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CHAOTIC ATTRACTORS FROM BORDER-COLLISION BIFURCATIONS: STABLE BORDER FIXED POINTS AND DETERMINANT-BASED LYAPUNOV EXPONENT BOUNDS.

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Abstract. The collision of a fixed point with a switching manifold (or border) in a piecewise-smooth map can create many different types of invariant sets. This paper explores two techniques that, combined, establish a chaotic attractor is created in a border-collision bifurcation in $\mathbb{R}^d$ ($d \geq 1$). First, asymptotic stability of the fixed point at the bifurcation is characterised and shown to imply a local attractor is created. Second, a lower bound on the maximal Lyapunov exponent is obtained from the determinants of the one-sided Jacobian matrices associated with the fixed point. Special care is taken to accommodate points whose forward orbits intersect the switching manifold as such intersections can have a stabilising effect. The results are applied to the two-dimensional border-collision normal form focusing on parameter values for which the map is piecewise area-expanding.

1. Introduction

A map on $\mathbb{R}^d$ ($d \geq 1$) is a discrete-time dynamical system

$$x \mapsto f(x), \tag{1.1}$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$. Given an initial state $x \in \mathbb{R}^d$, the $n^{\rm th}$ iterate $f^n(x)$ represents the state of the system after n time steps. We are usually most interested in the long-time (large n) behaviour of typical $x \in \mathbb{R}^d$ which is governed by the *attractors* of f. By attractors we mean topological attractors: invariant sets that attract a neighbourhood of initial points and satisfy some indivisibility property (e.g. contain a dense orbit) [**16, 25**]. This paper concerns attractors of maps that are piecewise-smooth, see for instance Fig. 1. Such maps have different functional forms in different subsets of $\mathbb{R}^d$ and arise naturally when modelling physical phenomena with abrupt events, such as mechanical systems with impacts [**6**], control systems with relays [**49**], and social processes with decisions [**40**].

As the parameters of a map are varied, attractors (and invariant sets more generally) undergo fundamental changes at *bifurcations* [**31**]. The simplest type of bifurcation novel to piecewise-smooth maps is a border-collision bifurcation (BCB). A BCB occurs when a fixed point collides with a switching manifold (where the functional form of the map changes). This paper concerns BCBs for maps f that,

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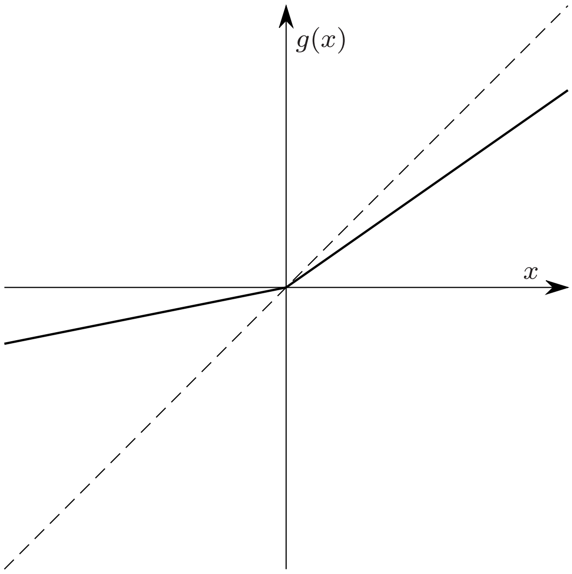

FIGURE 1. A sketch of the one-dimensional piecewise-linear map (3.13), considered briefly in §3.2. The map is continuous but non-differentiable at $x = 0$.

at least locally, are continuous across a smooth switching manifold and away from the switching manifold Df is continuous and bounded, see [**42**] for a review.

BCBs can create periodic, quasiperiodic, and chaotic attractors. The internal dynamics of DC/DC power converters, for example, can exhibit a sudden transition to chaos via a BCB [**7**]. Fig. 2 shows an example for a two-dimensional map. As the value of a parameter $\mu \in \mathbb{R}$ is increased a stable fixed point collides with the switching manifold (green line) when $\mu = 0$ at which a chaotic attractor is born.

Naturally one would like to know what attractors are created in a given BCB, but in general this is an extremely difficult problem. There are infinitely many possibilities even simply for the number of stable periodic solutions [**41**]. For this reason it seems that in order for us to usefully expand upon the state-of-the-art theory for BCBs, rather than pursue a detailed classification we should search for conditions for certain types of behaviour to occur [**22**], and this philosophy directs the present work.

This paper addresses the following problem: *under what conditions is a chaotic attractor created in a BCB*? Arguably the simplest geometric tool that can help us here is a trapping region (a compact set $\Omega \subset \mathbb{R}^d$ for which $f(\Omega)$ is contained in the interior of Ω) as such regions necessarily contain attractors. To then demonstrate chaos a wide variety of techniques have been employed. For two-dimensional maps, Misiurewicz [**36**] used the expansion and folding behaviour of unstable manifolds to prove there exists an attractor on which the map is transitive (and hence chaotic in this sense). Collet and Levy [**14**] subsequently constructed an invariant measure on this attractor and derived its basic ergodic properties. Recently Misiurewicz's

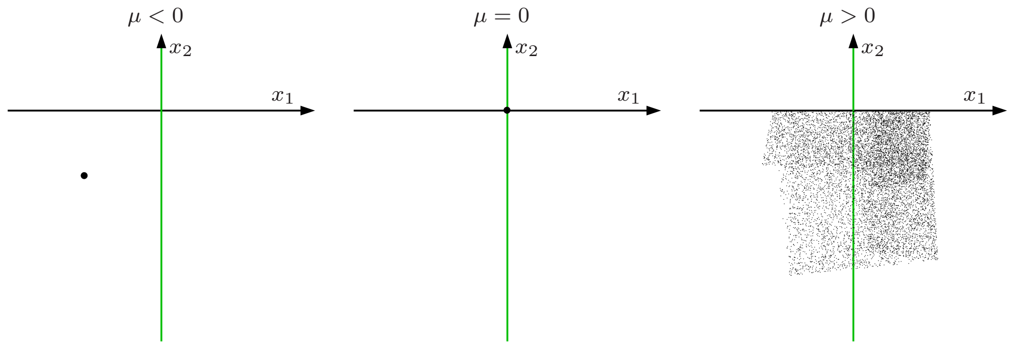

FIGURE 2. Phase portraits of the two-dimensional border-collision normal form (1.4) with $(\tau_L, \delta_L, \tau_R, \delta_R) = (0.05, -0.85, 0.1, 1.95)$, as in [**21**]. The map is continuous but non-differentiable on the switching manifold $x_1 = 0$ (green). As the value of the parameter μ is increased a stable fixed point collides with the switching manifold when $\mu = 0$ and a chaotic attractor is created (we have shown 10^4 points of a typical orbit).

result was generalised to the well-known robust chaos parameter regime of [**3**] via the construction of a forward invariant expanding cone in tangent space [**24**]. Glendinning [**23**] used a theorem of Young [**46**] to show that in part of this regime the map has a Sinai-Ruelle-Bowen measure (a certain type of chaotic attractor with nice ergodic properties [**47**]). Also in [**19, 20**], Glendinning applied a dimensionality result for piecewise-expanding maps [**13, 44**] to show that BCBs can create chaotic attractors of dimension equal to that of the phase space, as in Fig. 2.

In this paper chaotic attractors are obtained through two alternate techniques. First the existence of a local attractor is established from the stability of the fixed point at the BCB. Then a lower bound on the maximal Lyapunov exponent is given in terms of the determinants of the associated one-sided Jacobian matrices. It is hoped that these techniques will allow us to formally verify the existence of chaotic attractors in parameter regions where other methods are inconclusive.

1.1. Main ideas and overview. The remainder of this paper is organised as follows. We first briefly summarise key notation in §1.2. Then in §2 we explain how the stability of a fixed point at a BCB in a map f on $\mathbb{R}^d$ can be addressed by studying the stability of the origin for a piecewise-linear map. In general the stability problem for piecewise-linear maps is remarkably complicated [**11**]. Following [**1**] we show how stability can be characterised by iterating a compact set Ω that contains the origin in its interior, Theorem 2.2. Essentially we determine whether or not Ω is a trapping region for some iterate of the map. By choosing Ω to be a polytope this provides an effective numerical procedure by which stability can be established. We also prove that if the origin is asymptotically stable then the corresponding BCB must create a local attractor, Theorem 2.4.

To show that attractors are chaotic we use *Lyapunov exponents*. As discussed in §3.1, for a smooth map f on $\mathbb{R}^d$ Lyapunov exponents are defined by

$$\lambda(x,v) = \lim_{n \to \infty} \frac{1}{n} \ln(\|Df^n(x)v\|), \tag{1.2}$$

assuming the limit exists. The Lyapunov exponent $\lambda(x,v)$ represents rate at which the forward orbits of x and $x + \delta v$ diverge, in the $\delta \to 0$ limit. On an attractor we may have $\lambda(x,v) > 0$ (almost everywhere) in which case the dynamics is locally expanding—this is the essence of chaos.

For a piecewise-smooth map f, $\mathrm{D}f$ is undefined at points on switching manifolds. In this case the limiting rate of divergence is instead given by

$$\lambda(x,v) = \lim_{n \to \infty} \frac{1}{n} \ln(\|C_n(x,v)v\|), \tag{1.3}$$

where C_n is a matrix defined in §3.2. The key difference between (1.2) and (1.3) is that C_n depends on v, whereas $Df^n(x)$ does not. This is because if x lies on a switching manifold then the smooth component of f that we must use depends on which side the of the switching manifold the point $x + \delta v$ lies. Lyapunov exponents have also been described for more general classes of continuous but non-differentiable maps, see [**5, 9, 28**]. In §3.3 we derive a lower bound on the maximal value of (1.3) based on the determinants of the two one-sided Jacobian matrices associated with a BCB. For random matrix products, lower bounds have recently been obtained using analytic functions [**38, 39**].

Then in §3.4 we demonstrate the results with the two-dimensional border-collision normal form. This is the map

$$f_\mu(x) = \begin{cases} A_L x + b\mu, & x_1 \le 0, \\ A_R x + b\mu, & x_1 \ge 0, \end{cases} \tag{1.4}$$

where

$$A_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix}, \qquad A_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1.5}$$

and $\tau_L, \delta_L, \tau_R, \delta_R \in \mathbb{R}$. This map and its importance was established originally in [**37**]. If $\delta_L, \delta_R > 0$ (1.4) is orientation-preserving [**2, 3**] while if $\delta_L, \delta_R < 0$ (1.4) is orientation-reversing [**36**]. If $\delta_L \delta_R \le 0$ (1.4) is non-invertible [**21, 30, 35**]. If $\tau_L = -\tau_R$ and $\delta_L = \delta_R$ (1.4) reduces to the well-known Lozi map [**34**].

Finally §4 provides concluding remarks.

1.2. Notation. We study maps on $\mathbb{R}^d$ ($d \ge 1$) with the Euclidean norm $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$. The origin or zero vector is denoted $\mathbf{0}$. Open balls are written as $B_\varepsilon(x) = \left\{ y \in \mathbb{R}^d \,\middle|\, \|x - y\| < \varepsilon \right\}$. The tangent space to a point x is denoted $T\mathbb{R}^d$. Here we omit the dependence on x because $T\mathbb{R}^d$ is isomorphic to $\mathbb{R}^d$ for all x; indeed we simply treat tangent vectors as elements of $\mathbb{R}^d$. The unit sphere is denoted $\mathbb{S}^{d-1} = \left\{ v \in T\mathbb{R}^d \,\middle|\, \|v\| = 1 \right\}$.

If X is a subset of Y we write $X \subset Y$. We write $\mathrm{int}(X)$ and $\mathrm{cl}(X)$ for the interior and closure of X. The d-dimensional Lebesgue measure of a set $\Omega \subset \mathbb{R}^d$ is written as $\mathrm{meas}(\Omega)$. For a set $\gamma \subset \mathbb{S}^{d-1}$, we use the *spherical measure* which is defined by

$$\text{sph-meas}(\gamma) = \frac{\mathrm{meas}(\{\alpha v \,|\, v \in \gamma,\, 0 \le \alpha \le 1\})}{\mathrm{meas}(B_1(\mathbf{0}))}\,. \tag{1.6}$$

We use o for little-o notation [**12**]. In particular a function $\phi : \mathbb{R} \to \mathbb{R}$ is said to be $o(\delta)$ if $\lim_{\delta \to 0} \frac{\phi(\delta)}{\delta} = 0$.

2. The Stability of a Fixed Point at a Border-collision Bifurcation

We first recall some basic definitions. A fixed point x^* of a continuous map f on $\mathbb{R}^d$ is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $f^i(x) \in B_\varepsilon(x^*)$ for all $x \in B_\delta(x^*)$ and all $i \geq 0$. The point x^* is *asymptotically stable* if it is Lyapunov stable and there exists $\delta > 0$ such that $f^i(x) \to x^*$ as $i \to \infty$ for all $x \in B_\delta(x^*)$.

2.1. Reduction to a piecewise-linear map. Let f_μ be a continuous piecewise-C^1 map on $\mathbb{R}^d$ with parameter $\mu \in \mathbb{R}$. Suppose f_μ undergoes a BCB at a point on a single smooth switching manifold Σ that, locally, divides phase space into two regions. Only two pieces of f_μ influence the local dynamics and we assume coordinates can be chosen so that Σ is the plane $x_1 = 0$, that is

$$f_\mu(x) = \begin{cases} f_{L,\mu}(x), & x_1 \leq 0, \\ f_{R,\mu}(x), & x_1 \geq 0, \end{cases} \tag{2.1}$$

where $f_{L,\mu}$ and $f_{R,\mu}$ are C^1.

We further assume the BCB occurs at $x = \mathbf{0}$ when $\mu = 0$ and that $A_L = \mathrm{D}f_{L,0}(\mathbf{0})$ and $A_R = \mathrm{D}f_{R,0}(\mathbf{0})$ are well-defined. By assumption f is continuous on Σ thus A_R can only differ from A_L in its first column. We use these matrices to form the piecewise-linear map

$$g(x) = \begin{cases} A_L x, & x_1 \leq 0, \\ A_R x, & x_1 \geq 0. \end{cases} \tag{2.2}$$

This map approximates f_0 near $x = \mathbf{0}$, specifically $f_0(x) = g(x) + o(\|x\|)$, and we have the following result.

Lemma 2.1. *Consider* (2.1) *with* $\mu = 0$ *and suppose* $x = \mathbf{0}$ *is a fixed point. If* $\mathbf{0}$ *is an asymptotically stable fixed point of its piecewise-linear approximation* (2.2) *then* $\mathbf{0}$ *is an asymptotically stable fixed point of* (2.1).

Lemma 2.1 justifies our subsequent study of g (although its converse is not true in general). Lemma 2.1 is an immediate consequence of Theorem 1 of [**43**] (also a similar result is achieved in [**10**]) so we do not provide a proof.

2.2. Stability for piecewise-linear maps. The problem of the stability of $\mathbf{0}$ for the piecewise-linear map (2.2) is a special case of a common and well-studied problem in control theory. Many control systems are well modelled by piecewise-linear maps of the form $x(i+1) \mapsto A_{\sigma(i)} x(i)$, where $A_1, \ldots, A_N$ are real-valued $d \times d$ matrices and $\sigma(i) \in \{1, \ldots, N\}$ indicates which matrix is applied at the i^{th} time step. The function σ may be state-dependent (as in our case), a pre-determined signal (discretised wave-form), or determined by a particular control strategy. One would like to know if $\mathbf{0}$ is asymptotically stable for all possible σ, or, more weakly, if there exists σ for which $\mathbf{0}$ is asymptotically stable [**17, 18, 32**].

Broadly speaking the mathematical tool of choice to address these problems is the *Lyapunov function*. We instead follow [**1**] and consider the images of a compact region Ω for which $\mathbf{0} \in \mathrm{int}(\Omega)$. As discussed below this approach is well-suited to numerical implementation, allows any such Ω, and, unlike commonly used classes of

Lyapunov functions, provides an exact characterisation of the asymptotic stability of $\mathbf{0}$.

2.3. Stability for linearly homogeneous maps. The map (2.2) is an example of a *linearly homogeneous* map: a continuous map g on $\mathbb{R}^d$ for which $g(\alpha x) = \alpha g(x)$ for all $x \in \mathbb{R}^d$ and all $\alpha \geq 0$. Notice that every linearly homogeneous map has $\mathbf{0}$ as a fixed point.

Theorem 2.2. *Let g be a continuous, linearly homogeneous map on $\mathbb{R}^d$. Let $\Omega \subset \mathbb{R}^d$ be compact with $\mathbf{0} \in \mathrm{int}(\Omega)$. The following are equivalent.*

i) $\mathbf{0}$ *is an asymptotically stable fixed point of g,*
ii) there exists $m \geq 1$ such that $g^m(\Omega) \subset \mathrm{int}(\Omega)$, and
iii) there exists $m \geq 1$ such that $g^m(\Omega) \subset \mathrm{int}\left(\bigcup_{i=0}^{m-1} g^i(\Omega)\right)$.

Theorem 2.2, proved below, generalises a result of [**43**] and is motivated by [**1**] which focuses on convex subsets of $\mathbb{R}^d$. While condition (ii) provides a simpler characterisation of the asymptotic stability of $\mathbf{0}$ than condition (iii), the latter condition is more effective numerically because the minimum value of m for which condition (iii) is satisfied is often significantly smaller than the minimum value of m for which condition (ii) is satisfied.

To prove Theorem 2.2 we use the following result that was proved in [**43**].

Lemma 2.3. *Let g be a continuous, linearly homogeneous map on $\mathbb{R}^d$. Suppose there exists $\delta > 0$ such that $g^n(x) \to \mathbf{0}$ as $n \to \infty$ for all $x \in B_\delta(\mathbf{0})$. Then*

i) $\mathbf{0}$ *is an asymptotically stable fixed point of g, and*
ii) the convergence $g^n(x) \to \mathbf{0}$ is uniform on $B_\delta(\mathbf{0})$.

Proof of Theorem 2.2. Since $\mathbf{0} \in \mathrm{int}(\Omega)$ there exists $r > 0$ such that $B_r(\mathbf{0}) \subset \mathrm{int}(\Omega)$. Since Ω is bounded there exists $R > r$ such that $\Omega \subset B_R(\mathbf{0})$, see Fig. 3.

We first show (i) $\Rightarrow$ (ii). If $\mathbf{0}$ is asymptotically stable there exists $\delta > 0$ such that $g^i(x) \to \mathbf{0}$ as $i \to \infty$ for all $x \in B_\delta(\mathbf{0})$. Since g is linearly homogeneous, this

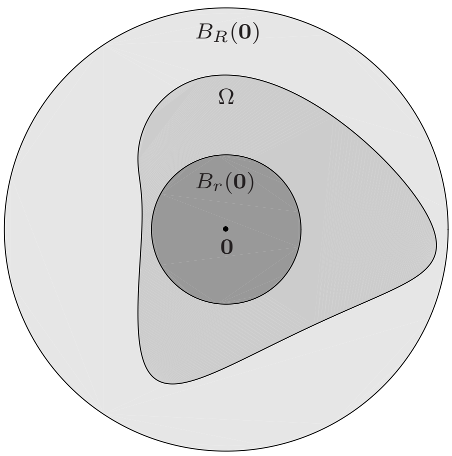

FIGURE 3. A sketch of sets introduced in the proof of Theorem 2.2.

is also true for all $x \in B_R(\mathbf{0})$. By Lemma 2.3(ii) there exists $m \geq 1$ such that $g^m(B_R(\mathbf{0})) \subset B_r(\mathbf{0})$. Then (ii) is satisfied because $B_r(\mathbf{0}) \subset \mathrm{int}(\Omega)$.

Observe (ii) $\Rightarrow$ (iii) is trivial because $\mathrm{int}(\Omega) \subset \mathrm{int}\left(\bigcup_{i=0}^{m-1} g^i(\Omega)\right)$. Thus it remains to show (iii) $\Rightarrow$ (i). Let $\Xi = \bigcup_{i=0}^{m-1} g^i(\Omega)$. Statement (iii) implies there exists $0 < \alpha < 1$ such that $g^m(\Omega) \subset \alpha\Xi$ (where $\alpha\Xi = \{\alpha x \,|\, x \in \Xi\}$). This is because if there does not exist such an α, then for each $\alpha_j = 1 - \frac{1}{j}$ there exists $x_j \in g^m(\Omega)$ with $x_j \notin \alpha_j\Xi$. The sequence $\{x_j\}$ is bounded so has a subsequence converging to some x^* (by the Bolzano-Weierstrass Theorem) and $x^* \in g^m(\Omega)$ because $g^m(\Omega)$ is closed and $x^* \notin \mathrm{int}(\Xi)$ which contradicts (iii).

We now show $g^m(\Xi) \subset \alpha\Xi$. Choose any $x \in \Xi$. Then $x \in g^i(\Omega)$ for some $i \in \{0, \ldots, m-1\}$ and $g^{m-i}(x) \in g^m(\Omega)$. Let $y = \frac{x}{\alpha}$. Since g is linearly homogeneous, $g^{m-i}(y) = \frac{1}{\alpha} g^{m-i}(x) \in \frac{1}{\alpha} g^m(\Omega) \subset \Xi$. But Ξ is forward invariant by (iii), thus $g^m(y) \in \Xi$. Then $g^m(x) = \alpha g^m(y) \in \alpha\Xi$ as required.

Then $g^{km}(\Xi) \subset \alpha^k\Xi$ for all $k \geq 1$ (again using the linear homogeneity of g). Thus $g^n(x) \to \mathbf{0}$ as $n \to \infty$ for all $x \in \Xi$. In particular, $g^n(x) \to \mathbf{0}$ for all $x \in B_r(\mathbf{0})$, thus $\mathbf{0}$ is asymptotically stable by Lemma 2.3(i). $\square$

2.4. Numerical implementation. Here we outline a numerical procedure, based on Theorem 2.2, that can verify the stability of a fixed point at a BCB. For a given BCB this could provide a computed-assisted proof of stability.

We first form the piecewise-linear approximation (2.2). Under such a map the image of a *polytope* (a region bounded by flat surfaces, i.e. a generalisation of polygons to more than two dimensions) is another polytope. If a polytope has finitely many vertices (so can be encoded with finitely many points) its image will also have finitely many vertices. Hence we choose Ω to be a reasonably simple polytope. We then iteratively compute $g^m(\Omega)$ and $\bigcup_{i=0}^{m-1} g^i(\Omega)$ (eliminating redundant vertices at each step) and if we find m for which condition (iii) of Theorem 2.2 is satisfied we conclude that $\mathbf{0}$ is an asymptotically stable fixed point of (2.2). Moreover, $\mathbf{0}$ will be an asymptotically stable fixed point of (2.1) at the BCB (by Lemma 2.1).

If we cannot find m satisfying condition (iii) by iterating up to some maximum permitted value $m_{\max}$, then we cannot make a conclusion about the stability of $\mathbf{0}$. However if the value of $m_{\max}$ is reasonably large this may provide useful evidence that $\mathbf{0}$ is in fact unstable.

The encoding of Ω and its images can further be aided by the following observation. A set $\Omega \subset \mathbb{R}^d$ is *radially convex* (or *star-shaped* with respect to the origin) if for every $x \in \Omega$ and $0 < \alpha < 1$ we have $\alpha x \in \Omega$. It is a simple exercise to show that the property of radial convexity is preserved under linearly homogeneous maps.

As an example we consider (2.2) in two-dimensions with A_L and A_R given by (1.5). We fix

$$\tau_L = 2, \qquad \delta_L = 1.4, \qquad \tau_R = -0.8, \tag{2.3}$$

and consider several different values of δ_R. We let Ω be the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$. Fig. 4 shows the smallest value of m for which condition (iii) is satisfied for 1000 different values of δ_R. Formally we can only conclude that $\mathbf{0}$ is asymptotically stable for the discrete set of 1000 parameter values that were simulated, but the results strongly suggest that $\mathbf{0}$ is asymptotically stable for all $-1.46 \leq \delta_R \leq -0.41$.

With $\delta_R = -1.4$, for example, condition (iii) is first satisfied with $m = 28$ and the sets $g^m(\Omega)$ and $\bigcup_{i=0}^{m-1} g^i(\Omega)$ are shown in Fig. 5a for this value of m. With instead $\delta_R = -0.5$ we obtain $m = 12$, see Fig. 5b. The 'spikes' in Fig. 5b are not numerical inaccuracies; by iterating $g^5(\Omega)$ near its intersection with the switching manifold at $x_2 \approx 2.44$ we generate a vertex with an acute angle that carries through to higher iterates. Certainly with alternate Ω condition (iii) can be satisfied at smaller values of m but we do not attempt this here. An understanding of the bifurcations at $\delta_R \approx -1.46$ and $\delta_R \approx -0.41$ at which stability is lost is beyond the scope of this paper.

2.5. Persistence of an local attractor. Here we show that if the fixed point at a BCB is asymptotically stable then there exists a local attractor for nearby values of the parameters.

Theorem 2.4. *Let f_μ be a continuous map on $\mathbb{R}^d$ with parameter $\mu \in \mathbb{R}$. Suppose $\mathbf{0}$ is an asymptotically stable fixed point of f_0 and let $\Omega \subset \mathbb{R}^d$ be a compact set in its basin of attraction. Then for all $\varepsilon > 0$ there exists $\delta > 0$ and $N \in \mathbb{Z}$ such that*

$$f_\mu^n(x) \in B_\varepsilon(\mathbf{0}), \quad \text{for all } x \in \Omega,\ n \geq N, \text{ and } \mu \in (-\delta, \delta). \tag{2.4}$$

Theorem 2.4 is the discrete-time analogue of Theorem 5.4 of [**8**] for ordinary differential equations and is proved below using standard techniques in real analysis. Specifically it shows that all points in a compact set $\Omega \subset \mathbb{R}^d$ map into, and never escape from, the ball $B_\varepsilon(\mathbf{0})$ within at most N iterations of f. Therefore Ω is a trapping region for f^N, assuming $B_\varepsilon(\mathbf{0}) \subset \Omega$, and thus f has an attractor in $B_\varepsilon(\mathbf{0})$.

Proof of Theorem 2.4. Choose any $\varepsilon > 0$. Since $\mathbf{0}$ is an asymptotically stable fixed point of f_0 there exists $\delta_1 > 0$ (with $\delta_1 \leq \varepsilon$) such that

$$f_0^n(x) \in B_{\frac{\varepsilon}{2}}(\mathbf{0}), \quad \text{for all } x \in B_{\delta_1}(\mathbf{0}), \text{ and all } n \geq 0, \tag{2.5}$$

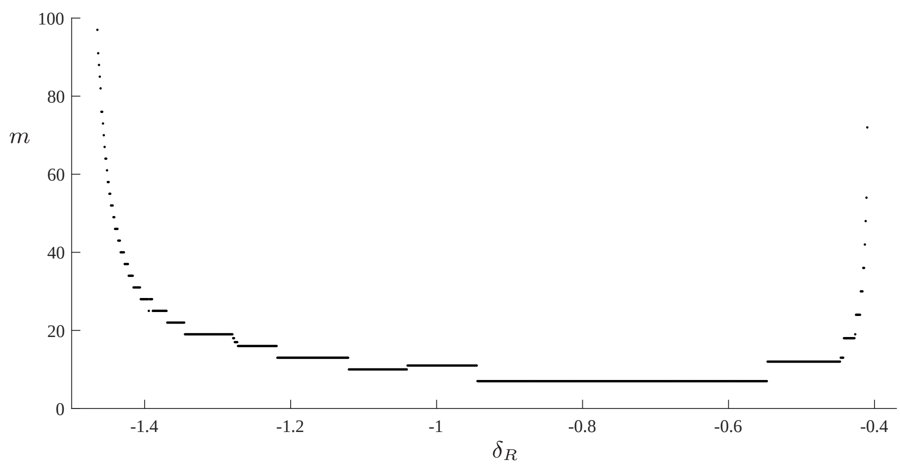

FIGURE 4. The smallest value of m for which condition (iii) of Theorem 2.2 is satisfied for (2.2) with (1.5) and (2.3), where Ω is the square with vertices $(\pm 1, 0)$ and $(0, \pm 1)$.

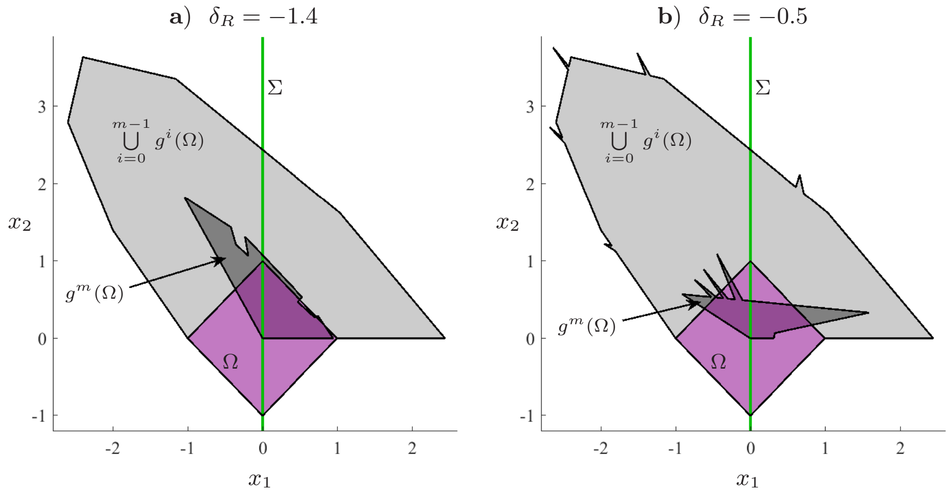

FIGURE 5. Phase portraits illustrating condition (iii) of Theorem 2.2 for the map of Fig. 4 for two different values of δ_R. In the left plot $m = 28$; in the right plot $m = 12$.

and $f_0^n(x) \to \mathbf{0}$ as $n \to \infty$, for all $x \in \mathrm{cl}(B_{\delta_1}(\mathbf{0}))$. This convergence is uniform because $\mathrm{cl}(B_{\delta_1}(\mathbf{0}))$ is compact (this is a consequence of the Arzelà-Ascoli theorem; for details refer to the proof of Lemma 3 of [**43**]). Thus there exists $M \in \mathbb{Z}$ such that

$$f_0^M(x) \in B_{\frac{\delta_1}{2}}(\mathbf{0}), \quad \text{for all } x \in B_{\delta_1}(\mathbf{0}). \tag{2.6}$$

Also $f_0^n(x) \to \mathbf{0}$ as $n \to \infty$, for all $x \in \Omega$, because Ω belongs to the basin of attraction of $\mathbf{0}$. This convergence is similarly uniform because Ω is compact. Thus there exists $N \in \mathbb{Z}$ such that

$$f_0^N(x) \in B_{\frac{\delta_1}{2}}(\mathbf{0}), \quad \text{for all } x \in \Omega. \tag{2.7}$$

Since f_μ^N is continuous and Ω is compact, f_μ^N is uniformly continuous on Ω. Thus there exists $\delta_2 > 0$ such that

$$\left\| f_\mu^N(x) - f_0^N(x) \right\| < \frac{\delta_1}{2}\,, \quad \text{for all } x \in \Omega, \text{ and all } \mu \in (-\delta_2, \delta_2). \tag{2.8}$$

Similarly f_μ^i is uniformly continuous on $B_{\delta_1}(\mathbf{0})$ for all $i \geq 1$, thus there exists $\delta > 0$ (with $\delta \leq \delta_2$) such that

$$\left\| f_\mu^i(x) - f_0^i(x) \right\| < \frac{\delta_1}{2}\,, \quad \text{for all } x \in B_{\delta_1}(\mathbf{0}), \text{ all } i \in \{1, \ldots, M\}, \text{ and all } \mu \in (-\delta, \delta). \tag{2.9}$$

Now choose any $x \in \Omega$ and any $\mu \in (-\delta, \delta)$. By (2.7) and (2.8), we have

$$\left\| f_\mu^N(x) \right\| \leq \left\| f_\mu^N(x) - f_0^N(x) \right\| + \left\| f_0^N(x) \right\| \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1\,.$$

This verifies (2.4) for $n = N$ (because $\delta_1 \leq \varepsilon$). By applying (2.5) and (2.9) to the point $f_\mu^N(x)$, for any $i \in \{1, \ldots, M\}$ we have

$$\left\| f_\mu^{N+i}(x) \right\| \leq \left\| f_\mu^{N+i}(x) - f_0^{N+i}(x) \right\| + \left\| f_0^{N+i}(x) \right\| \leq \frac{\delta_1}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

This verifies (2.4) for $n = N+1, \ldots, N+M$. By applying (2.6) and (2.9) to $f_\mu^N(x)$, we have

$$\left\| f_\mu^{N+M}(x) \right\| \leq \left\| f_\mu^{N+M}(x) - f_0^{N+M}(x) \right\| + \left\| f_0^{N+M}(x) \right\| \leq \frac{\delta_1}{2} + \frac{\delta_1}{2} = \delta_1 .$$

Thus we can repeat the last two steps indefinitely to verify (2.4) for all $n \geq N$. □

3. Lyapunov Exponents

3.1. Lyapunov exponents for smooth maps. Let f be a C^1 map on $\mathbb{R}^d$. Given $x \in \mathbb{R}^d$, $v \in \mathrm{T}\mathbb{R}^d$, and $0 < \delta \ll 1$, we are interested in how the forward orbit of the perturbed point $x + \delta v$ compares to the forward orbit of x. After one application of f the difference between the iterates is

$$f(x + \delta v) - f(x) = \delta \mathrm{D}f(x)v + o(\delta). \tag{3.1}$$

After n applications of f the difference is

$$f^n(x + \delta v) - f^n(x) = \delta \mathrm{D}f^n(x)v + o(\delta). \tag{3.2}$$

For large n the distance between the iterates is $\|f^n(x + \delta v) - f^n(x)\| \sim \delta \mathrm{e}^{\lambda(x,v)n}$, where

$$\lambda(x, v) = \lim_{n\to\infty} \frac{1}{n} \ln(\|\mathrm{D}f^n(x)v\|), \tag{3.3}$$

if this limit exists. The *Lyapunov exponent* $\lambda(x, v)$ represents the asymptotic rate of expansion of f at $x \in \mathbb{R}^d$ in the direction $v \in \mathrm{T}\mathbb{R}^d$. Oseledets' theorem [**4, 15, 45**] gives conditions under which, for almost all x in an invariant set, $\lambda(x, v)$ is well-defined and takes at most d values independent of x.

Before we consider piecewise-smooth maps, it is helpful to consider the evolution of points and tangent vectors together. Define a map h on the tangent bundle $\mathbb{R}^d \times \mathrm{T}\mathbb{R}^d$ by

$$h(x, v) = \big(f(x), \mathrm{D}f(x)v\big). \tag{3.4}$$

Since $\mathrm{D}f^n(x) = \mathrm{D}f\big(f^{n-1}(x)\big)\mathrm{D}f\big(f^{n-2}(x)\big) \cdots \mathrm{D}f(x)$, the composition of h with itself n times is

$$h^n(x, v) = \big(f^n(x), \mathrm{D}f^n(x)v\big). \tag{3.5}$$

The first component of (3.5) is the n^{th} iterate of x under f. The second component provides the vector in (3.3). The matrix $\mathrm{D}f^n(x)$ is a *cocycle* because $\mathrm{D}f^{m+n}(x) = \mathrm{D}f^m(f^n(x))\mathrm{D}f^n(x)$ for all $m, n \geq 0$, [**27**].

3.2. Lyapunov exponents for piecewise-smooth maps. Now let f be a continuous, piecewise-smooth map of the form (2.1) (here we ignore the dependency on μ). We assume that f_L and f_R are C^1 throughout $\left\{x \in \mathbb{R}^d \,\middle|\, x_1 \leq 0\right\}$ and $\left\{x \in \mathbb{R}^d \,\middle|\, x_1 \geq 0\right\}$, respectively, and first generalise (3.1) to this piecewise-smooth setting.

Lemma 3.1. *For any* $x \in \mathbb{R}^d$ *and* $v \in \mathrm{T}\mathbb{R}^d$,

$$f(x+\delta v) - f(x) = \delta C(x,v)v + o(\delta), \tag{3.6}$$

where

$$C(x,v) = \begin{cases} \mathrm{D}f_L(x), & x_1 < 0, \text{ or } x_1 = 0 \text{ and } v_1 < 0, \\ \mathrm{D}f_R(x), & x_1 > 0, \text{ or } x_1 = 0 \text{ and } v_1 \ge 0, \end{cases} \tag{3.7}$$

Proof. If $x_1 > 0$ then $(x+\delta v)_1 > 0$ for sufficiently small $\delta > 0$. Thus $f(x) = f_R(x)$, $f(x+\delta v) = f_R(x+\delta v)$, and the differentiability of f_R gives (3.6) with $C(x,v) = \mathrm{D}f_R(x)$. If $x_1 < 0$ the same arguments apply to f_L.

If $x_1 = 0$ and $v_1 \ge 0$ then $(x+\delta v)_1 \ge 0$. Since f is continuous on Σ, we again have $f(x) = f_R(x)$, $f(x+\delta v) = f_R(x+\delta v)$, and thus (3.6) with $C(x,v) = \mathrm{D}f_R(x)$. If $x_1 = 0$ and $v_1 < 0$ the same arguments apply to f_L. □

In (3.7) our choice of $\mathrm{D}f_R(x)$ in the special case $x_1 = v_1 = 0$ is immaterial because although in general $\mathrm{D}f_L(x)$ and $\mathrm{D}f_R(x)$ are different matrices, the continuity of f implies $\mathrm{D}f_L(x)v = \mathrm{D}f_R(x)v$. As an example, Fig. 6 shows the set $\left\{ C(\mathbf{0},v)v \,\middle|\, v \in \mathbb{S}^1 \right\}$ for the border-collision normal form with the parameter values of Fig. 5a.

Analogous to (3.4), we define

$$h(x,v) = \big(f(x), C(x,v)v\big). \tag{3.8}$$

Evidently we have

$$h^n(x,v) = \big(f^n(x), C_n(x,v)v\big), \tag{3.9}$$

where

$$C_n(x,v) = C\big(h^{n-1}(x,v)\big)C\big(h^{n-2}(x,v)\big)\cdots C(x,v). \tag{3.10}$$

This is a cocycle because $C_{m+n}(x,v) = C_m(h^n(x,v))C_n(x,v)$ for all $m,n \ge 0$. We now show that $C_n(x,v)$ plays the desired role of $\mathrm{D}f^n(x)$.

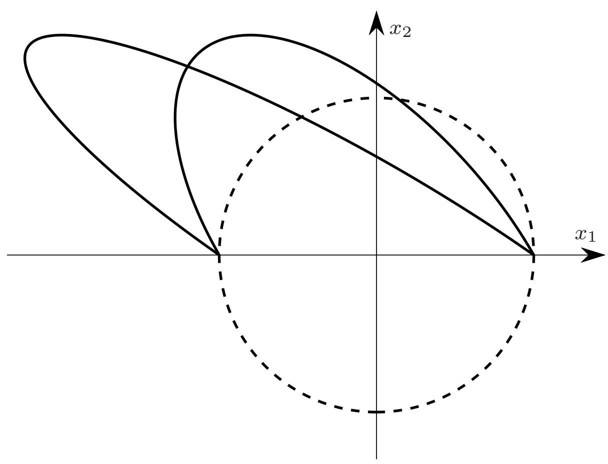

FIGURE 6. The unit circle (dashed) and its image (solid) under the tangent map $v \mapsto C(x,v)v$ with $x = \mathbf{0}$ for the map (2.2) with (1.5), (2.3), and $\delta_R = -1.4$.

Lemma 3.2. *For any $x \in \mathbb{R}^d$ and $v \in \mathrm{T}\mathbb{R}^d$,*

$$f^n(x+\delta v) - f^n(x) = \delta C_n(x,v)v + o(\delta), \tag{3.11}$$

for all $n \geq 1$.

Proof. The result is true for $n=1$ by Lemma 3.1. Suppose the result is true for some $n=k \geq 1$. Then

$$f^k(x+\delta v) - f^k(x) = \delta C_k(x,v)v + o(\delta),$$

which implies

$$f^{k+1}(x+\delta v) - f^{k+1}(x) = f\big(f^k(x) + \delta C_k(x,v)v\big) - f^{k+1}(x) + o(\delta),$$

using also the continuity of $\mathrm{D}f$. By then applying Lemma 3.1 to $h^k(x,v)$ we obtain

$$f^{k+1}(x+\delta v) - f^{k+1}(x) = \delta C(h^k(x,v))C_k(x,v)v + o(\delta),$$

and thus, by the cocycle property, the result is also true for $n=k+1$. Hence the result is true for all $n \geq 1$ by induction. □

In view of Lemma 3.2, we define

$$\lambda(x,v) = \lim_{n\to\infty} \frac{1}{n} \ln(\|C_n(x,v)v\|), \tag{3.12}$$

if the limit exists. Oseledets' theorem does not apply to (3.12) because the cocycle $C_n(x,v)$ is dependent on v. Indeed it is not difficult to find examples for which $\lambda(x,v)$ takes more than d distinct values. Consider for instance the one-dimensional map

$$g(x) = \begin{cases} a_L x, & x \leq 0, \\ a_R x, & x \geq 0, \end{cases} \tag{3.13}$$

where $a_L, a_R > 0$, see Fig. 1. For the fixed point $x=0$, we have $\lambda(0,-1) = \ln(a_L)$ and $\lambda(0,1) = \ln(a_R)$. Thus if $a_L \neq a_R$ then (3.12) takes two different values at $x=0$.

For the two-dimensional map (2.2) with (1.5) it was found in [**43**] that the normalised tangent map $v \mapsto \frac{C(x,v)v}{\|C(x,v)v\|}$ on $\mathbb{S}^1$ appears to be chaotic with infinitely many ergodic invariant probability measures each generating a potentially distinct value for (3.12) at $x = \mathbf{0}$.

3.3. Determinant-based bounds on the maximal Lyapunov exponent. Let f_μ be a continuous, piecewise-C^1 map of the form (2.1). We have shown that if $x=\mathbf{0}$ is an asymptotically stable fixed point of f_0 then f_μ has an attractor Λ near $x=\mathbf{0}$ when $|\mu|$ is small (see Theorem 2.4). Here we construct a lower bound for the maximum value of the Lyapunov exponent (3.12) for $x \in \Lambda$.

Let $\Omega \subset \mathbb{R}^d$ be a compact set and let

$$\begin{aligned} a_L &= \min\big\{|\det(\mathrm{D}f_{L,\mu}(x))| \,\big|\, x \in \Omega,\, x_1 \leq 0\big\}, \\ a_R &= \min\big\{|\det(\mathrm{D}f_{R,\mu}(x))| \,\big|\, x \in \Omega,\, x_1 \geq 0\big\}. \end{aligned} \tag{3.14}$$

The idea is that Ω is small, so a_L and a_R are usefully approximated by $|\det(A_L)|$ and $|\det(A_R)|$, and $\Lambda \subset \Omega$, so results for $x \in \Omega$ immediately apply to $x \in \Lambda$. In fact for the purposes of showing that a chaotic attractor is created in the BCB at $\mu = 0$, by Theorem 2.4 we can make Ω as small as we like (and still allow $\mu \neq 0$) hence a_L and a_R can be made to be arbitrarily close to $|\det(A_L)|$ and $|\det(A_R)|$.

Given $x \in \Omega$, let ℓ_n [resp. r_n] be the number of iterates $i \in \{0, \ldots, n-1\}$ for which $f^i(x)_1 < 0$ [resp. $f^i(x)_1 > 0$]. Also let

$$\ell = \liminf_{n \to \infty} \frac{\ell_n}{n}, \qquad r = \liminf_{n \to \infty} \frac{r_n}{n}. \tag{3.15}$$

These quantities are x-dependent but in our notation we have omitted this dependency for brevity.

Theorem 3.3. *Suppose $f^i(x) \in \Omega$ for all $i \geq 0$. Let*

$$a = \min\left[a_L^{\ell} a_R^{1-\ell}, a_L^{1-r} a_R^{r}\right], \tag{3.16}$$

and suppose $a > 0$. Let

$$\lambda_{\rm bound} = \frac{1}{d} \ln\left(2^{\ell+r-1} a\right). \tag{3.17}$$

Then

$$\liminf_{n \to \infty} \frac{1}{n} \ln(\|C_n(x,v)v\|) \geq \lambda_{\rm bound}\,, \tag{3.18}$$

for almost all $v \in T\mathbb{R}^d$.

A proof of Theorem 3.3 follows some technical remarks. If $\lambda_{\rm bound} > 0$ and the limit (3.12) exists, then $\lambda(x,v) > 0$ for some (in fact almost all) v, and so the maximal Lyapunov exponent is positive. If $\lambda_{\rm bound} > 0$ but the limit (3.12) does not exist, (3.18) still ensures the dynamics is locally expanding by Lemma 3.2.

Remark 3.4. If ν is the invariant probability measure associated with an attractor $\Lambda \subset \Omega$, we often have $\nu(\Sigma) = 0$ (exceptions include periodic solutions with one or more points on Σ). In this case $\frac{\ell_n + r_n}{n} \to 1$ as $n \to \infty$ for typical $x \in \Lambda$. Then usually $\ell + r = 1$ which implies $a = a_L^{\ell} a_R^{r}$. Here a is the weighted geometric mean of a_L and a_R induced by the forward orbit of x. Then $\lambda_{\rm bound} = \frac{1}{d}\ln(a)$, therefore $a > 1$ implies $\lambda(x,v) > 0$ (assuming (3.12) exists).

Remark 3.5. In practice, for a given map one may be able to obtain estimates on ℓ and r throughout a set Ω [**20**]. For example if it is not possible for three consecutive iterates to lie in $x_1 < 0$ (i.e. *LLL* is a *forbidden word* [**33**]) then $\ell \leq \frac{2}{3}$. If it is only known that $\ell + r = 1$, then $a = a_L^{\ell} a_R^{r} \geq \min(a_L, a_R)$. So if a_L and a_R are both greater than 1 we must have $\lambda(x,v) > 0$ (assuming (3.12) exists) which makes sense because in this case the map is area-expanding on both sides of Σ.

Remark 3.6. It is instructive to apply Theorem 3.3 to $x = \mathbf{0}$ when this point is a fixed point. In this case $\ell = r = 0$, so $a = \min(a_L, a_R)$ and $\lambda_{\rm bound} = \frac{1}{d}\ln\left(\frac{a}{2}\right)$. Thus even if a_L and a_R are both greater than 1 (but not both greater than 2) it is possible to have $\lambda(x,v) < 0$ and thus possible for the fixed point to be stable (as in Fig. 5a).

Our proof of Theorem 3.3 is based on the following upper bound for the measure of the set of unit tangent vectors v for which $\|Av\|$ is 'small', where A is a matrix. The bound is crude but far-reaching and well-suited for Theorem 3.3 because the only information of A that is used is its determinant.

Lemma 3.7. *Let A be a real-valued $d \times d$ matrix with* $\det(A) \neq 0$*, and let $c > 0$. Then*

$$\text{sph-meas}\left(\left\{v \in \mathbb{S}^{d-1} \,\middle|\, \|Av\| \leq c\right\}\right) \leq \frac{c^d}{|\det(A)|}. \tag{3.19}$$

Proof. In view of the substitution $A \mapsto cA$ it suffices to consider $c = 1$. The set under consideration is then $\gamma = \left\{ v \in \mathbb{S}^{d-1} \,\middle|\, \|Av\| \le 1 \right\}$. The left hand-side of (3.19) is

$$\text{sph-meas}(\gamma) = \frac{\text{meas}(\Gamma)}{\text{meas}(B_1(0))}, \tag{3.20}$$

where $\Gamma = \{\alpha v \,|\, v \in \gamma,\, 0 \le \alpha \le 1\}$. Notice $\|Au\| \le 1$ for all $u \in \Gamma$. Therefore Γ is a subset of $A^{-1}B_1(\mathbf{0}) = \left\{ A^{-1}v \,\middle|\, v \in B_1(\mathbf{0}) \right\}$, see Fig. 7. Under multiplication by A^{-1} the Lebesgue measure of a measurable set is scaled by the factor $|\det(A^{-1})| = \frac{1}{|\det(A)|}$, thus

$$\text{meas}\big(A^{-1}B_1(0)\big) = \frac{\text{meas}(B_1(0))}{|\det(A)|}. \tag{3.21}$$

Since $\Gamma \subset A^{-1}B_1(0)$, by combining (3.20) and (3.21) we obtain (3.19) with $c = 1$, as required. □

Proof of Theorem 3.3. Choose any $\varepsilon > 0$. Let

$$\gamma_\varepsilon = \left\{ v \in \mathbb{S}^{d-1} \,\middle|\, \liminf_{n\to\infty} \frac{1}{n} \ln(\|C_n(x,v)v\|) \le \lambda_{\text{bound}} - 2\varepsilon \right\}, \tag{3.22}$$

and

$$\gamma_{\varepsilon,n} = \left\{ v \in \mathbb{S}^{d-1} \,\middle|\, \frac{1}{n} \ln(\|C_n(x,v)v\|) \le \lambda_{\text{bound}} - \varepsilon \right\}, \tag{3.23}$$

for all $n \ge 1$. If $v \in \gamma_\varepsilon$, then $v \in \gamma_{\varepsilon,n}$, for infinitely many values of $n \ge 1$. Thus for all $N \ge 1$ we have $\gamma_\varepsilon \subset \bigcup_{n \ge N} \gamma_{\varepsilon,n}$. Below we show that

$$\text{sph-meas}(\gamma_{\varepsilon,n}) \le \mathrm{e}^{-\frac{d\varepsilon n}{3}}, \tag{3.24}$$

for sufficiently large values of n. This will complete the proof because it implies $\text{sph-meas}\Big(\bigcup_{n \ge N} \gamma_{\varepsilon,n}\Big) \le \frac{\mathrm{e}^{\frac{-d\varepsilon N}{3}}}{1-\mathrm{e}^{\frac{-d\varepsilon}{3}}} \to 0$ as $N \to \infty$. Hence $\text{sph-meas}(\gamma_\varepsilon) = 0$ and so,

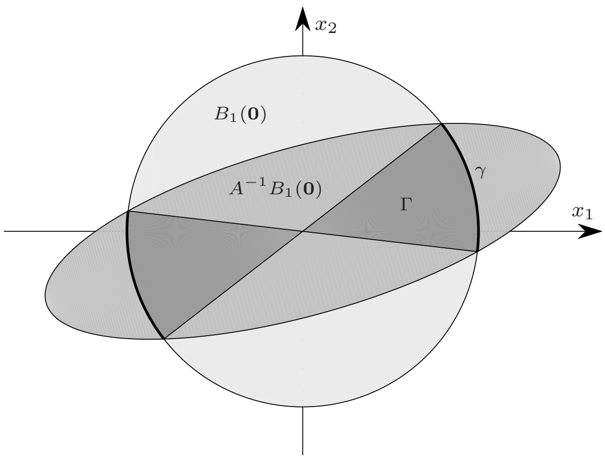

FIGURE 7. A sketch of the geometric elements introduced in the proof of Lemma 3.7.

because we can take $\varepsilon > 0$ arbitrarily small, (3.18) holds for almost all $v \in \mathbb{S}^{d-1}$. The left hand-side of (3.18) is independent of $\|v\|$, thus (3.18) also holds for almost all $v \in T\mathbb{R}^d$.

Given $n \geq 1$, let $\{L, R\}^n$ denote the set of words of length n involving the symbols L and R. Below we index the elements of any word $\mathcal{S} \in \{L, R\}^n$ from $i = 0$ to $i = n-1$ and write $\mathcal{S} = \mathcal{S}_0 \mathcal{S}_1 \cdots \mathcal{S}_{n-1}$. Let

$$\Xi_n = \big\{\mathcal{S} \in \{L, R\}^n \,\big|\, \mathcal{S}_i = L \text{ if } f^i(x)_1 < 0 \text{ and } \mathcal{S}_i = R \text{ if } f^i(x)_1 > 0, \\ \text{for all } i \in \{0, \ldots, n-1\}\big\}. \tag{3.25}$$

The set Ξ_n contains $2^{n-(\ell_n+r_n)}$ words because to create a word $\mathcal{S} \in \Xi_n$, we are free to choose either $\mathcal{S}_i = L$ or $\mathcal{S}_i = R$ only if $f^i(x) = 0$. The number of indices $i \in \{0, \ldots, n-1\}$ for which $f^i(x) = 0$ is $n - (\ell_n + r_n)$. Thus we have $n - (\ell_n + r_n)$ independent choices between two symbols, so a total of $2^{n-(\ell_n+r_n)}$ words.

Given $v \in \mathbb{S}^{d-1}$, by (3.7) and (3.10) we have

$$C_n(x, v) = Df_{\mathcal{S}_{n-1}}\big(f^{n-1}(x)\big) \cdots Df_{\mathcal{S}_0}(x)$$

for some $\mathcal{S} \in \{L, R\}^n$. We can write this as

$$C_n(x, v) = Df_{\mathcal{S}}(x),$$

where $f_{\mathcal{S}} = f_{\mathcal{S}_{n-1}} \circ \cdots \circ f_{\mathcal{S}_0}$ denotes the composition of f_L and f_R in the order determined by $\mathcal{S}$. But $\mathcal{S} \in \Xi_n$ because by (3.7) and (3.10) we must have $\mathcal{S}_i = L$ if $f^i(x)_1 < 0$ and $\mathcal{S}_i = R$ if $f^i(x)_1 > 0$, for all $i \in \{0, \ldots, n-1\}$.

We cannot apply Lemma 3.7 to the set $\gamma_{\varepsilon,n}$ by using $A = C_n(x, v)$ because this matrix depends on v. For this reason we introduce the set

$$\gamma_{\varepsilon,n}^{\mathcal{S}} = \left\{ v \in \mathbb{S}^{d-1} \,\middle|\, \frac{1}{n} \ln(\|Df_{\mathcal{S}}(x)v\|) \leq \lambda_{\text{bound}} - \varepsilon \right\},$$

for a given word $\mathcal{S} \in \Xi_n$. Then $\gamma_{\varepsilon,n} \subset \bigcup_{\mathcal{S} \in \Xi_n} \gamma_{\varepsilon,n}^{\mathcal{S}}$, and so

$$\text{sph-meas}(\gamma_{\varepsilon,n}) \leq \sum_{\mathcal{S} \in \Xi_n} \text{sph-meas}\big(\gamma_{\varepsilon,n}^{\mathcal{S}}\big). \tag{3.26}$$

By Lemma 3.7,

$$\text{sph-meas}\big(\gamma_{\varepsilon,n}^{\mathcal{S}}\big) \leq \frac{\mathrm{e}^{dn(\lambda_{\text{bound}} - \varepsilon)}}{|\det(Df_{\mathcal{S}}(x))|}. \tag{3.27}$$

For the remainder of the proof we assume $a_L \geq a_R$ without loss of generality. By (3.14), $|\det(Df_{\mathcal{S}}(x))| \geq a_L^k a_R^{n-k}$, where k is the number of L's in the word $\mathcal{S}$. Since $\ell_n \leq k \leq n - r_n$ and $a_L \geq a_R$ we have

$$|\det(Df_{\mathcal{S}}(x))| \geq a_L^{\ell_n} a_R^{n-\ell_n}. \tag{3.28}$$

By substituting (3.17) and (3.28) into (3.27) we obtain

$$\text{sph-meas}\big(\gamma_{\varepsilon,n}^{\mathcal{S}}\big) \leq \frac{2^{(\ell+r-1)n} a^n \mathrm{e}^{-dn\varepsilon}}{a_L^{\ell_n} a_R^{n-\ell_n}}.$$

Since Ξ_n contains $2^{n-(\ell_n+r_n)}$ words, (3.26) then implies

$$\text{sph-meas}(\gamma_{\varepsilon,n}) \leq \frac{2^{(\ell+r)n-(\ell_n+r_n)} a^n \mathrm{e}^{-dn\varepsilon}}{a_L^{\ell_n} a_R^{n-\ell_n}} = \left(\frac{2^{\ell+r-\frac{\ell_n+r_n}{n}} a \, \mathrm{e}^{-d\varepsilon}}{a_L^{\frac{\ell_n}{n}} a_R^{1-\frac{\ell_n}{n}}} \right)^n. \tag{3.29}$$

Since $\liminf_{n \to \infty} \frac{\ell_n + r_n}{n} = \ell + r$, there exists $N_1 \in \mathbb{Z}$ such that for all $n \geq N_1$ we have $\frac{\ell_n + r_n}{n} \geq \ell + r - \frac{d\varepsilon}{3\ln(2)}$, that is $2^{\ell + r - \frac{\ell_n + r_n}{n}} \leq \mathrm{e}^{\frac{d\varepsilon}{3}}$. Similarly, since $\liminf_{n \to \infty} \frac{\ell_n}{n} = \ell$, there exists $N_2 \in \mathbb{Z}$ such that for all $n \geq N_2$ we have $\left(\frac{a_L}{a_R}\right)^{\frac{\ell_n}{n}} \geq \left(\frac{a_L}{a_R}\right)^{\ell} \mathrm{e}^{\frac{-d\varepsilon}{3}}$, that is $\frac{a}{a_L^{\frac{\ell_n}{n}} a_R^{1-\frac{\ell_n}{n}}} \leq \mathrm{e}^{\frac{d\varepsilon}{3}}$. By inserting these bounds into (3.29) we obtain (3.24) (valid for all $n \geq \max(N_1, N_2)$) as required. $\square$

3.4. Numerical simulations. We illustrate in this section the results with the two-dimensional border-collision normal form (1.4). Our goal is to establish the existence of a chaotic attractor by obtaining $\lambda_{\rm bound} > 0$ and compare this bound to a numerically computed value for the Lyapunov exponent $\lambda(x, v)$. We found in §2.4 that with (2.3) and $\mu = 0$ the fixed point $x = \mathbf{0}$ appears to be asymptotically stable for all $-1.46 \leq \delta_R \leq -0.41$. Theorem 2.4 then implies the map has an attractor near $x = \mathbf{0}$ for sufficiently small $|\mu|$. But (1.4) is piecewise-linear—the structure of the dynamics is independent of the magnitude of μ—hence there exists a bounded attractor for all $\mu \in \mathbb{R}$.

Numerical investigations suggest that this attractor is unique. For the two particular values $\delta_R = -1.4$ and $\delta_R = -0.5$ the attractor is shown in Fig. 8 for $\mu = 1$

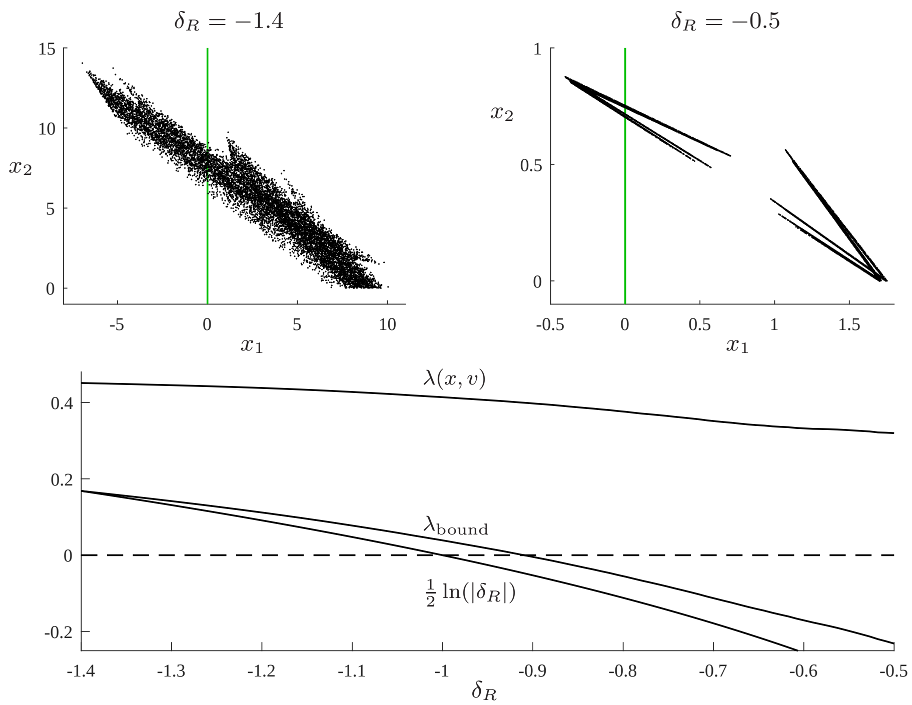

FIGURE 8. The lower plot shows the maximal Lyapunov exponent and two lower bounds for the two-dimensional border-collision normal form (1.4) with (2.3) and $\mu = 1$. The upper plots are phase portraits showing the numerically computed attractor.

and Fig. 9 for $\mu = -1$. These figures also show numerically computed values for the Lyapunov exponent $\lambda(x, v)$ (3.12) and the lower bound $\lambda_{\rm bound}$ (3.17). For each value of δ_R these were computed from 10^6 iterates of the forward orbit of $x = \mathbf{0}$ with the first 100 (transient) iterates removed. For the computation of $\lambda(x, v)$ we used $v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

By Theorem 3.3 we expect $\lambda(x, v) > \lambda_{\rm bound}$, and this is indeed the case. Indeed $\lambda(x, v) > 0$ for all values of δ_R in Figs. 8 and 9 suggesting the map has a chaotic attractor for all such δ_R and all $\mu \neq 0$.

As discussed in Remark 3.5 we can construct a simpler bound that does not require knowledge of the forward orbit of x. Assuming $\ell + r = 1$ we have $\lambda_{\rm bound} \geq \frac{1}{d} \ln(\min(a_L, a_R))$. Since (1.4) is piecewise-linear, $a_L = |\delta_L|$ and $a_R = |\delta_R|$. Here $|\delta_L| \geq |\delta_R|$ thus $\lambda_{\rm bound}$ is bounded by $\frac{1}{d} \ln(|\delta_R|)$ which we have also plotted.

4. Discussion

Chaotic attractors of piecewise-smooth maps are useful in cryptography [**29**] but undesirable in most engineering and control applications [**48**]. In both settings it is helpful to understand parameter regions where chaotic attractors exist.

In this paper we have shown how the existence of a topological attractor follows from the asymptotic stability of a fixed point on a switching manifold. It is well

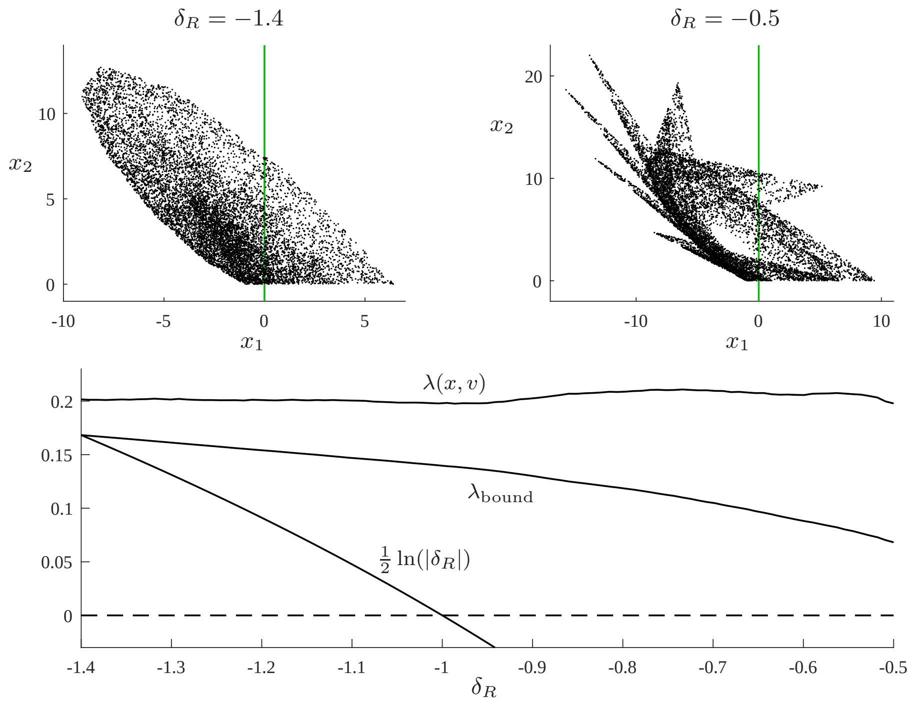

FIGURE 9. This repeats Fig. 8 for $\mu = -1$.

known that stability can often be established by constructing a Lyapunov function. In particular there are well-established methods by which the existence of a piecewise-quadratic Lyapunov function can be verified [**26, 32**]. However, these methods fail in some instances for which the fixed point is stable because only a limited class of Lyapunov functions is considered.

For this reason, following [**1**], here we advocate condition (iii) of Theorem 2.2 for demonstrating stability. This condition characterises asymptotic stability exactly and, as discussed in §2.4, is readily amenable to an accurate and efficient numerical implementation for piecewise-linear maps. We achieved this here for the two-dimensional border-collision normal form with $\mu = 0$ and showed that the origin can be stable even if both pieces of the map are area-expanding. This is possible because the map is non-invertible over the given parameter range and the expansion competes with the contractive effect of folding at the switching manifold.

In §3.3 we obtained the lower bound (3.17) on the maximal Lyapunov exponent. For the two-dimensional border-collision normal form with $\mu \neq 0$, if $\nu(\Sigma) = 0$ (where ν is the invariant probability measure of an attractor), which appears to be the case for the four numerically computed attractors shown in Figs. 8 and 9, then we expect to have $\lambda_{\rm bound} \geq \frac{1}{2} \ln \big(\min(|\delta_L|, |\delta_R|) \big)$. This immediately gives $\lambda_{\rm bound} > 0$ in the area-expanding case and thus chaos in the sense of a positive Lyapunov exponent. The necessity of the assumption $\nu(\Sigma) = 0$ is seen by simply putting $\mu = 0$. In this case $\nu(\Sigma) = 1$, where ν is the Dirac measure corresponding the fixed point $x = \mathbf{0}$, and this point may be stable.

In order to reveal the full power of Theorem 3.3 it remains to apply bounds on the values of ℓ and r obtained from restrictions to the possible symbolic dynamics and apply the bound (3.17) to attractors for which $0 < \nu(\Sigma) < 1$, but examples of this are not known for the border-collision normal form. It also remains to obtain tighter bounds on the maximal Lyapunov exponent by using more information about A_L and A_R than simply their determinants.

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