\,,$$

see for example, [**12**].

The next result shows how properties of $I\,(+)\,IM$ are related to those of I.

Proposition 2.2. *Let R be a ring, M an R-module and I ideal of R.*

(1) *Let M be faithful multiplication. Then $I\,(+)\,IM$ is a comultiplication in $R(M)$ if and only if I is a comultiplication in R.*
(2) *Let M be faithful multiplication. Then $I\,(+)\,IM$ is coidempotent in $R(M)$ if and only if I is coidempotent in R.*
(3) *If $I\,(+)\,IM$ is copure in $R(M)$ then I is copure in R.*

Proof. (1) Let M be faithful multiplication and $I\,(+)\,IM$ is a comultiplication. Then

$$\begin{aligned}I\,(+)\,IM &= \operatorname{ann}(\operatorname{ann}(I\,(+)\,IM))\\ &= \operatorname{ann}(\operatorname{ann} I \cap \operatorname{ann}(IM)\,(+)\,[0 :_M I])\\ &= \operatorname{ann}(\operatorname{ann} I\,(+)\,(\operatorname{ann} I)\,M))\\ &= \operatorname{ann}(\operatorname{ann} I) \cap \operatorname{ann}((\operatorname{ann} I)\,M)\,(+)\,[0 :_M \operatorname{ann} I]\\ &= \operatorname{ann}(\operatorname{ann} I)\,(+)\,\operatorname{ann}(\operatorname{ann} I)\,M.\end{aligned}$$

This shows that $I = \operatorname{ann}(\operatorname{ann} I)$ and I is a comultiplication in R. The statement is reversible.

(2) Assume M is faithful multiplication and $I\,(+)\,IM$ is coidempotent in $R\,(M)$. Then

$$\begin{aligned} I\,(+)\,IM &= \left(\operatorname{ann}\left(I\,(+)\,IM\right)\right)^2 \\ &= \operatorname{ann}\left(\operatorname{ann} I\,(+)\,[0:_M I]\right)^2 \\ &= \operatorname{ann}\Big(\left(\operatorname{ann} I\right)^2 (+)\left(\operatorname{ann} I\right)\left(\operatorname{ann} I\right) M\Big) \\ &= \operatorname{ann}\Big(\left(\operatorname{ann} I\right)^2 (+)\left(\operatorname{ann} I\right)^2 M\Big) \\ &= \operatorname{ann}\left(\operatorname{ann} I\right)^2 \cap \operatorname{ann}\Big(\left(\operatorname{ann} I\right)^2 M\Big)\,(+)\left[0:_M \left(\operatorname{ann} I\right)^2\right] \\ &= \operatorname{ann}\left(\operatorname{ann} I\right)^2 (+)\operatorname{ann}\left(\operatorname{ann} I\right)^2 M. \end{aligned}$$

Hence $I = \operatorname{ann}\left(\operatorname{ann} I\right)^2$ and I is coidempotent in R. The statement is reversible.

(3) Suppose $I\,(+)\,IM$ is copure in $R\,(M)$. Let J be an ideal of R. Then

$$\begin{aligned} \left[I\,(+)\,IM :_{R(M)} J\,(+)\,JM\right] &= I\,(+)\,IM + \operatorname{ann}\left(J\,(+)\,JM\right) \\ &= I\,(+)\,IM\,(+)\operatorname{ann} J\,(+)\,[0:_M J] \\ &= \left(I + \operatorname{ann} J\right)(+)\,IM + [0:_M J]\,. \end{aligned}$$

On the other hand,

$$\begin{aligned} \left[I\,(+)\,IM :_{R(M)} J\,(+)\,JM\right] &= [I:J] \cap [IM:JM]\,(+\,[IM:_M J]) \\ &= [I:J]\,(+)\,[IM:_M J] \end{aligned}$$

It follows that $[I:J] = I + \operatorname{ann} J$ and I copure in R. Note that $[IM:_M J] = IM\,(+)\,[0:_M J]$ means that IM is also copure in M.

□

An immediate consequence is the following corollary. It follows by [**10**] Proposition 2.1 and Proposition 2.2.

Corollary 2.3. *Let R be a ring, M a finitely generated faithful multiplication R-module and N a submodule of M.*

(1) *N is a comultiplication in M if and only if $[N:M]\,(+)\,N$ is a comultiplication in $R\,(M)$.*
(2) *N is coidempotent in M if and only if $[N:M]\,(+)\,N$ is coidempotent in $R\,(M)$.*
(3) *If $[N:M]\,(+)\,N$ is copure in $R\,(M)$ then N is copure in M.*

The following three results show how properties of the ideal $I\,(+)\,N$ can be tranferred to its components and conversely.

Theorem 2.4. *Let R be a ring, M an R-module and $I\,(+)\,N$ a homogeneous ideal of $R\,(M)$.*

(1) *If M is faithful and $I\,(+)\,N$ is a comultiplication in $R\,(M)$ then N is a comultiplication in M. Assuming further that M is multiplication then I is a comultiplication in R.*
(2) *If M is faithful and $I\,(+)\,N$ is coidempotent in $R\,(M)$ then N is coidempotent in M. Assuming further that M is multiplication then I is coidempotent in R.*
(3) *If $I\,(+)\,N$ is copure in $R\,(M)$ then N is copure in M and I copure in R.*

Proof. (1) Suppose $I\,(+)\,N$ is a comultiplication. Then

$$I\,(+)\,N = \operatorname{ann}(\operatorname{ann}(I\,(+)\,N)) = \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} N)\,(+)\,[0 :_M I])$$

But $IM \subseteq N$, and hence $\operatorname{ann} N \subseteq \operatorname{ann}(IM) = \operatorname{ann} I$, So,

$$\begin{aligned} I\,(+)\,N &= \operatorname{ann}(\operatorname{ann} N\,(+)\,[0 :_M I]) \\ &= \operatorname{ann}(\operatorname{ann} N \cap \operatorname{ann}[0 :_M I])\,(+)\,[0 :_M \operatorname{ann} N]\,. \end{aligned}$$

It follows that $N = [0 :_M \operatorname{ann} N]$ and N is a comultiplication. Suppose now M is faithful and multiplication. Then $[0 :_M I] = (\operatorname{ann} I)\,M$ and hence

$$\operatorname{ann}[0 :_M I] = \operatorname{ann}((\operatorname{ann} I)\,M) = \operatorname{ann}(\operatorname{ann} I)$$

because M is faithful. Moreover, note that $\operatorname{ann} N \subseteq \operatorname{ann} I$ and therefore $\operatorname{ann}(\operatorname{ann} I) \subseteq \operatorname{ann}(\operatorname{ann} N)$. This gives

$$I = \operatorname{ann}(\operatorname{ann} N) \cap \operatorname{ann}[0 :_M I] = \operatorname{ann}(\operatorname{ann} I)\,,$$

and I is a comultiplication.

(2) Assume M is faithful and $I\,(+)\,N$ is coidempotent. Then

$$\begin{aligned} I\,(+)\,N &= \operatorname{ann}(\operatorname{ann}(I\,(+)\,N))^2 \\ &= \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} N)\,(+)\,[0 :_M I])^2 \\ &= \operatorname{ann}(\operatorname{ann} N\,(+)\,[0 :_M I])^2 \\ &= \operatorname{ann}\Big((\operatorname{ann} N)^2\,(+)\,(\operatorname{ann} N)\,[0 :_M I]\Big) \\ &= \operatorname{ann}(\operatorname{ann} N)^2 \cap (\operatorname{ann}(\operatorname{ann} N\,[0 :_M I])\,(+)\,\Big[0 :_M (\operatorname{ann} N)^2\Big]\,. \end{aligned}$$

It follows that $N = \Big[0 :_M (\operatorname{ann} N)^2\Big]$ and N is coidempotent in M. Suppose now that M is faithful multiplication. Then

$$\begin{aligned} I &= \operatorname{ann}(\operatorname{ann} N)^2 \cap \operatorname{ann}((\operatorname{ann} N)\,(\operatorname{ann} I)\,M) \\ &= \operatorname{ann}(\operatorname{ann} N)^2 \cap \operatorname{ann}((\operatorname{ann} N)\operatorname{ann} I)\,. \end{aligned}$$

But $\operatorname{ann} N \subseteq \operatorname{ann} I$. Thus $(\operatorname{ann} N)^2 \subseteq (\operatorname{ann} N)\,(\operatorname{ann} I)$, and hence

$$\operatorname{ann}((\operatorname{ann} N)\,(\operatorname{ann} I)) \subseteq \operatorname{ann}\Big((\operatorname{ann} N)^2\Big)\,.$$

Therefore, $I = \operatorname{ann}((\operatorname{ann} N)\,(\operatorname{ann} I))$. Again, since

$$\operatorname{ann} N \subseteq \operatorname{ann} I,\ (\operatorname{ann} N)\,(\operatorname{ann} I) \subseteq (\operatorname{ann} I)^2\,,$$

and hence

$$\operatorname{ann}\Big((\operatorname{ann} I)^2\Big) \subseteq \operatorname{ann}(\operatorname{ann} N \operatorname{ann} I)\,.$$

It follows that

$$I \supseteq \operatorname{ann}(\operatorname{ann} I)^2 \supseteq \operatorname{ann}(\operatorname{ann} I) \supseteq I,$$

so that $I = \operatorname{ann}(\operatorname{ann} I)^2$ and I is coidempotent.

(3) Let $I\,(+)\,N$ be a copure ideal of $R\,(M)$. Let J be an ideal of R. Then

$$\begin{aligned}\left[I\,(+)\,N :_{R(M)} J\,(+)\,JM\right] &= I\,(+)\,N\,(+)\operatorname{ann}(J\,(+)\,JM)\\ &= I\,(+)\,N\,(+)\,(\operatorname{ann}J\,(+)\,[0 :_M J])\\ &= I + \operatorname{ann}J\,(+)\,N + [0 :_M J]\,.\end{aligned}$$

On the other hand,

$$\left[I\,(+)\,N :_{R(M)} J\,(+)\,JM\right] = [I : J] \cap [N : JM]\,(+)\,[N :_M J]\,.$$

Since $I\,(+)\,N$ is a homogeneous ideal of $R\,(M)$, $IM \subseteq N$, hence $I \subseteq [N : M]$, and this gives that $[I : J] \subseteq [[N : M] : J] \subseteq [N : JM]$.
It follows that

$$\left[I\,(+)\,N :_{R(M)} J\,(+)\,JM\right] = [I : J]\,(+)\,[N :_M J]\,.$$

This shows that

$$[I : J] = I + \operatorname{ann}J, \text{ and } [N :_M J] = N + [0 :_M J]\,.$$

This gives that I is copure in R and N copure in M, and this completes the proof of the theorem.

□

Proposition 2.5. *Let R be a ring, M a faithful multiplication R-module, and $I\,(+)\,N$ a homogeneous ideal of $R\,(M)$. Then the following holds.*

(1) *If I is a comultiplication and $\operatorname{ann}N \subseteq \operatorname{ann}I$ then $I\,(+)\,N$ is a comultiplication.*
(2) *If I is coidempotent and $\operatorname{ann}N \subseteq \operatorname{ann}I$ then $I\,(+)\,N$ is coidempotent.*
(3) *Let I be a comultiplication in R and N a comultiplication in M such that $\operatorname{ann}I + [IM : N] = R$ then $I\,(+)\,N$ is a comultiplication.*
(4) *Let I be coidempotent in R and N coidempotent in M such that $\operatorname{ann}I + [IM : N] = R$ then $I\,(+)\,N$ is coidempotent.*

Proof. We only prove parts (1) and (3). Parts (2) and (4) are similar.

(1) Since M is a faithful multiplication and I is a comultiplication, it follows by Proposition 2.2 that $I\,(+)\,IM$ is a comultiplication in $R\,(M)$. Now,

$$\operatorname{ann}(I\,(+)\,N) = (\operatorname{ann}I \cap \operatorname{ann}N)\,(+)\,[0 :_M I]\,.$$

But $\operatorname{ann}N \subseteq \operatorname{ann}I$. Thus $\operatorname{ann}(I\,(+)\,N) = \operatorname{ann}I\,(+)\,[0 :_M I]$. On the other hand $\operatorname{ann}(I\,(+)\,IM) = \operatorname{ann}I\,(+)\,[0 :_M I]$. Since $I\,(+)\,IM \subseteq I\,(+)\,N$, the result follows from [**10**] Proposition 3.2.

(3) Since I is a comultiplication, $I\,(+)\,IM$ is a comultiplication. Also N is comultiplication in M, therefore $0\,(+)\,N$ is a comultiplication in $R\,(M)$. As

$$\operatorname{ann}I + [IM : N] = R,$$

we obtain that

$$\left[0\,(+)\,N :_{R(M)} I\,(+)\,IM\right] + \left[I\,(+)\,IM :_{R(M)} 0\,(+)\,N\right] = R\,(M)\,,$$

and the result now follows by [**10**] Proposition 3.2.

□

Unfortunately, the converse of statement (3) of each of Proposition 2.1, Proposition 2.2 and Theorem 2.4 are not true in general.

The next theorem gives sufficient conditions for $I\,(+)\,IM$, $0\,(+)\,N$ and $I\,(+)\,N$ to be copure ideals in $R\,(M)$.

Theorem 2.6. *Let R be a ring and M be a finitely generated faithful multiplication R-module. Let $R(M)$ be a homogeneous ring.*

(1) *If I is copure in R then $I\,(+)\,IM$ is copure in $R\,(M)$.*

(2) *Let every ideal $J\,(+)\,K$ of $R\,(M)$ has the property $\operatorname{ann} J \subseteq \operatorname{ann} K$. If N is copure in M then $0\,(+)\,N$ is copure in $R\,(M)$.*

(3) *Let every ideal $J\,(+)\,K$ of $R\,(M)$ has the property $\operatorname{ann} J \subseteq \operatorname{ann} K$. If I is copure in R and N copure in M then $I\,(+)\,N$ is copure in $R\,(M)$.*

Proof. (1) Let $J\,(+)\,K$ be an ideal of $R\,(M)$. Then $JM \subseteq K$, and hence $\operatorname{ann} K \subseteq \operatorname{ann}(JM) = \operatorname{ann} J$. Suppose that I is copure in R. Then by [**10**] Proposition 2.1 IM copure in M. Now

$$\begin{aligned}
\left[I\,(+)\,IM :_{R(M)} J\,(+)\,K\right] &= [I : J] \cap [IM : K]\,(+)\,[IM :_M J] \\
&\subseteq [I : J] \cap [I : [K : M]]\,(+)\,IM + [0 :_M J] \\
&= (I + \operatorname{ann} J) \cap (I + \operatorname{ann}[K : M])\,(+)\,IM + [0 :_M J] \\
&= (I + \operatorname{ann} J) \cap (I + \operatorname{ann} K)\,(+)\,IM + [0 :_M J] \\
&= I + \operatorname{ann} K\,(+)\,IM\,(+)\,[0 :_M J] \\
&= I\,(+)\,IM + \operatorname{ann} K\,(+)\,[0 :_M J] \\
&= I\,(+)\,IM + (\operatorname{ann} J \cap \operatorname{ann} K) + [0 :_M J] \\
&= I\,(+)\,IM + \operatorname{ann}(J\,(+)\,K) \subseteq \left[I\,(+)\,IM :_{R(M)} J\,(+)\,K\right],
\end{aligned}$$

so that

$$\left[I\,(+)\,IM :_{R(M)} J\,(+)\,K\right] = I\,(+)\,IM + \operatorname{ann}(J\,(+)\,K),$$

and $I\,(+)\,IM$ is copure in $R\,(M)$.

(2) Suppose N is copure in M. By [**10**] Proposition 2.1, $[N : M]$ is copure in R. Let $J\,(+)\,K$ be an ideal of $R\,(M)$ such that $\operatorname{ann} J = \operatorname{ann} K$. Then

$$\begin{aligned}
\left[0\,(+)\,N :_{R(M)} J\,(+)\,K\right] &= \operatorname{ann} J \cap [N : K]\,(+)\,[N :_M J]) \\
&= \operatorname{ann} J \cap [[N : M] : [K : M])\,(+)\,N + [0 :_M J] \\
&= \operatorname{ann} J \cap [[N : M] + \operatorname{ann}[K : M])\,(+)\,N + [0 :_M J] \\
&= \operatorname{ann} K \cap ([N : M] + \operatorname{ann} K)\,(+)\,N + [0 :_M J] \\
&= \operatorname{ann} K + N + [0 :_M J] = 0\,(+)\,N + \operatorname{ann} K\,(+)\,[0 :_M J] \\
&= 0\,(+)\,N + (\operatorname{ann} J \cap \operatorname{ann} K)\,(+)\,[0 :_M J] \\
&= 0\,(+)\,N + \operatorname{ann}(J\,(+)\,K).
\end{aligned}$$

Hence $0\,(+)\,N$ is copure in $R\,(M)$.

(3) Let I be copure in R and N copure in M. By [**10**] Proposition 2.1, $[N : M]$ is copure in R. Let $J\,(+)\,K$ be an ideal of $R\,(M)$ with $\operatorname{ann} J = \operatorname{ann} K$. Then

$$\begin{aligned}\left[I\,(+)\,N :_{R(M)} J\,(+)\,K\right] &= [I : J] \cap [N : K]\,(+)\,[N :_M J] \\ &= [I : J] \cap [[N : M] : [K : M]]\,(+)\,[N :_M J] \\ &= (I + \operatorname{ann} J) \cap ([N : M] + \operatorname{ann}[K : M])\,(+)\,N + [0 :_M J] \\ &= (I + \operatorname{ann} J) \cap ([N : M] + \operatorname{ann} K)\,(+)\,N + [0 :_M J]\end{aligned}$$

But $\operatorname{ann} J = \operatorname{ann} K$ and $I \subseteq [N : M]$. Thus

$$\begin{aligned}\left[I\,(+)\,N :_{R(M)} J\,(+)\,K\right] &= (I + \operatorname{ann} K)\,(+)\,N + [0 :_M J] \\ &= I\,(+)\,N + (\operatorname{ann} J \cap \operatorname{ann} K)\,(+)\,[0 :_M J] \\ &= I\,(+)\,N + \operatorname{ann}(J\,(+)\,K).\end{aligned}$$

This gives that $I\,(+)\,N$ is copure ideal of $R\,(M)$.

□

The following result shows how the comultiplication and copure properties of an ideal I of R transfer to the ideal $I\,(+)\,M$ of $R\,(M)$.

Proposition 2.7. *Let R be a ring, M an R-module and I a non-zero ideal of R.*
(1) *If M is faithful and $I\,(+)\,M$ is a comultiplication then I is a comultiplication. The converse is true if we assume further that M is multiplication.*
(2) *If $I\,(+)\,M$ is copure then I is copure. The converse is true if* $\operatorname{ann} I \subseteq \operatorname{ann} M$.

Proof. (1) The first part of the statement is true by Theorem 2.4. Now let M be faithful multiplication and I a comultiplication. Then

$$\begin{aligned}\operatorname{ann}(\operatorname{ann}(I\,(+)\,M)) &= \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} M)\,(+)\,[0 :_M I]) \\ &= \operatorname{ann}(0\,(+)\,(\operatorname{ann} I)\,M) \\ &= \operatorname{ann}((\operatorname{ann} I)\,M)\,(+)\,M \\ &= \operatorname{ann}(\operatorname{ann} I)\,(+)\,M \\ &= I\,(+)\,M.\end{aligned}$$

(2) If $I\,(+)\,M$ is copure then I is copure follows by Theorem 4. Suppose now that I is copure. Let H be ideal of $R\,(M)$. Then

$$\left[I\,(+)\,M :_{R(M)} H\right] = \left[I\,(+)\,M :_{R(M)} H + I\,(+)\,M\right].$$

Since $0\,(+)\,M \subseteq H + I\,(+)\,M$, $H + I\,(+)\,M = J\,(+)\,M$ for some ideal $I \subseteq J$. It follows that

$$\begin{aligned}\left[I\,(+)\,M :_{R(M)} H\right] &= \left[I\,(+)\,M :_{R(M)} J\,(+)\,M\right] = [I : J]\,(+)\,M \\ &= (I + \operatorname{ann} J)\,(+)\,M = I\,(+)\,M + \operatorname{ann} J\,(+)\,M \\ &= I\,(+)\,M + (\operatorname{ann} J \cap \operatorname{ann} M)\,(+)\,[0 :_M J] \\ &= I\,(+)\,M + \operatorname{ann}(J\,(+)\,M) = I\,(+)\,M + \operatorname{ann}(H + I\,(+)\,M) \\ &\subseteq I\,(+)\,M + \operatorname{ann} H \subseteq \left[I\,(+)\,M :_{R(M)} H\right].\end{aligned}$$

This gives that $\left[I\,(+)\,M :_{R(M)} H\right] = I\,(+)\,M + \operatorname{ann} H$, and $I\,(+)\,M$ is copure.

□

The next result shows how properties of the ideal $I\,(+)\,M$ of $R\,(M)$ are related to those of I when M is a projective R-module.

Proposition 2.8. *Let R be a ring and M a projective R-module. Let I be an ideal of R.*

(1) *$I\,(+)\,M$ is a comultiplication ideal of $R\,(M)$ if and only if I is a comultiplication ideal of R and $M = IM$.*

(2) *$I\,(+)\,M$ is a coidempotent ideal of $R\,(M)$ if and only if I is a coidempotent ideal of R and $M = IM$.*

Proof. (1) Let $I\,(+)\,M$ be a comultiplication. Then

$$\begin{aligned}
&\operatorname{ann}(\operatorname{ann}(I\,(+)\,M))\\
&\quad= \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} M)\,(+)\,[0 :_M I])\\
&\quad= \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} M)\,(+)\,(\operatorname{ann} I)\,M)\\
&\quad= \operatorname{ann}(\operatorname{ann} I \cap \operatorname{ann} M) \cap \operatorname{ann}((\operatorname{ann} I)\,M)\,(+)\,\operatorname{ann}(\operatorname{ann} I \cap \operatorname{ann} M)\,M\\
&\quad\subseteq \operatorname{ann}(\operatorname{ann} I) \cap \operatorname{ann}((\operatorname{ann} I)\,M)\,(+)\,\operatorname{ann}(\operatorname{ann} I)\,M\\
&\quad= \operatorname{ann}(\operatorname{ann} I)\,(+)\,\operatorname{ann}(\operatorname{ann} I)\,M = I\,(+)\,IM\\
&\quad\subseteq I\,(+)\,M \subseteq \operatorname{ann}(\operatorname{ann}(I\,(+)\,M)),
\end{aligned}$$

so that $I\,(+)\,M = I\,(+)\,IM = \operatorname{ann}(\operatorname{ann}(I\,(+)\,M))$. Hence $I\,(+)\,M$ is a comultiplication and $M = IM$. The converse follows by Proposition 2.2.

(2) Assume that $I\,(+)\,M$ is coidempotent. Then

$$\begin{aligned}
\operatorname{ann}(\operatorname{ann}(I\,(+)\,M))^2 &= \operatorname{ann}((\operatorname{ann} I \cap \operatorname{ann} M)\,(+)\,(\operatorname{ann} I)\,M)^2\\
&= \operatorname{ann}\Big((\operatorname{ann} I \cap \operatorname{ann} M)^2\,(+)\,\operatorname{ann} I\,(\operatorname{ann} I \cap \operatorname{ann} M)\,M\Big)\\
&= \operatorname{ann}(\operatorname{ann} I \cap \operatorname{ann} M)^2 \cap \operatorname{ann}(\operatorname{ann} I\,(\operatorname{ann} I \cap \operatorname{ann} M)\,M)\\
&\qquad (+)\,\operatorname{ann}(\operatorname{ann} I\,(\operatorname{ann} I \cap \operatorname{ann} M)\,M)\,M\\
&\subseteq \operatorname{ann}(\operatorname{ann} I)^2\,(+)\,\operatorname{ann}(\operatorname{ann} I)^2\,M\\
&= I\,(+)\,IM \subseteq I\,(+)\,M \subseteq \operatorname{ann}(\operatorname{ann}(I\,(+)\,M))^2.
\end{aligned}$$

Thereore $I\,(+)\,M$ is coidempotent and $M = IM$. The converse follows by Proposition 2.2.

□

We close our work by a result which determines when the ring $R\,(M)$ is a comultiplication ring.

Theorem 2.9. *Let R be ring and M an R-module.*

(1) *Let M be a faithful multiplication or finitely generated projective. If $R\,(M)$ is a comultiplication then R is a comultiplication and M a comultiplication. The converse is true if $\left[0 :_{R(M)} H\right] + \left[0 :_{R(M)} 0\,(+)\,M\right] = \left[0 :_{R(M)} H \cap 0\,(+)\,M\right]$ for every ideal of H of $R\,(M)$.*

(2) *Let M be faithful multiplication. If $R\,(M)$ is coidempotent, then R is coidempotent and M is coidempotent.*

(3) *If $R\,(M)$ is copure then R is copure and M copure.*

Proof. We first prove (1). Assume that M is faithful multiplication or finitely generated projective. Suppose $R(M)$ is a comultiplication. Let I be an ideal of R and N a submodule of M. Then $I(+)M$ and $0(+)N$ are comultiplication ideals of $R(M)$. It follows by Propositions 2.1 and 2.8, I is a comultiplication ideal of R, and hence R is a comultiplication. Then N is also a comultiplication which implies that M is a comultiplication. For the converse assume M is faithful multiplication or finitely generated projective and the equality

$$\left[0:_{R(M)} H\right]+\left[0:_{R(M)} 0(+)M\right]=\left[0:_{R(M)} H\cap 0(+)M\right]$$

is satisfied. Let H be an ideal of $R(M)$. Let $H+0(+)M=I(+)M$ and $H\cap 0(+)M=0(+)N$ for some ideal I of R and some submodule N of M. Since I is a comultiplication, it follows by Propositions 2.7 and 2.8 that $I(+)M$ is a comultiplication. Then

$$\begin{aligned} H+0(+)M=I(+)M &= \operatorname{ann}(\operatorname{ann}(I(+)M)) \\ &= \operatorname{ann}(\operatorname{ann}(H+0(+)M)) \\ &= \operatorname{ann}(\operatorname{ann} H\cap \operatorname{ann}(0(+)M)) \\ &\supseteq \operatorname{ann}(\operatorname{ann} H)+\operatorname{ann}(\operatorname{ann}(0(+)M)) \\ &\supseteq \operatorname{ann}(\operatorname{ann} H)+0(+)M\supseteq H+0(+)M, \end{aligned}$$

so that $H+0(+)M=I(+)M=\operatorname{ann}(\operatorname{ann} H)+0(+)M$. Now, in case M is faithful multiplication or finitely generated projective and N is a comultiplication, then $0(+)N$ is a comultiplication by Proposition 2.1 and the remark after it. Next

$$\begin{aligned} H\cap 0(+)M=0(+)N &= \operatorname{ann}(\operatorname{ann}(0(+)N)) \\ &= \operatorname{ann}(\operatorname{ann}(H\cap 0(+)M)) \\ &= \operatorname{ann}(\operatorname{ann} H+\operatorname{ann}(0(+)M)) \\ &= \operatorname{ann}(\operatorname{ann} H)\cap \operatorname{ann}(\operatorname{ann}(0(+)M)) \\ &\supseteq \operatorname{ann}(\operatorname{ann} H)\cap 0(+)M\supseteq H\cap 0(+)M, \end{aligned}$$

so that $H\cap 0(+)M=\operatorname{ann}(\operatorname{ann} H)\cap 0(+)M$. Apply the modular law, one gets that

$$\begin{aligned} H &= (H+0(+)M)\cap H=(\operatorname{ann}(\operatorname{ann} H)+0(+)M)\cap H \\ &= \operatorname{ann}(\operatorname{ann} H)+(H\cap 0(+)M) \\ &= \operatorname{ann}(\operatorname{ann} H)+(\operatorname{ann}(\operatorname{ann} H)\cap 0(+)M) \\ &= \operatorname{ann}(\operatorname{ann} H), \end{aligned}$$

and hence $R(M)$ is a comultiplication.

The proofs of (2) and (3) are similar to the first part of (1) using Propositions 2.1 and 2.2. This finishes the proof of the theorem. □

References

[1] Y. Al Shaniafi and P. F. Smith, *Comultiplication modules over commutative rings*, J. Commut. Algebra **3** (1) (2011), 1–29.

[2] M. M. Ali, *Idealization and theorems of D. D. Anderson*, Commun. Algebra **34** (12) (2006), 4479–4501.

[3] M. M. Ali, *Idealization and theorems of D. D. Anderson II*, Commun. Algebra **35** (9) (2007), 2767–2792.
[4] M. M. Ali, *Multiplication modules and homogeneous idealization*, Beitr. Algebra Geom. **47** (1) (2006), 249–270.
[5] M. M. Ali, *Multiplication modules and homogeneous idealization II*, Beitr. Algebra Geom. **48** (2) (2007), 321–343.
[6] M. M. Ali, *Multiplication modules and homogeneous idealization III*, Beitr. Algebra Geom. **49** (2) (2008), 449–479.
[7] M. M. Ali, $\frac{1}{2}$-*Cancellation modules and homogeneous idealization*, Commun. Algebra **35** (11) (2007), 3524–3543.
[8] M. M. Ali, $\frac{1}{2}$-*Cancellation modules and homogeneous idealization II*, Commun. Algebra **36** (10) (2008), 3842–3864.
[9] M. M. Ali, *Idempotent and nilpotent submodules of multiplication modules*, Commun. Algebra **36** (12) (2008), 4620–4642.
[10] M. M. Ali and F. Al Kharousi, *Comultplication submodules of multiplication modules*, Commun. Algebra **48** (11) (2020), 4832–4845.
Accepted for publication in Commun. Algebra.
[11] M. M. Ali and D. J. Smith, *Pure submodules of multiplication modules*, Beitr. Algebra Geom. **45** (1) (2004), 61–74.
[12] M. M. Ali and D. J. Smith, *Some remarks on multiplication and projective modules*, Commun. Algebra **32** (10) (2004), 3897–3909.
[13] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (1) (2009), 3–56.
[14] H. Ansari-Toroghy and F. Farshadifar, *The dual notion of multiplication modules*, Taiwanese J. Math., **11** (4) (2007), 1189–1201.
[15] H. Ansari-Toroghy and F. Farshadifar, *Fully idempotent and coidempotent modules*, Bull. Iran. Math. Soc. **38** (4) (2012), 987–1005.
[16] A. Barnard, *Multiplication modules*, J. Algebra **71** (1981), 174–178.
[17] J. A. Huckaba, *Commutative Rings with Zero Divisors*, Monographs and Textbooks in Pure and Applied Mathematics 117, Marcel Dekker, New York, 1988.
[18] H. Matsumara, *Commutative Ring Theory*, Studies in advanced mathematics 8, Cambridge University Press, New York, 1986.
[19] M. Nagata, *Local Rings*, Tracts in pure and applied mathematics 13, Interscience Publishers, New York, 1962.

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