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THE BEHAVIOUR UNDER ITERATION OF A CLASS OF SPATIALLY-DISCRETISED QUADRATIC MAPS

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Abstract. Potentially drastic changes in the dynamical behaviour of iterated maps due to rounding errors in computer arithmetic have motivated the study of spatially-discretised maps, whereby the dynamical behaviour of a map is compared to that of a variant obtained by composing the map with a spatial discretisation operator, such as the floor function. In this paper, we study the dynamical behaviour of the spatially-discretised quadratic map $x \mapsto \lfloor \lambda x^2 \rfloor$, for all values of $\lambda \in \mathbb{R}$. Specifically, we prove that the map possesses at most three non-zero fixed points, and that every orbit of the map either diverges or becomes eventually constant at one of the existing fixed points.

1. Introduction

The decision of using a computer to generate the orbits of an iterated map carries a violent consequence: spatial discretisation [**3, 4, 9, 10, 7, 8, 12**]. Indeed, whether one approves or not, the computer, constrained by its finite capability, could only regard the map's domain as being discrete, or even finite, thereby possibly periodicising all existing orbits. Such a phenomenon has drawn considerable attention not only in mathematics [**1, 2, 11, 13, 25, 22**] but also in computer science [**6, 21, 26, 20**], at times leading to intriguing open conjectures [**1, 13, 25**].

Despite the abundance of studies promising a general theory [**2, 3, 4, 9, 10, 7, 8, 11, 12**], the presently limited understanding of spatial discretisation could in part be attributed to the literature's scarcity of toy models: modest spatially-discretised maps whose dynamical behaviour admits complete analytical description. Studies on such maps appear to be non-existent, apart from some recent ones on the map $x \mapsto \lfloor \lambda x \rfloor$ [**24**] and its generalisation $x \mapsto \lfloor \lambda x + \mu \rfloor$ [**17**], where $\lambda, \mu \in \mathbb{R}$. Here, the decision to use the floor function implies no loss of generality since, defining the following general discretisation operators for every $k > 0$:

$$\lfloor x \rfloor_k := \max \{ m \in k\mathbb{Z} : m \leqslant x \},$$
$$\lceil x \rceil_k := \min \{ m \in k\mathbb{Z} : m \geqslant x \},$$
$$\lfloor x \rceil_k := \left\lfloor x + \frac{k}{2} \right\rfloor_k ,$$

one verifies that, for every $k > 0$, the maps

$$x \mapsto \lfloor \lambda x + \mu \rfloor_k , \qquad x \mapsto \lceil \lambda x + \mu \rceil_k , \qquad \text{and} \qquad x \mapsto \lfloor \lambda x + \mu \rceil_k$$

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are conjugate via the homeomorphisms

$$x \mapsto \frac{x}{k}, \qquad x \mapsto -\frac{x}{k}, \qquad \text{and} \qquad x \mapsto \frac{x}{k} \tag{1}$$

to the maps

$$x \mapsto \left\lfloor \lambda x + \frac{\mu}{k} \right\rfloor, \qquad x \mapsto \left\lfloor \lambda x + \left(-\frac{\mu}{k}\right) \right\rfloor, \qquad \text{and} \qquad x \mapsto \left\lfloor \lambda x + \left(\frac{\mu}{k} + \frac{1}{2}\right) \right\rfloor,$$

respectively.[1] As a result, the dynamical behaviour of spatially-discretised first-degree polynomial maps is already largely understood.

In this paper, we thus turn our attention to spatially-discretised second-degree polynomial maps. Specifically, we consider

$$f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = \left\lfloor \lambda x^2 \right\rfloor. \tag{2}$$

As in the previous studies, we aim to describe the limiting behaviour of every orbit of the map, for all values of $\lambda \in \mathbb{R}$. The case of $\lambda = 0$ is trivial. In the cases of $\lambda > 0$ and $\lambda < 0$, respectively, every orbit of the map eventually enters the invariant subdomains $[0, \infty)$ and $(-\infty, 0]$, in which the map displays a sequence of plateaus stepping upwards as its argument increases (see Figure 1), implying that every periodic point is a fixed point. In this paper, we show that, depending on the value of λ, the map has at most three non-zero fixed points, and that each of its orbits either diverges or stabilises at a fixed point.[2]

More precisely, letting

$$\operatorname{Fix}(f) := \{x \in \mathbb{R} : f(x) = x\}$$

be the set of all fixed points of f, and

$$\omega_f(x) := \Big\{\ell \in \mathbb{R} \cup \{-\infty, \infty\} : \text{there exist } n_1, n_2, \ldots \in \mathbb{N}_0 \text{ with } n_1 < n_2 < \cdots \text{ such that } f^{n_k}(x) \xrightarrow{k\to\infty} \ell\Big\}$$

be the set of all limit points of the orbit of f with initial condition $x \in \mathbb{R}$, we shall prove the following theorem.

Theorem 1.

(i) For $\lambda < -1$, we have $\operatorname{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-\infty\}, & \text{if } x \neq 0. \end{cases}$$

(ii) For $-1 \leqslant \lambda < -\frac{1}{2}$, we have $\operatorname{Fix}(f) = \{0, -1\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right) \cup \left(0, \sqrt{-\frac{1}{\lambda}}\right]; \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{1}{\lambda}}\right) \cup \left(\sqrt{-\frac{1}{\lambda}}, \infty\right). \end{cases}$$

[1] The discretisation operators $x \mapsto \lfloor x \rceil_k$, $x \mapsto \lceil x \rfloor_k$, and $x \mapsto \lfloor x \rceil_k$ manifest a computer's rounding to d decimal places in the case $k = 10^{-d}$.

[2] An orbit that stabilises at a fixed point is one that eventually becomes constant at the fixed point. The same terminology is also used in studies of other maps such as the mean-median map [**5, 14, 15, 16, 18, 19**].

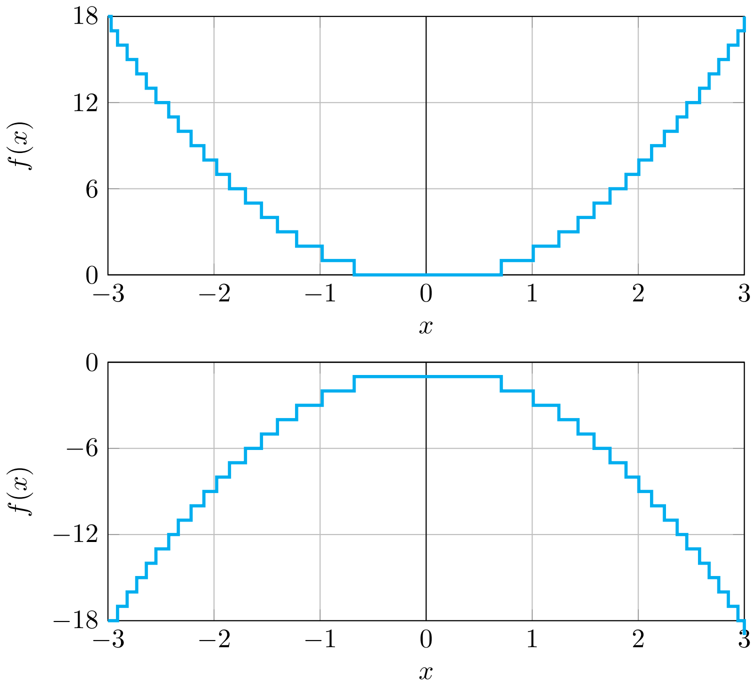

FIGURE 1. Plots of the map (2) for $\lambda = 2$ (top) and for $\lambda = -2$ (bottom).

(iii) For $-\frac{1}{2} \leqslant \lambda < -\frac{1}{3}$, we have $\mathrm{Fix}(f) = \{0, -1, -2\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right) \cup \left(0, \sqrt{-\frac{1}{\lambda}}\right]; \\ \{-2\}, & \text{if } x \in \left[-\sqrt{-\frac{2}{\lambda}}, -\sqrt{-\frac{1}{\lambda}}\right) \cup \left(\sqrt{-\frac{1}{\lambda}}, \sqrt{-\frac{2}{\lambda}}\right]; \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{2}{\lambda}}\right) \cup \left(\sqrt{-\frac{2}{\lambda}}, \infty\right). \end{cases}$$

(iv) For $-\frac{1}{3} \leqslant \lambda < -\frac{1}{4}$, we have $\mathrm{Fix}(f) = \{0, -1, -2, -3\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right) \cup \left(0, \sqrt{-\frac{1}{\lambda}}\right]; \\ \{-2\}, & \text{if } x \in \left[-\sqrt{-\frac{2}{\lambda}}, -\sqrt{-\frac{1}{\lambda}}\right) \cup \left(\sqrt{-\frac{1}{\lambda}}, \sqrt{-\frac{2}{\lambda}}\right]; \\ \{-3\}, & \text{if } x \in \left[-\sqrt{-\frac{3}{\lambda}}, -\sqrt{-\frac{2}{\lambda}}\right) \cup \left(\sqrt{-\frac{2}{\lambda}}, \sqrt{-\frac{3}{\lambda}}\right]; \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{3}{\lambda}}\right) \cup \left(\sqrt{-\frac{3}{\lambda}}, \infty\right). \end{cases}$$

(v) For $-\frac{1}{4} \leqslant \lambda < 0$, we have the following for every integer $a \leqslant -4$.

- If $\frac{1}{a} \leqslant \lambda < \frac{a+2}{(a+1)^2}$, then $\mathrm{Fix}(f) = \{0, -1, a+1, a\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{\frac{a+2}{\lambda}}, 0\right) \cup \left(0, \sqrt{\frac{a+2}{\lambda}}\right]; \\ \{a+1\}, & \text{if } x \in \left[-\sqrt{\frac{a+1}{\lambda}}, -\sqrt{\frac{a+2}{\lambda}}\right) \cup \left(\sqrt{\frac{a+2}{\lambda}}, \sqrt{\frac{a+1}{\lambda}}\right]; \\ \{a\}, & \text{if } x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right) \cup \left(\sqrt{\frac{a+1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right]; \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right) \cup \left(\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

- If $\frac{a+2}{(a+1)^2} \leqslant \lambda < \frac{1}{a-1}$, then $\mathrm{Fix}(f) = \{0, -1, a\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{\frac{a+1}{\lambda}}, 0\right) \cup \left(0, \sqrt{\frac{a+1}{\lambda}}\right]; \\ \{a\}, & \text{if } x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right) \cup \left(\sqrt{\frac{a+1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right]; \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right) \cup \left(\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

(vi) For $\lambda = 0$, we have $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \{0\} \qquad \text{for all } x \in \mathbb{R}.$$

(vii) For $0 < \lambda < 2$, we have the following for every integer $a \geqslant 2$.

- If $\frac{1}{a-1} \leqslant \lambda < \frac{a}{(a-1)^2}$, then $\mathrm{Fix}(f) = \{0, a-1\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left(-\sqrt{\frac{a-1}{\lambda}}, \sqrt{\frac{a-1}{\lambda}}\right); \\ \{a-1\}, & \text{if } x \in \left(-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a-1}{\lambda}}\right] \cup \left[\sqrt{\frac{a-1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right] \cup \left[\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

- If $\frac{a+1}{a^2} \leqslant \lambda < \frac{1}{a-1}$, then $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left(-\sqrt{\frac{a}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right] \cup \left[\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

(viii) For $\lambda \geqslant 2$, we have $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left(-\sqrt{\frac{1}{\lambda}}, \sqrt{\frac{1}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{1}{\lambda}}\right] \cup \left[\sqrt{\frac{1}{\lambda}}, \infty\right). \end{cases}$$

The fact that Theorem 1 provides a complete description of the fixed points of the map (2) allows us to construct the fixed-point bifurcation diagram

$$\left\{(\lambda, x) \in \mathbb{R}^2 : \left\lfloor \lambda x^2 \right\rfloor = x\right\}$$

of the map (2), displayed in Figure 2. Notice that the trivial fixed point 0 exists for all $\lambda \in \mathbb{R}$, and that for $\lambda > 0$, the map possesses at most one non-zero fixed point, whereas for $\lambda < 0$, up to three non-zero fixed points may coexist. Notice also that

$$\min \mathrm{Fix}(f) \xrightarrow{\lambda \to 0^-} -\infty \qquad \text{and} \qquad \max \mathrm{Fix}(f) \xrightarrow{\lambda \to 0^+} \infty.$$

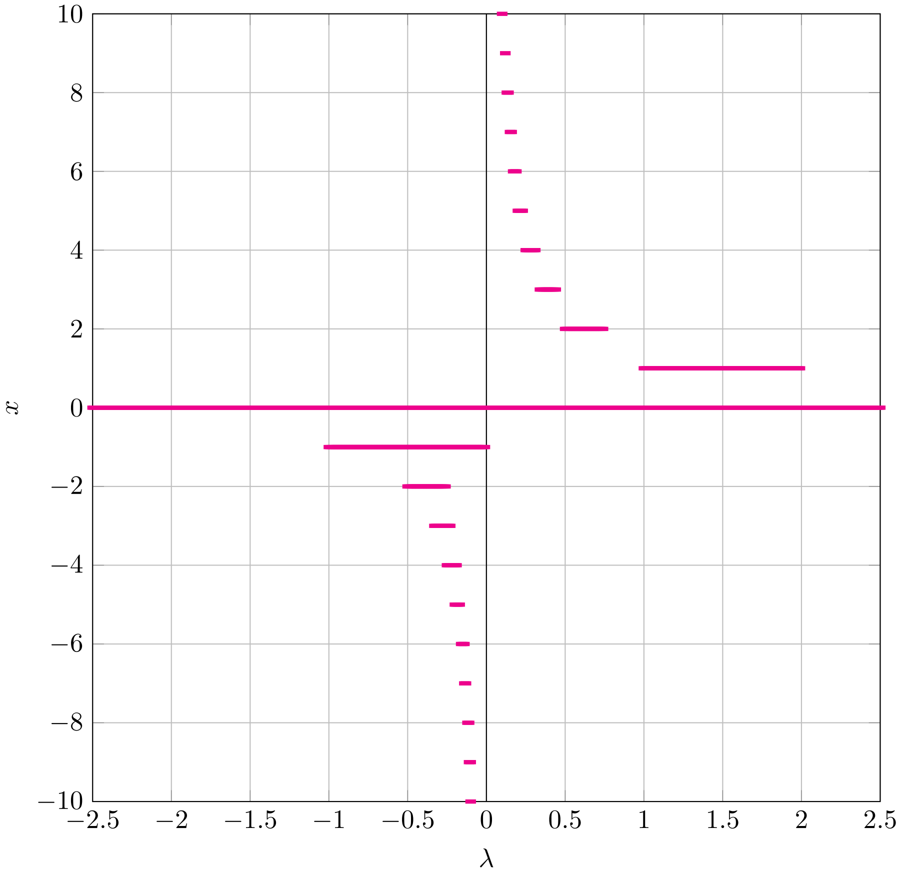

FIGURE 2. The fixed-point bifurcation diagram of the map (2).

The rest of the paper contains our proof of the above theorem. More precisely, omitting the trivial case $\lambda = 0$, we shall deal with the case $\lambda > 0$ in the upcoming section 2, and with the case $\lambda < 0$ in the final section 3.

As in the case of first-degree polynomial maps, we note that the homeomorphisms (1) conjugate the maps

$$x \mapsto \lfloor \lambda x^2 \rfloor_k, \qquad x \mapsto \lceil \lambda x^2 \rceil_k, \qquad \text{and} \qquad x \mapsto \lfloor \lambda x^2 \rceil_k$$

to the maps

$$x \mapsto \lfloor (\lambda k)\, x^2 \rfloor, \qquad x \mapsto \lfloor (-\lambda k)\, x^2 \rfloor, \qquad \text{and} \qquad x \mapsto \left\lfloor (\lambda k)\, x^2 + \frac{1}{2} \right\rfloor, \tag{3}$$

respectively. From the perspective of computer arithmetic, Theorem 1 and Figure 2 show that using fewer decimal places —which implies potentially larger rounding errors— tends to reduce the number of fixed points in these quadratic maps, the extreme case being all bounded orbits stabilise at the unique zero fixed point, which is the superstable fixed point in the absence of spatial discretisation. Conversely, using more decimal places —which implies smaller rounding errors— tends to increase the number of fixed points, thereby allowing the bounded orbits to exhibit more varied limiting values.

The third map in (3) provides motivation for a further study of the more general quadratic family $x \mapsto \lfloor \lambda x^2 + \mu \rfloor$. In particular, it would be interesting to investigate how spatial discretisation periodicises the quadratic family's chaotic orbits (Figure 3). Ideas along these lines have been proposed by Misiurewicz [**23**].

2. The Case $\lambda > 0$

In this section we shall prove our assertions in Theorem 1 in the case $\lambda > 0$. In this case, all fixed points belong to the invariant subdomain $[0, \infty)$. Moreover, the fact that the map (2) is even implies that for all $x \in \mathbb{R}$ we have $\omega_f(-x) = \omega_f(x)$, so that it suffices to determine $\omega_f(x)$ for all $x \geqslant 0$. We shall divide our work into two propositions: Proposition 2 on the subcase $\lambda \geqslant 2$ (subsection 2.1), and Proposition 3 on the subcase $0 < \lambda < 2$ (subsection 2.2).

2.1. The subcase $\lambda \geqslant 2$. Our assertions in the subcase $\lambda \geqslant 2$ in Theorem 1 follow from this next proposition.

Proposition 2. For $\lambda \geqslant 2$, we have $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left[0, \sqrt{\frac{1}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left[\sqrt{\frac{1}{\lambda}}, \infty\right). \end{cases}$$

Proof. It is plain that 0 is a fixed point.

- Let $x \in \left[0, \sqrt{\frac{1}{\lambda}}\right)$. Then $0 \leqslant \lambda x^2 < 1$, and so $f(x) = 0$.
- Let $x \in \left[\sqrt{\frac{1}{\lambda}}, \sqrt{\frac{2}{\lambda}}\right)$. Then $1 \leqslant \lambda x^2 < 2$, and so $f(x) = 1 \geqslant \sqrt{\frac{2}{\lambda}} > x$.
- Let $x \in \left[\sqrt{\frac{2}{\lambda}}, \infty\right)$. Then $\lambda x^2 \geqslant 2$, and so $f(x) \geqslant 2$. Consequently,

$$f(x) > \sqrt{f(x)+1} = \sqrt{\lfloor \lambda x^2 \rfloor + 1} > \sqrt{\lambda x^2} > x.$$

The proposition follows. ☐

2.2. The subcase $0 < \lambda < 2$. The following proposition takes care of the more involved subcase $0 < \lambda < 2$.

Proposition 3. For $0 < \lambda < 2$, we have the following for every integer $a \geqslant 2$.

- If $\frac{1}{a-1} \leqslant \lambda < \frac{a}{(a-1)^2}$, then $\mathrm{Fix}(f) = \{0, a-1\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left[0, \sqrt{\frac{a-1}{\lambda}}\right); \\ \{a-1\}, & \text{if } x \in \left[\sqrt{\frac{a-1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left[\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

- If $\frac{a+1}{a^2} \leqslant \lambda < \frac{1}{a-1}$, then $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x \in \left[0, \sqrt{\frac{a}{\lambda}}\right); \\ \{\infty\}, & \text{if } x \in \left[\sqrt{\frac{a}{\lambda}}, \infty\right). \end{cases}$$

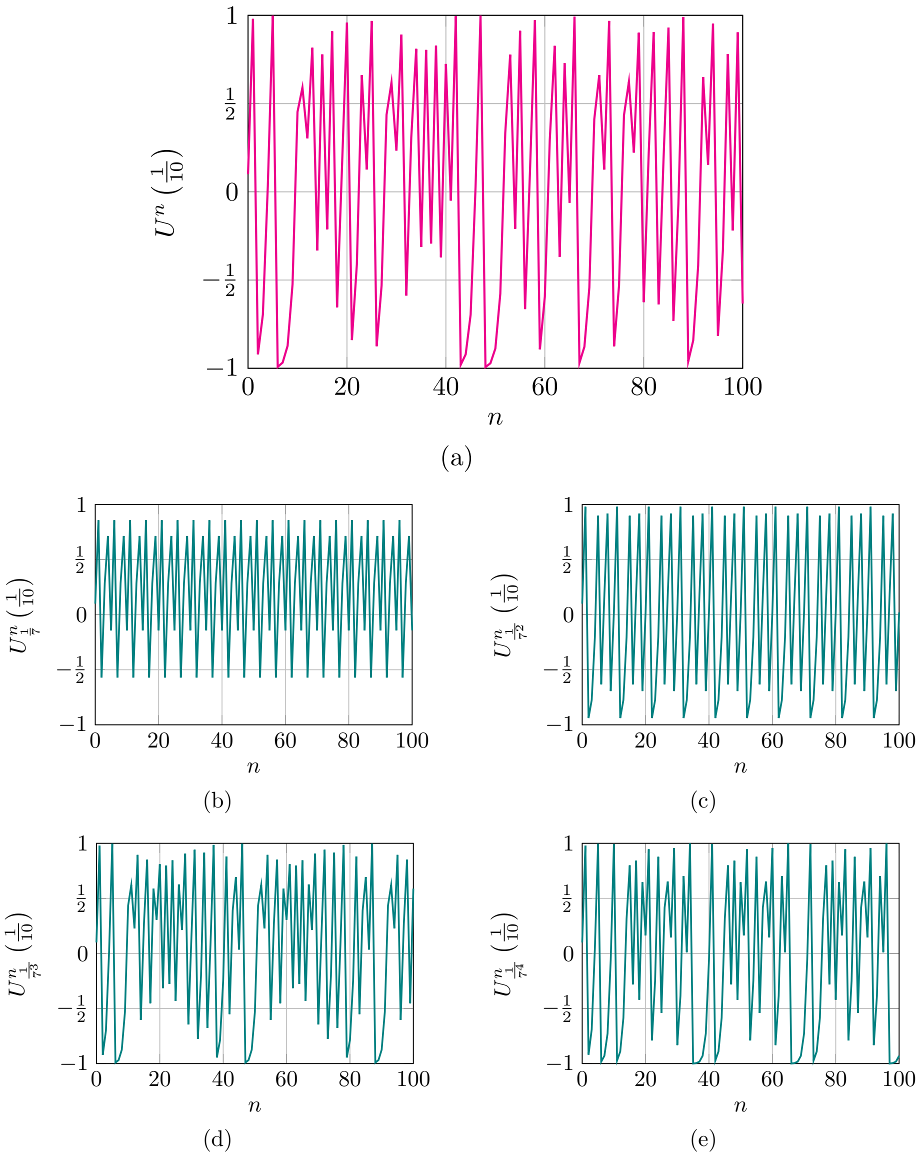

FIGURE 3. (a) The chaotic orbit with initial condition $\frac{1}{10}$ of the Ulam map $U(x) = 1 - 2x^2$; (b)–(e) The orbits with initial condition $\frac{1}{10}$ of the spatially discretised maps $U_{\frac{1}{7^k}}(x) = \lfloor 1 - 2x^2 \rceil_{\frac{1}{7^k}}$ for $k \in \{1, 2, 3, 4\}$, which are preperiodic with periods 5, 10, 41, 31, respectively.

Proof. Again, it is obvious that 0 is a fixed point. Let $a \geqslant 2$ be an integer. First, suppose that $\frac{1}{a-1} \leqslant \lambda < \frac{a}{(a-1)^2}$. Then

$$a - 1 \leqslant \lambda(a-1)^2 < a, \qquad \text{and so} \qquad f(a-1) = a - 1,$$

proving that $a-1$ is a fixed point.

- Let $x \in \left(0, \sqrt{\frac{a-2}{\lambda}}\right)$. Then $a > 2$, and so
$$\lambda < \frac{a}{(a-1)^2} < \frac{1}{a-2}.$$
Thus,
$$f(x) \leqslant \lambda x^2 < \sqrt{\lambda^2}\sqrt{\frac{a-2}{\lambda}}\,x = \sqrt{\lambda(a-2)}\,x < \sqrt{\frac{1}{a-2}(a-2)}\,x < x.$$
- Let $x \in \left[\sqrt{\frac{a-2}{\lambda}}, \sqrt{\frac{a-1}{\lambda}}\right)$. Then $a-2 \leqslant \lambda x^2 < a-1$, and so
$$f(x) = a-2 \leqslant \sqrt{(a-2)^2 + \frac{a-2}{a}} = \sqrt{\frac{a-2}{\left[\frac{a}{(a-1)^2}\right]}} < \sqrt{\frac{a-2}{\lambda}} \leqslant x.$$
- Let $x \in \left[\sqrt{\frac{a-1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right)$. Then $a-1 \leqslant \lambda x^2 < a$, and so $f(x) = a-1$.
- Let $x \in \left[\sqrt{\frac{a}{\lambda}}, \sqrt{\frac{a+1}{\lambda}}\right)$. Then $a \leqslant \lambda x^2 < a+1$, and so
$$f(x) = a > \sqrt{a^2-1} = \sqrt{\frac{a+1}{\left(\frac{1}{a-1}\right)}} \geqslant \sqrt{\frac{a+1}{\lambda}} > x.$$
- Let $x \in \left[\sqrt{\frac{a+1}{\lambda}}, \infty\right)$. Since $\frac{a+1}{a^2} < \frac{1}{a-1}$, then
$$\frac{a+1}{a^2} < \lambda, \qquad \text{and so} \qquad \sqrt{\frac{a+1}{\lambda}} > \frac{1}{\sqrt{\lambda(a+1)}-1}.$$
Next, the fact that $x \geqslant \sqrt{\frac{a+1}{\lambda}}$ implies
$$x > \frac{1}{\sqrt{\lambda(a+1)}-1}, \qquad \text{i.e.,} \qquad \sqrt{\lambda(a+1)}\,x - 1 > x.$$
Therefore,
$$f(x) > \lambda x^2 - 1 \geqslant \sqrt{\lambda^2}\sqrt{\frac{a+1}{\lambda}}\,x - 1 = \sqrt{\lambda(a+1)}\,x - 1 > x.$$

The first assertion in Proposition 3 follows.

Next, suppose that $\frac{a+1}{a^2} \leqslant \lambda < \frac{1}{a-1}$.

- Let $x \in \left(0, \sqrt{\frac{a-1}{\lambda}}\right)$. Then
$$f(x) \leqslant \lambda x^2 < \sqrt{\lambda^2}\sqrt{\frac{a-1}{\lambda}}\,x = \sqrt{\lambda(a-1)}\,x < \sqrt{\frac{1}{a-1}(a-1)}\,x < x.$$
- Let $x \in \left[\sqrt{\frac{a-1}{\lambda}}, \sqrt{\frac{a}{\lambda}}\right)$. Then $a-1 \leqslant \lambda x^2 < a$, and so
$$f(x) = a-1 = \sqrt{\frac{a-1}{\left(\frac{1}{a-1}\right)}} < \sqrt{\frac{a-1}{\lambda}} \leqslant x.$$

- Let $x \in \left[\sqrt{\frac{a}{\lambda}}, \sqrt{\frac{a+1}{\lambda}}\right)$. Then $a \leqslant \lambda x^2 < a+1$, and so

$$f(x) = a = \sqrt{\frac{a+1}{\left(\frac{a+1}{a^2}\right)}} \geqslant \sqrt{\frac{a+1}{\lambda}} > x.$$

- Let $x \in \left[\sqrt{\frac{a+1}{\lambda}}, \infty\right)$. Since

$$\frac{a+1}{a^2} \leqslant \lambda, \qquad \text{then} \qquad \sqrt{\frac{a+1}{\lambda}} \geqslant \frac{1}{\sqrt{\lambda(a+1)}-1}.$$

  Next, the fact that $x \geqslant \sqrt{\frac{a+1}{\lambda}}$ implies

$$x \geqslant \frac{1}{\sqrt{\lambda(a+1)}-1}, \qquad \text{i.e.,} \qquad \sqrt{\lambda(a+1)}\,x - 1 \geqslant x.$$

  Therefore,

$$f(x) > \lambda x^2 - 1 \geqslant \sqrt{\lambda^2}\sqrt{\frac{a+1}{\lambda}}\,x - 1 = \sqrt{\lambda(a+1)}\,x - 1 \geqslant x.$$

The second assertion in Proposition 3 follows. □

3. The Case $\lambda < 0$

Let us now turn to the case $\lambda < 0$, in which all fixed points belong to the invariant subdomain $(-\infty, 0]$, and the even property of our map implies that it suffices to determine $\omega_f(x)$ for all $x \leqslant 0$. We deal with the easier subcase $\lambda < -\frac{1}{4}$ in Proposition 4 (subsection 3.1) and with the more intricate subcase $-\frac{1}{4} \leqslant \lambda < 0$ in Proposition 5 (subsection 3.2).

3.1. The subcase $\lambda < -\frac{1}{4}$.

The subcase $\lambda < -\frac{1}{4}$ of Theorem 1 is a consequence of the following proposition.

Proposition 4.

- For $\lambda < -1$, we have $\mathrm{Fix}(f) = \{0\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-\infty\}, & \text{if } x \in (-\infty, 0). \end{cases}$$

- For $-1 \leqslant \lambda < -\frac{1}{2}$, we have $\mathrm{Fix}(f) = \{0, -1\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right); \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{1}{\lambda}}\right). \end{cases}$$

- For $-\frac{1}{2} \leqslant \lambda < -\frac{1}{3}$, we have $\mathrm{Fix}(f) = \{0, -1, -2\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right); \\ \{-2\}, & \text{if } x \in \left[-\sqrt{-\frac{2}{\lambda}}, -\sqrt{-\frac{1}{\lambda}}\right); \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{2}{\lambda}}\right). \end{cases}$$

- For $-\frac{1}{3} \leqslant \lambda < -\frac{1}{4}$, we have $\mathrm{Fix}(f) = \{0, -1, -2, -3\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right); \\ \{-2\}, & \text{if } x \in \left[-\sqrt{-\frac{2}{\lambda}}, -\sqrt{-\frac{1}{\lambda}}\right); \\ \{-3\}, & \text{if } x \in \left[-\sqrt{-\frac{3}{\lambda}}, -\sqrt{-\frac{2}{\lambda}}\right); \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{-\frac{3}{\lambda}}\right). \end{cases}$$

Proof. That 0 is a fixed point is trivial. Now suppose that $\lambda < -1$.

- Let $x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right)$. Then $-1 \leqslant \lambda x^2 < 0$, and so $f(x) = -1 < -\sqrt{-\frac{1}{\lambda}} \leqslant x$.
- Let $x \in \left(-\infty, -\sqrt{-\frac{1}{\lambda}}\right)$. Then $x < -\sqrt{-\frac{1}{\lambda}} < \frac{1}{\lambda}$, and so $f(x) \leqslant \lambda x^2 < x$.

Next, suppose that $-\frac{1}{a} \leqslant \lambda < -\frac{1}{a+1}$, for some $a \in \{1, 2, 3\}$. Then for every $i \in \{1, \ldots, a\}$ we have

$$-\frac{i^2}{a} \leqslant \lambda i^2 < -\frac{i^2}{a+1}.$$

Since

$$-i \leqslant -\frac{i^2}{a} \qquad \text{and} \qquad -\frac{i^2}{a+1} < \frac{(-i+1)i}{a+1} < -i+1,$$

then $f(-i) = \lfloor \lambda i^2 \rfloor = -i$, which means that $-i$ is a fixed point.

- Let $x \in \left[-\sqrt{-\frac{i}{\lambda}}, -\sqrt{-\frac{i-1}{\lambda}}\right)$ for some $i \in \{1, \ldots, a\}$. Then $-i \leqslant \lambda x^2 < -i+1$, and so $f(x) = -i$.
- Let $x \in \left(-\infty, -\sqrt{-\frac{a}{\lambda}}\right)$. If $x \in \left[\frac{1}{\lambda}, -\sqrt{-\frac{a}{\lambda}}\right)$, then $\lambda x^2 < -a$, and so $f(x) \leqslant -a-1 < \frac{1}{\lambda} \leqslant x$. Otherwise, we have $x \in \left(-\infty, \frac{1}{\lambda}\right)$, in which case $f(x) \leqslant \lambda x^2 < x$.

The proposition is proved. □

3.2. The subcase $-\frac{1}{4} \leqslant \lambda < 0$. Finally, it remains to address the subcase $-\frac{1}{4} \leqslant \lambda < 0$.

Proposition 5. For $-\frac{1}{4} \leqslant \lambda < 0$, we have the following for every integer $a \leqslant -4$.

- If $\frac{1}{a} \leqslant \lambda < \frac{a+2}{(a+1)^2}$, then $\mathrm{Fix}(f) = \{0, -1, a+1, a\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{\frac{a+2}{\lambda}}, 0\right); \\ \{a+1\}, & \text{if } x \in \left[-\sqrt{\frac{a+1}{\lambda}}, -\sqrt{\frac{a+2}{\lambda}}\right); \\ \{a\}, & \text{if } x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right); \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right). \end{cases}$$

- If $\frac{a+2}{(a+1)^2} \leqslant \lambda < \frac{1}{a-1}$, then $\mathrm{Fix}(f) = \{0, -1, a\}$ and

$$\omega_f(x) = \begin{cases} \{0\}, & \text{if } x = 0; \\ \{-1\}, & \text{if } x \in \left[-\sqrt{\frac{a+1}{\lambda}}, 0\right); \\ \{a\}, & \text{if } x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right); \\ \{-\infty\}, & \text{if } x \in \left(-\infty, -\sqrt{\frac{a}{\lambda}}\right). \end{cases}$$

Proof. Since $-\frac{1}{4} \leqslant \lambda < 0$, it is apparent that 0 and -1 are fixed points. Let $a \leqslant -4$ be an integer. First, suppose that $\frac{1}{a} \leqslant \lambda < \frac{a+2}{(a+1)^2}$. Then

$$\frac{(a+1)^2}{a} \leqslant \lambda(a+1)^2 < a+2.$$

Since $a < -1$, then $a+1 < \frac{(a+1)^2}{a}$, and so $f(a+1) = \lfloor \lambda(a+1)^2 \rfloor = a+1$, which means that $a+1$ is a fixed point. Furthermore,

$$a \leqslant \lambda a^2 < \frac{(a+2)a^2}{(a+1)^2}.$$

Since $a \leqslant -4$, then

$$(a+1)^2 + a = \left(a + \frac{3}{2}\right)^2 - \frac{5}{4} \geqslant 5 > 0, \qquad \text{and so} \qquad -\frac{a}{(a+1)^2} < 1,$$

the latter being equivalent to

$$\frac{(a+2)a^2}{(a+1)^2} < a+1.$$

Therefore, $f(a) = \lfloor \lambda a^2 \rfloor = a$, which means that a is a fixed point.

- Let $x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right)$. Then $-1 \leqslant \lambda x^2 < 0$, and so $f(x) = -1$.
- Let $x \in \left[-\sqrt{\frac{i}{\lambda}}, -\sqrt{\frac{i+1}{\lambda}}\right)$ for some $i \in \{a+2, \ldots, -2\}$. Since the map $x \mapsto \frac{x+1}{x^2}$ is monotonically decreasing in $(-\infty, -2)$, then

$$\frac{a+3}{(a+2)^2} \geqslant \frac{i+1}{i^2}.$$

On the other hand, the inequality $a \leqslant -4$ is equivalent to

$$\frac{1}{a} \geqslant \frac{a+3}{(a+2)^2}.$$

It follows that $\lambda \geqslant \frac{i+1}{i^2}$. Together with the fact that $i \leqslant \lambda x^2 < i+1$, this implies

$$f(x) = i = -\sqrt{\frac{i+1}{\left(\frac{i+1}{i^2}\right)}} \geqslant -\sqrt{\frac{i+1}{\lambda}} > x.$$

- Let $x \in \left[-\sqrt{\frac{a+1}{\lambda}}, -\sqrt{\frac{a+2}{\lambda}}\right)$. Then $a+1 \leqslant \lambda x^2 < a+2$, and so $f(x) = a+1$.
- Let $x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right)$. Then $a \leqslant \lambda x^2 < a+1$, and so $f(x) = a$.
- Let $x \in \left[-\sqrt{\frac{a-1}{\lambda}}, -\sqrt{\frac{a}{\lambda}}\right)$. Then $a-1 \leqslant \lambda x^2 < a$, and so

$$f(x) = a-1 = -\sqrt{\frac{a-1}{\left(\frac{1}{a-1}\right)}} < -\sqrt{\frac{a-1}{\left(\frac{1}{a}\right)}} \leqslant -\sqrt{\frac{a-1}{\lambda}} \leqslant x.$$

- Let $x \in \left(-\infty, -\sqrt{\frac{a-1}{\lambda}}\right)$. Since the inequality $a < -3$ is equivalent to

$$\frac{a+2}{(a+1)^2} < \frac{1}{a-1},$$

we have that $\lambda < \frac{1}{a-1}$. Thus,

$$f(x) \leqslant \lambda x^2 < \left(-\sqrt{\lambda^2}\right)\left(-\sqrt{\frac{a-1}{\lambda}}\right) x = \sqrt{\lambda(a-1)}x < \sqrt{\frac{1}{a-1}(a-1)}\, x = x.$$

This establishes the first assertion in Proposition 5.

Next, suppose that $\frac{a+2}{(a+1)^2} \leqslant \lambda < \frac{1}{a-1}$. Then

$$\frac{(a+2)a^2}{(a+1)^2} \leqslant \lambda a^2 < \frac{a^2}{a-1}.$$

The fact that $a < -1$ implies

$$-\frac{a}{(a+1)^2} > 0 \qquad \text{and} \qquad \frac{a}{a-1} < 1,$$

which are equivalent to

$$a < \frac{(a+2)a^2}{(a+1)^2} \qquad \text{and} \qquad \frac{a^2}{a-1} < a+1,$$

respectively. Thus, $f(a) = \lfloor \lambda a^2 \rfloor = a$, which means that a is a fixed point.

- Let $x \in \left[-\sqrt{-\frac{1}{\lambda}}, 0\right)$. Then $-1 \leqslant \lambda x^2 < 0$, and so $f(x) = -1$.
- Let $x \in \left[-\sqrt{\frac{i}{\lambda}}, -\sqrt{\frac{i+1}{\lambda}}\right)$ for some $i \in \{a+1, \dots, -2\}$. As before, since the map $x \mapsto \frac{x+1}{x^2}$ is monotonically decreasing in $(-\infty, -2)$, then

$$\frac{a+2}{(a+1)^2} \geqslant \frac{i+1}{i^2}.$$

Since $\lambda \geqslant \frac{a+2}{(a+1)^2}$, then $\lambda \geqslant \frac{i+1}{i^2}$. Together with the fact that $i \leqslant \lambda x^2 < i+1$, this implies

$$f(x) = i = -\sqrt{\frac{i+1}{\left(\frac{i+1}{i^2}\right)}} \geqslant -\sqrt{\frac{i+1}{\lambda}} > x.$$

- Let $x \in \left[-\sqrt{\frac{a}{\lambda}}, -\sqrt{\frac{a+1}{\lambda}}\right)$. Then $a \leqslant \lambda x^2 < a+1$, and so $f(x) = a$.
- Let $x \in \left[-\sqrt{\frac{a-1}{\lambda}}, -\sqrt{\frac{a}{\lambda}}\right)$. Then $a-1 \leqslant \lambda x^2 < a$, and so

$$f(x) = \lfloor \lambda x^2 \rfloor = a - 1 = -\sqrt{\frac{a-1}{\left(\frac{1}{a-1}\right)}} < -\sqrt{\frac{a-1}{\lambda}} \leqslant x.$$

- Let $x \in \left(-\infty, -\sqrt{\frac{a-1}{\lambda}}\right)$. Then

$$f(x) \leqslant \lambda x^2 < \left(-\sqrt{\lambda^2}\right)\left(-\sqrt{\frac{a-1}{\lambda}}\right) x = \sqrt{\lambda(a-1)}\, x < \sqrt{\frac{1}{a-1}(a-1)}\, x = x.$$

This establishes the second assertion in Proposition 5. □

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