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SPECTRUM OF k-QUASI-CLASS A_n OPERATORS

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Abstract. In this paper, we introduce a new class of operators, called k-quasi-class A_n operators, which is a superclass of class A and a subclass of (n,k)-quasiparanormal operators. We will show basic structural properties and some spectral properties of this class of operators. We show that, if T is of k-quasi-class A_n then $T-\lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. Also, we will prove T is polaroid and Weyl's theorem holds for T and $f(T)$, where f is an analytic function in a neighborhood of the spectrum of T. Moreover, we show that if λ is an isolated point of $\sigma(T)$ and E is the Riesz idempotent of the spectrum of a k-quasi-class A_n operator T, then $E\mathscr{H} = \ker(T-\lambda)$ if $\lambda \neq 0$ and $E\mathscr{H} = \ker(T^{n+1})$ if $\lambda = 0$.

1. Introduction

Let $\mathscr{H}$ be an infinite dimensional complex Hilbert and $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. Every operator T can be decomposed into $T = U|T|$ with a partial isometry U, where $|T|$ is the square root of T^*T. If U is determined uniquely by the kernel condition $\ker(U) = \ker(|T|)$, then this decomposition is called the *polar decomposition*, which is one of the most important results in operator theory (for example see [**6**], [**9**], [**12**] and [**13**]). In this paper, $T = U|T|$ denotes the polar decomposition satisfying the kernel condition $\ker(U) = \ker(|T|)$.

An easy extension of normal operators, hyponormal operators have been studied by many researchers. Though there are many unsolved interesting problems for this class (for example the invariant subspace problem), one of recent trends in operator theory is to study natural extensions of hyponormal operators. Here we introduce some of these non-hyponormal operators. Following [**8**], an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be hyponormal if $T^*T \geq TT^*$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be paranormal if $\|Tx\|^2 \leq \|T^2x\|$ for every unit vector $x \in \mathscr{H}$ ([**7**]). Furthermore, $T \in \mathscr{B}(\mathscr{H})$ is said to be n-paranormal operator, if $\|Tx\|^{n+1} \leq \|T^{n+1}x\| \|x\|^n$ for all $x \in \mathscr{H}$ ([**5**]). An operator T is said to be (n,k)-quasiparanormal as in [**19**], if

$$\|T(T^kx)\| \leq \|T^{n+k+1}x\|^{\frac{1}{n+1}} \|T^kx\|^{\frac{n}{n+1}}$$

for all $x \in \mathscr{H}$. T. Furuta et al. introduced in [**8**] a very interesting class of bounded linear operators in Hilbert space: operators satisfying defined by $|T^2| \geq |T|^2$, which

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we call class A here. They showed that class A is a subclass of paranormal operators. Hence, we have

$$\begin{aligned}\{\text{Hyponormal}\} &\subset \{\text{class } A\} \subset \{\text{paranormal}\}\\ &\subset \{n\text{-paranormal}\} \subset \{(n,k)\text{-quasiparanormal}\}.\end{aligned}$$

Throughout this paper, we shall denote the spectrum, the point spectrum and the isolated points of the spectrum of $T \in \mathscr{B}(\mathscr{H})$ by $\sigma(T), \sigma_p(T)$ and $iso\sigma(T)$ respectively. The range and the kernel of $T \in \mathscr{B}(\mathscr{H})$ will be denoted by $\mathscr{R}(T)$ and $\ker(T)$, respectively. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set S will be denoted by $\overline{S}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$.

In this paper, we introduce a new class of operators, called k-quasi-class A_n operators, which is a superclass of class A and a subclass of (n,k)-paranormal operators. We will show basic structural properties and some spectral properties of this class of operators. We show that, if T is k-quasi-class A_n then $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$. Also, we will prove T is isoloid and Weyl's theorem holds for T and $f(T)$, where f is an analytic function in a neighborhood of the spectrum of T. It is also shown that if E is the Riesz idempotent for a non-zero isolated point of the spectrum of ak-quasi-class A_n operator T, then $E\mathscr{H} = \ker(T - \lambda)$ if $\lambda \neq 0$ and $E\mathscr{H} = \ker(T^{n+1})$ if $\lambda = 0$.

2. Spectral Properties of k-quasi-class A_n Operators

Definition 2.1. An operator $T \in \mathscr{B}(\mathscr{H})$ is said be of class A_n (equivalently $T \in \mathbb{A}_n$) if

$$|T^{n+1}|^{\frac{2}{n+1}} \geq |T|^2$$

for some positive integer n.

Remark 2.2. From the previous definitions, we have that
(i) for $n = 1$, then $\mathbb{A}_1$ coincides with the set of class A operators.
(ii) if $T \in \mathbb{A}_n$, then T is n-paranormal.

Theorem 2.3. *If $T \in \mathbb{A}_n$, then T is normaloid.*

Proof. Let T be of class A_n. We may assume $\|T\| = 1$. Let $\|x\| = 1$. Then

$$\begin{aligned}\|Tx\|^{n+1} &= \left\langle |T|^2 x, x\right\rangle^{\frac{n+1}{2}} \leq \left\langle |T^{n+1}|^{\frac{2}{n+1}} x, x\right\rangle^{\frac{n+1}{2}}\\ &\leq \left\langle |T^{n+1}|^2 x, x\right\rangle^{\frac{1}{2}} = \left\|T^{n+1}x\right\| \leq \left\|T^{n-1}\right\| \left\|T^2 x\right\| \leq \left\|T^2 x\right\| \leq 1.\end{aligned}$$

Hence

$$\frac{\|Tx\|^{n+1}}{\|x\|^n} \leq \left\|T^2 x\right\| \leq \|x\| \text{ for } x \neq 0. \tag{2.1}$$

Since $\|T\| = 1$, there exists a sequence of unit vectors $\{x_j\}$ such that $\|Tx_j\| \to 1$. Then (2.1) implies

$$1 \leftarrow \|Tx_j\|^{n+1} \leq \left\|T^2 x_j\right\| \leq \|x_j\| = 1.$$

Hence $\left\|T^2x_j\right\| \to 1$ and $\left\|T^2\right\| = 1 = \|T\|^2$. By taking Tx instead of x in (2.1), we have

$$\frac{\left\|T^2x\right\|^{n+1}}{\|Tx\|^n} \le \left\|T^3x\right\| \le \|Tx\|. \tag{2.2}$$

Hence

$$1 \leftarrow \frac{\left\|T^2x_j\right\|^{n+1}}{\|Tx_j\|^n} \le \left\|T^3x_j\right\| \le \|Tx_j\| \to 1.$$

Hence $\left\|T^3x_j\right\| \to 1$ and $\left\|T^3\right\| = 1 = \|T\|^3$. Similarly, we have $\|T^n\| = \|T\|^n$. Thus T is normaloid. □

Definition 2.4. We say that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be of k-quasi-class A_n (equivalently $T \in \mathbb{Q}(A_n, k)$) if

$$T^{*k}(|T^{n+1}|^{\frac{2}{n+1}} - |T|^2)T^k \ge 0$$

for some positive integers m and k.

Note that for $n = 1$, the set of k-quasi-class A_n operators coincides with the set of k-quasi-class A operators.

Example 2.5. Suppose that $\mathscr{H}$ is the direct sum of a denumerable number of copies of two dimensional Hilbert space $\mathbb{R} \times \mathbb{R}$ and let A and B be two positive operators on $\mathbb{R}\times\mathbb{R}$. For any fixed positive integer m, define an operator $T = T_{A,B,m}$ on $\mathscr{H}$ as follows:

$$T((x_1, x_2, \cdots)) = (0, A(x_1), A(x_2), \cdots, A(x_m), B(x_{m+1}), \cdots).$$

It's adjoint T^* is given by

$$T^*((x_1, x_2, \cdots)) = (A(x_2), A(x_3), \cdots, A(x_m), B(x_{m+1}), \cdots).$$

For any $m \ge n$, $T = T_{A,B,m}$ is of k-quasi-class A_n if and only if A and B satisfies

$$A^k(A^{n+1-i}B^{2i}A^{n+1-i})^{\frac{2}{n+1}}A^k \ge A^{2+2k} \text{ for } i = 1, \cdots, n.$$

If $A = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then T is of k-quasi-class A_2.

Since $S \ge 0$ implies $R^*SR \ge 0$, the following result is trivial. The converse is also true if T has dense range.

Lemma 2.6. *If $T \in \mathbb{A}_n$ for some integer $n \ge 1$, then $T \in \mathbb{Q}(A_n, k)$ for every integer $k \ge 1$.*

Lemma 2.7. [**10**, Hansen's Inequality] *If $A, B \in \mathscr{B}(\mathscr{H})$ satisfy $A \ge 0$ and $\|B\| \le 1$, then*

$$(B^*AB)^\alpha \ge B^*A^\alpha B \qquad \text{for all } \alpha \in (0, 1].$$

Lemma 2.8. *Let $T \in \mathbb{Q}(A_n, k)$ and T^k does not have a dense range. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k}),$$

where $T_1 = T|_{\overline{\Re(T^k)}}$ is the restriction of T to $\overline{\Re(T^k)}$, $T_1 \in \mathbb{A}_n$ and T_3 is nilpotent of nilpotency k. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Consider the matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\Re(T^k)} \oplus \ker(T^{*k})$:

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}.$$

Let P be the orthogonal projection onto $\overline{\Re(T^k)}$. Then $\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} = TP = PTP$. Since $T \in \mathbb{Q}(A_n, k)$, we have

$$P\left(|T^{n+1}|^{\frac{2}{n+1}} - |T|^2\right)P \geq 0.$$

We remark that

$$P|T|^2P = PT^*TP = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then by Lemma 2.7

$$\begin{aligned} P|T^{n+1}|^{\frac{2}{n+1}}P &= P(T^{*(n+1)}T^{n+1})^{\frac{1}{n+1}}P \\ &\leq \left(PT^{*(n+1)}T^{n+1}P\right)^{\frac{1}{n+1}} \leq \left((TP)^{*(n+1)}(TP)^{n+1}\right)^{\frac{1}{n+1}} \\ &= \begin{pmatrix} |T_1^{n+1}|^2 & 0 \\ 0 & 0 \end{pmatrix}^{\frac{1}{n+1}} = \begin{pmatrix} |T_1^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{pmatrix} |T_1^{n+1}|^{\frac{2}{n+1}} & 0 \\ 0 & 0 \end{pmatrix} \geq P|T^{n+1}|^{\frac{2}{n+1}}P \geq P|T|^2P = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., $T_1 \in \mathbb{A}_n$. On the other hand if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \in \mathscr{H}$, then

$$\langle T_3^k u_2, u_2\rangle = \langle T^k(I-P)u, (I-P)u\rangle = \langle (I-P)u, T^{*k}(I-P)u\rangle = 0,$$

which implies that $T_3^k = 0$. It is well known that $\sigma(T_1)\cup\sigma(T_3) = \sigma(T)\cup\mathcal{C}$, where $\mathcal{C}$ is the union of certain of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1)\cap\sigma(T_3)$ and $\sigma(T_1)\cap\sigma(T_3)$ has no interior points. Therefore, we have

$$\sigma(T) = \sigma(T_1)\cup\sigma(T_3) = \sigma(T_1)\cup\{0\}.$$

□

Lemma 2.9. *Let $T \in \mathscr{B}(\mathscr{H})$ be of k-quasi-class A_n and $\mathscr{M}$ be its invariant subspace. Then the restriction $T|_{\mathscr{M}}$ of T to $\mathscr{M}$ is also of k-quasi-class A_n.*

Proof. Let Q be the orthogonal projection onto $\mathscr{M}$. Decompose

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp.$$

Then

$$|T_1|^2 = (Q|T|^2Q)|_{\mathscr{M}}$$

and

$$|T_1^{n+1}|^{\frac{2}{n+1}} = (Q|T^{n+1}|^2Q)^{\frac{1}{n+1}}|_{\mathscr{M}}.$$

Let $x \in \mathscr{M}$. Then

$$\begin{aligned}\left\langle T_1^{*k}|T_1|^2T_1^kx, x\right\rangle &= \left\langle T_1^{*k}(Q|T|^2Q)|_{\mathscr{M}}T_1^kx, x\right\rangle \\ &= \left\langle |T|^2T^kx, T^kx\right\rangle \le \left\langle |T^{n+1}|^{\frac{2}{n+1}}T^kx, T^kx\right\rangle \\ &= \left\langle T_1^{*k}(Q|T^{n+1}|^2Q)^{\frac{1}{n+1}}|_{\mathscr{M}}T_1^kx, x\right\rangle \\ &\le \left\langle T_1^{*k}|T_1^{n+1}|^{\frac{2}{n+1}}T_1^kx, x\right\rangle.\end{aligned}$$

□

Recall from [**19**] that an operator $T \in \mathscr{B}(\mathscr{H})$ is said to be (n,k)-quasiparanormal if

$$\left\|T^{k+1}x\right\| \le \left\|T^{n+k+1}x\right\|^{\frac{1}{n+1}} \left\|T^kx\right\|^{\frac{n}{n+1}}$$

for all $x \in \mathscr{H}$. We also need the following lemma in the sequel.

Lemma 2.10. [**3**, Hölder-McCarthy inequality] *Let $T \in \mathscr{B}(\mathscr{H})$ be a positive operator. Then the following inequalities hold for all $x \in \mathscr{H}$:*

(i) $\langle T^rx, x\rangle \le \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for* $0 < r < 1$,

(ii) $\langle T^rx, x\rangle \ge \langle Tx, x\rangle^r \|x\|^{2(r-1)}$ *for* $r \ge 1$.

Theorem 2.11. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{Q}(A_n, k)$, then T is (n,k)-quasiparanormal operator.*

Proof. Since $T \in \mathbb{Q}(A_n, k)$, by the Hölder-McCarthy inequality, we have

$$\begin{aligned}\left\|T(T^kx)\right\|^2 &= \left\langle T^{*k}|T|^2T^kx, x\right\rangle \le \left\langle T^{*k}|T^{n+1}|^{\frac{2}{n+1}}T^kx, x\right\rangle \\ &\le \left\langle |T^{n+1}|^2T^kx, T^kx\right\rangle^{\frac{1}{n+1}} \left\|T^kx\right\|^{\frac{2n}{n+1}} \\ &= \left\|T^{n+k+1}x\right\|^{\frac{2}{n+1}} \left\|T^kx\right\|^{\frac{2n}{n+1}}\end{aligned}$$

and so

$$\left\|T^{k+1}x\right\| \le \left\|T^{n+k+1}x\right\|^{\frac{1}{n+1}} \left\|T^kx\right\|^{\frac{n}{n+1}},$$

thus T is (n,k)-quasiparanormal operator. □

Theorem 2.12. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{Q}(A_n, k)$ has a dense range, then T is of class A_n.*

Proof. Since T has dense range, $\overline{\mathscr{R}(T^k)} = \mathscr{H}$. Then for any $y \in \mathscr{H}$ there exists a sequence $\{x_m\} \subset \mathscr{H}$ such that $\lim\limits_{n\longrightarrow\infty} T^kx_m = y$. Since $T \in \mathbb{Q}(A_n, k)$, we have

$$\left\langle T^{*k}|T^{n+1}|^{\frac{2}{n+1}}T^kx_m, x_m\right\rangle \ge \left\langle T^{*k}|T|^2T^kx_m, x_m\right\rangle$$
$$\left\langle |T^{n+1}|^{\frac{2}{n+1}}T^kx_m, T^kx_m\right\rangle \ge \left\langle |T|^2T^kx_m, T^kx_m\right\rangle$$

for all $m \in \mathbb{N}$. By the continuity of the inner product, we have

$$\left\langle (|T^{n+1}|^{\frac{2}{n+1}} - |T|^2)y, y\right\rangle \ge 0.$$

Therefore T is a class A_n operator. □

Corollary 2.13. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{Q}(A_n, k)$ but is not of class A_n, then T is not invertible.*

Lemma 2.14. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{A}_n$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda$.*

Proof. Since $T \in \mathbb{A}_n$, T is n-paranormal. Hence the result follows from [**17**]. □

Theorem 2.15. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{Q}(A_n, k)$ and $\sigma(T) = \{\lambda\}$, then $T = \lambda$ if $\lambda \neq 0$ and $T^{k+1} = 0$ if $\lambda = 0$.*

Proof. If the range of T^k is dense, then T is of class A_n. Hence $T = \lambda$ by Lemma 2.14. If the range of T^k is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^{*k})$$

where $T_1 \in \mathbb{A}_n$, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 2.8. In this case, $\lambda = 0$. Hence $T_1 = 0$ by Lemma 2.8 and Lemma 2.14. Thus

$$T^{k+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{k+1} = \begin{pmatrix} 0 & T_2 T_3^k \\ 0 & T_3^{k+1} \end{pmatrix} = 0.$$

□

By similar proof of Theorem 4.2 of [**14**], we have:

Theorem 2.16. *If $T \in \mathbb{Q}(A_n, k)$ and $(T - \lambda)x = 0$ for $\lambda \neq 0$, then $|T^{n+1}|x = |\lambda|^{n+1}x$, hence $(T^{n+1} - \lambda^{n+1})^* x = 0$.*

Proof. The proof of $|T^{n+1}|x = |\lambda|^{n+1}x$ is similar to the proof of Theorem 4.2 of [**14**]. Then

$$|\lambda|^{n+2}x = |T^{n+1}|^2 x = T^{*(n+1)} T^{n+1} x = \lambda^{n+1} T^{*(n+1)} x.$$

□

The theorem has the following implication.

Corollary 2.17. *If $T \in \mathbb{Q}(A_n, k)$ and $(T - \alpha)x = 0$, $(T - \beta)x = 0$ with $\alpha^{n+1} \neq \beta^{n+1}$, then $\langle x, y \rangle = 0$.*

Proof. We may assume $\beta \neq 0$. Then

$$\alpha^{n+1} \langle x, y \rangle = \left\langle T^{n+1} x, y \right\rangle = \left\langle x, T^{*(n+1)} \right\rangle = \beta^{n+1} \langle x, y \rangle$$

and so $\langle x, y \rangle = 0$. □

We say that $T \in \mathscr{B}(\mathscr{H})$ has the single valued extension property (SVEP) at $\lambda \in \mathbb{C}$, if for every open neighborhood U of λ the only analytic function $f : U \to \mathbb{C}$ which satisfies equation $(T - \lambda) f(\lambda) = 0$ is the constant function $f \equiv 0$. The operator T is said to have single valued extension property if T has SVEP at every $\lambda \in \mathbb{C}$. An operator $T \in \mathscr{B}(\mathscr{H})$ has SVEP at every point of the resolvent $\rho(T) = \mathbb{C} \setminus \sigma(T)$. Every operator T has SVEP at an isolated point of the spectrum.

Corollary 2.18. *If $T \in \mathbb{Q}(A_n, k)$, then T has SVEP.*

Proof. Let f be an analytic function on an open set D such that $(T - \alpha) f(\alpha) = 0$ for $\alpha \in D$. Let $\alpha = re^{i\theta} \neq 0$ and $\alpha_m = r^{1 + \frac{1}{m}} e^{i\theta}$. Then

$$\|f(\alpha)\|^2 = \lim \langle f(\alpha), f(\alpha_m) \rangle = 0$$

by Corollary 2.17. □

Corollary 2.19. *Suppose that T is non-zero, is of k-quasi-class A_n and has no nontrivial T-invariant closed subspace. Then T is a class A_n operator.*

Proof. Since T has no non-trivial invariant closed subspace, it has no non-trivial hyperinvariant subspace. But $\ker(T^k)$ and $\overline{\mathscr{R}(T^k)}$ are hyperinvariant subspaces where $T \neq 0$, hence we have $\ker(T^k) \neq \mathscr{H}$ and $\overline{\mathscr{R}(T^k)} \neq \{0\}$. Therefore $\ker(T^k) = \{0\}$ and $\overline{\mathscr{R}(T^k)} = \mathscr{H}$. In particular, T has dense range. It follows from Corollary 2.12 that T is of class A_n operator. □

Recall that $T \in \mathscr{B}(\mathscr{H})$ is said to have finite ascent if $\ker(T^n) = \ker(T^{n+1})$ for some positive integer n.

Theorem 2.20. *If $T \in \mathbb{Q}(A_n, k)$, then $\ker(T-\lambda) = \ker(T-\lambda)^2$ if $\lambda \neq 0$ and $\ker(T^k) = \ker(T^{k+1})$ if $\lambda = 0$. Consequently, $T - \lambda$ has finite ascent for all $\lambda \in \mathbb{C}$.*

Proof. Assume $0 \neq \sigma_p(T)$ because the case $\lambda \notin \sigma_p(T)$ is obvious. Let $0 \neq x \in \ker(T-\lambda)^2$, $x = x_1 \oplus x_2 \in \mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^k)$ and

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ on } \mathscr{H} = \overline{\mathscr{R}(T^k)} \oplus \ker(T^k).$$

Then

$$\begin{aligned} 0 = (T-\lambda)^2 x &= \begin{pmatrix} T_1 - \lambda & T_2 \\ 0 & T_3 - \lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} (T_1-\lambda)^2 x_1 + ((T_1-\lambda)T_2 + T_2(T_3-\lambda))x_2 \\ (T_3-\lambda)^2 x_2 \end{pmatrix}. \end{aligned}$$

Consequently, $x_2 = 0$ because $T_3 - \lambda$ is invertible by Lemma 2.8. Thus $(T_1-\lambda)^2 x_1 = 0$ and $(T_1 - \lambda)x_1 = 0$ by Theorem 4.3 of [**14**]. Therefore

$$(T-\lambda)x = (T-\lambda)(x_1 \oplus 0) = (T_1-\lambda)x_1 = 0.$$

If $\lambda = 0$ and $x \in \ker(T^{k+1})$, then it follows from Theorem 2.11 that

$$\left\|T(T^k x)\right\| \leq \left\|T^{n+1}(T^k x)\right\|^{\frac{1}{n+1}} \left\|T^k x\right\|^{\frac{n}{n+1}} = 0.$$

Hence $T(T^k x) = 0$. Then $x \in \ker(T^k)$. □

3. Riesz Idempotent for k-quasi-class A_n Operators

Let μ be an isolated point of $\sigma(T)$. Then the Riesz idempotent E of T with respect to μ is defined by

$$E := \frac{1}{2\pi i} \int_{\partial D} (\nu - T)^{-1} \, d\nu,$$

where D is a closed disc centred at μ which contains no other points from the spectrum of T. It is known that $E^2 = E, ET = TE, \sigma(T|_{\mathscr{R}(E)}) = \{\mu\}$ and $\ker(T-\mu) \subseteq \mathscr{R}(E)$. In [**16**], Stampfli showed that if T satisfies the growth condition G_1, then E is self-adjoint and $\mathscr{R}(E) = \ker(T-\mu)$. Recently, Jeon and Kim [**11**], Uchiyama [**18**] and Rashid [**15**] obtained Stampfli's result for quasi-class A operators, paranormal operators and k-quasi-$*$-paranormal operators. In general even if T is a paranormal operator, the Riesz idempotent E of T with respect to μ is not necessarily self-adjoint. Recall that $T \in \mathscr{B}(\mathscr{H})$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T.

Theorem 3.1. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathbb{Q}(A_n, k)$, then T is isoloid.*

Proof. Suppose that T has a representation given in Lemma 2.8. Let z be an isolated point in $\sigma(T)$. Since $\sigma(T) = \sigma(T_1) \cup \{0\}$, z is an isolated point in $\sigma(T_1)$ or $z = 0$. If z is an isolated point in $\sigma(T_1)$, then $z \in \sigma_p(T_1) \subset \sigma_p(T)$ by Lemma 4.1 of [**14**]. Assume that $z = 0$ and $z \notin \sigma(T_1)$. Then for $x \in \ker(T_3)$, $-T_1^{-1}T_2x \oplus x \in \ker(T)$. This achieves the proof. □

Theorem 3.2. *Let $T \in \mathbb{Q}(A_n, k)$. Then T is polaroid. Let λ be an isolated point of $\sigma(T)$ and E be Riesz idempotent for λ. Then $E\mathscr{H} = \ker(T - \lambda)$ if $\lambda \neq 0$ and $E\mathscr{H} = \ker(T^{n+1})$ if $\lambda = 0$.*

Proof. Since $E\mathscr{H}$ is an invariant subspace of T and $\sigma(T|_{E\mathscr{H}}) = \{\lambda\}$, we have $T|_{E\mathscr{H}} = \lambda$ if $\lambda \neq 0$ and $(T|_{E\mathscr{H}})^{k+1} = 0$ if $\lambda = 0$ by Theorem 2.15. Hence $E\mathscr{H} \subset \ker(T|_{E\mathscr{H}} - \lambda) \subset \ker(T - \lambda)$ if $\lambda \neq 0$ and $E\mathscr{H} \subset \ker(T|_{E\mathscr{H}})^{k+1} \subset \ker T^{k+1}$ if $\lambda = 0$. Hence $E\mathscr{H} = \ker(T - \lambda)$ if $\lambda \neq 0$ and $E\mathscr{H} = \ker T^{k+1}$ by Lemma 5.2 of [**19**]. Hence

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$$

where $\sigma(T_1) = \sigma(T|E\mathscr{H}) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Then $T_1 - \lambda$ is nilpotent and $T_2 - \lambda$ is invertible. Hence $T - \lambda$ has finite ascent and descent. Hence T is polaroid.

□

4. Weyl's Theorem for $\mathbb{Q}(A_n, k)$

Let $\alpha(T)$ and $\beta(T)$ be the nullity and the deficiency of T defined by $\alpha(T) := \dim \ker(T)$ and $\beta(T) := \operatorname{codim} \mathscr{R}(T)$. An operator $T \in \mathscr{B}(\mathscr{H})$ is called upper semi-Fredholm if it has a closed range and $\alpha(T) < \infty$, while T is called lower semi-Fredholm if $\beta(T) < \infty$. An operator T is semi-Fredholm if it is either upper or lower semi-Fredholm, and it is said to be a Fredholm if it is both upper and lower semi- Fredholm. If $T \in \mathscr{B}(\mathscr{H})$ is semi-Fredholm, then the index is defined by

$$\operatorname{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be upper semi-Weyl if it is upper semi-Fredholm and $\operatorname{ind}(T) \leq 0$ while T is said to be lower semi-Weyl if it is lower semi-Fredholm and $\operatorname{ind}(T) \geq 0$. An operator is Weyl if it is Fredholm and is of index zero.

The Weyl spectrum and the essential approximate spectrum are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

According to [**4**] we say that Weyl's theorem holds for T if T satisfies the equality

$$\sigma(T) \setminus \pi_{00}(T) = \sigma_w(T),$$

where $\pi_{00}(T)$ is the set of the isolated points of $\sigma(T)$ that are eigenvalues of finite multiplicity.

Theorem 4.1. *If $T \in \mathbb{Q}(A_n, k)$, then Weyl's theorem holds for $f(T)$, where f is analytic and locally non-constant on an open set containing $\sigma(T)$.*

Proof. Since T has SVEP and T is polaroid, this result follows by Theorem 3.14 of [**2**]. □

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$.

Theorem 4.2. *Let $T \in \mathscr{B}(\mathscr{H})$. If T or T^* is k-quasi-class A_n, then $\sigma_w(f(T)) = f(\sigma_w(T))$ for all $f \in Hol(\sigma(T))$.*

Proof. If either $T \in \mathbb{Q}(A_n, k)$ or $T^* \in \mathbb{Q}(A_n, k)$, then T has SVEP by Corollary 2.18 and so it follows from [**1**, Corollary 3.72] that $\sigma_w(f(T)) = f(\sigma_w(T))$ for all $f \in Hol(\sigma(T))$. □

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