

A NEW PROOF OF AN IDENTITY CONCERNING 5-CORE PARTITIONS

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Abstract. Let $a_5(n)$ be the number of partitions of n that are 5-cores. We provide a new elementary proof of an identity involving $a_5(n)$ due to Baruah and Berndt by employing 2-dissection formulas and identities involving the Ramanujan’s parameter $k(q)$ due to Cooper, Chern, and Tang.

1. Introduction

We denote $f_m := \prod_{n \geq 1} (1 - q^{mn})$ for $m \in \mathbb{N}$ and $q \in \mathbb{C}$ with $|q| < 1$ throughout this paper. A partition of a positive integer n is a finite nonincreasing sequence of positive integer parts λ_k whose sum is n . A partition of n can be visualized via its Ferrers–Young diagram consisting of nodes arranged in left-justified rows, where the k th row has λ_k nodes. For each node situated at a point (i, j) , we define its hook number as the total number of dots directly below and to the right of that node, including the node itself. We say that a partition of n is t -core for some $t \in \mathbb{N}$ if none of its hook numbers is divisible by t .

Granville and Ono [7] showed that for $t \geq 4$, there is at least one t -core partition of n for any positive integer n . Garvan, Kim, and Stanton [6] proved that the generating function for the number $a_t(n)$ of partitions of n that are t -cores with $a_t(0) := 1$ is given by

$$\sum_{n=0}^{\infty} a_t(n)q^n = \frac{f_t^t}{f_1}.$$

Baruah and Berndt [1] discovered a linear identity involving $a_5(n)$ that reads

$$a_5(4n + 3) = a_5(2n + 1) + 2a_5(n) \tag{1.1}$$

by employing the classical modular equation of degree 5 due to Ramanujan [2, p. 280, Entry 13(iii)]. Kim [11] found a systematic method via the action of Hecke operators on certain modular forms of weight $(p - 1)/2$ that yields linear identities involving $a_p(n)$ for primes $p \geq 5$, including (1.1). Park [13] deduced (1.1) by constructing modular equations between certain modular functions on the congruence subgroup $\Gamma_0(20)$ via the result of Ishida and Ishii [10]. For more important results about t -core partitions and other related partitions, we refer the interested reader to a recent survey paper of Cho et. al [4].

In this short note, we give a new elementary proof of the identity (1.1) different from that of Baruah and Berndt [1]. More precisely, we establish that (1.1) is a consequence of the following theta function identity.

Theorem 1.1. *We have the identity*

$$\frac{f_2^3 f_{10}^9}{f_1^3 f_4 f_5 f_{20}^3} - 4q^2 \frac{f_4 f_5^2 f_{20}^3}{f_1^2} = \frac{f_5^5}{f_1} + 2q \frac{f_{10}^5}{f_2}.$$

The rest of the paper is organized as follows. In Section 2, we first show that (1.1) follows from Theorem 1.1 using certain 2-dissection formulas. We then prove Theorem 1.1 by relying on some identities involving the Ramanujan's parameter

$$k(q) := q \prod_{n=1}^{\infty} \frac{(1 - q^{10n-9})(1 - q^{10n-8})(1 - q^{10n-2})(1 - q^{10n-1})}{(1 - q^{10n-7})(1 - q^{10n-6})(1 - q^{10n-4})(1 - q^{10n-3})}$$

due to Cooper [5] and Chern and Tang [3]. The proofs presented here are deductive derivations and do not require any modular equation of Ramanujan in advance.

2. Proofs of (1.1) and Theorem 1.1

We prove in this section (1.1) and Theorem 1.1. We begin with confirming (1.1) using Theorem 1.1 and the following 2-dissection formulas

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2}, \quad (2.1)$$

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \quad (2.2)$$

Identity (2.1) follows from replacing q with $-q$ in [8, Lemma 2.3]. Identity (2.2) is due to Hirschhorn and Sellers [9, Theorem 2.1] who examined the parity of 5-regular partitions.

Proof of (1.1). Using (2.1) and (2.2), we have

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(n) q^n &= \frac{f_5^5}{f_1} = \frac{f_5}{f_1} \cdot f_5^4 \\ &= \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right) \left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5 \frac{f_{10}^2 f_{40}^4}{f_{20}^2} \right). \end{aligned} \quad (2.3)$$

We expand the right-hand side of (2.3) and extract the terms involving q^{2n+1} , divide both sides by q , and then replace q^2 with q . In view of Theorem 1.1 and (2.3), we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(2n+1) q^n &= \frac{f_2^3 f_{10}^9}{f_1^3 f_4 f_5 f_{20}^3} - 4q^2 \frac{f_4 f_5^2 f_{20}^3}{f_1^2} \\ &= \frac{f_5^5}{f_1} + 2q \frac{f_{10}^5}{f_2} \\ &= \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}} \right) \left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5 \frac{f_{10}^2 f_{40}^4}{f_{20}^2} \right) + 2q \frac{f_{10}^5}{f_2}. \end{aligned} \quad (2.4)$$

We expand the right-hand side of (2.4) and extract the terms involving q^{2n+1} , divide both sides by q , and then replace q^2 with q . We see that

$$\begin{aligned} \sum_{n=0}^{\infty} a_5(4n+3)q^n &= \frac{f_2^3 f_{10}^9}{f_1^3 f_4 f_5 f_{20}^3} - 4q^2 \frac{f_4 f_5^2 f_{20}^3}{f_1^2} + 2 \frac{f_5^5}{f_1} \\ &= \sum_{n=0}^{\infty} a_5(2n+1)q^n + 2 \sum_{n=0}^{\infty} a_5(n)q^n, \end{aligned}$$

which yields (1.1) after comparing coefficients. \square

We devote the rest of the section to proving Theorem 1.1. We first require the following identities involving $k(q)$ due to Cooper [5] and Chern and Tang [3].

Lemma 2.1. *We have the identities*

$$\frac{f_2 f_5^5}{q f_1 f_{10}^5} = \frac{1}{k(q)} - k(q), \quad (2.5)$$

$$\frac{f_2^4 f_5^2}{q f_1^2 f_{10}^4} = \frac{1}{k(q)} + 1 - k(q), \quad (2.6)$$

$$\frac{f_1^3 f_5}{q f_2 f_{10}^3} = \frac{1}{k(q)} - 4 - k(q). \quad (2.7)$$

Proof. See [5, Theorem 10.4]. \square

Lemma 2.2. *We have the identity*

$$\frac{k(q)}{k(q^2)} - \frac{k(q^2)}{k(q)} = \frac{f_1 f_5^3}{q f_{10}^4}.$$

Proof. See [3, Theorem 3.3]. \square

As a byproduct of Lemma 2.2, we derive the following algebraic relation between $k(q)$ and $k(q^2)$, which will be crucial in the proof of Theorem 1.1. It is worth noting that Lee and Park [12] recently found this relation via modular functions.

Theorem 2.3. *We have the identity*

$$X^2 - Y + 2XY + X^2Y + Y^2 = 0,$$

where $X := k(q)$ and $Y := k(q^2)$.

Proof. Squaring the identity in Lemma 2.2 and applying (2.5) and (2.7), we obtain

$$\left(\frac{X}{Y} - \frac{Y}{X}\right)^2 = \frac{f_1^2 f_5^6}{q^2 f_{10}^8} = \frac{f_2 f_5^5}{q f_1 f_{10}^5} \cdot \frac{f_1^3 f_5}{q f_2 f_{10}^3} = \left(\frac{1}{X} - X\right) \left(\frac{1}{X} - 4 - X\right),$$

which simplifies to

$$\frac{(X^2 + Y - 2XY - X^2Y + Y^2)(X^2 - Y + 2XY + X^2Y + Y^2)}{X^2Y^2} = 0. \quad (2.8)$$

Using the q -expansion

$$k(q) = q - q^2 - q^3 + 2q^4 - 2q^6 + 2q^7 + O(q^8)$$

of $k(q)$, we see that the second factor of the numerator of (2.8) vanishes while the first factor does not. Hence, we arrive at the desired identity. \square

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. We first restate the given identity as

$$\frac{f_2^4 f_{10}^4}{q f_1^3 f_4 f_5 f_{20}^3} - 4q \frac{f_2 f_4 f_5^2 f_{20}^3}{f_1^2 f_{10}^5} = \frac{f_2 f_5^5}{q f_1 f_{10}^5} + 2. \quad (2.9)$$

Let $X := k(q)$ and $Y := k(q^2)$. Replacing q with q^2 in (2.7) yields

$$Z := \frac{1}{Y} - 4 - Y = \frac{f_2^3 f_{10}}{q^2 f_4 f_{20}^3}. \quad (2.10)$$

We know from Theorem 2.3 and (2.10) that

$$1 - 2X - X^2 = \frac{X^2}{Y} + Y = X^2(Z + 4 + Y) + Y = X^2(Z + 4) + Y(X^2 + 1).$$

Solving the above equation for Y , we have

$$Y = \frac{1 - 2X - X^2 - X^2(Z + 4)}{X^2 + 1}. \quad (2.11)$$

Plugging in (2.11) into Theorem 2.3 and clearing denominators, we arrive at

$$X^2 Z^2 + (-1 - 2X + X^2)(1 - 4X - X^2)Z - 4(1 + X - X^2)(1 - 4X - X^2) = 0. \quad (2.12)$$

Invoking Lemma 2.1, (2.10), and (2.12), we deduce that

$$\begin{aligned} \frac{f_2^4 f_{10}^4}{q f_1^3 f_4 f_5 f_{20}^3} - 4q \frac{f_2 f_4 f_5^2 f_{20}^3}{f_1^2 f_{10}^5} &= \frac{f_2^3 f_{10}}{q^2 f_4 f_{20}^3} \cdot \frac{q f_2 f_{10}^3}{f_1^3 f_5} - 4 \cdot \frac{q^2 f_4 f_{20}^3}{f_2^3 f_{10}} \cdot \frac{f_2^4 f_5^2}{q f_1^2 f_{10}^4} \\ &= \frac{ZX}{1 - 4X - X^2} - 4 \cdot \frac{1 + X - X^2}{ZX} \\ &= \frac{Z^2 X^2 - 4(1 + X - X^2)(1 - 4X - X^2)}{ZX(1 - 4X - X^2)} \\ &= \frac{(1 + 2X - X^2)(1 - 4X - X^2)Z}{ZX(1 - 4X - X^2)} \\ &= \frac{1}{X} - X + 2 \\ &= \frac{f_2 f_5^5}{q f_1 f_{10}^5} + 2, \end{aligned}$$

which is precisely (2.9). This completes the proof. \square

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