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A NOTE ON THE REGULARITY CRITERION OF WEAK SOLUTIONS FOR THE MICROPOLAR FLUID EQUATIONS

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Abstract. The aim of this paper is to investigate the regularity criterion of Leray-Hopf weak solutions to the 3D incompressible micropolar fluid equations. It is shown that if

$$\int_0^T \frac{\|\nabla\pi(t)\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u(\cdot,t)\|_{L^3}^{\alpha}+\|\omega(\cdot,t)\|_{L^3}^{\alpha}}dt<\infty \quad \text{with} \quad \alpha=\begin{cases} 3, & 1<r\leq\frac{9}{7}, \\ \frac{2r}{3(r-1)}, & \frac{9}{7}<r<3, \end{cases}$$

then the corresponding weak solution (u,ω) is regular on $[0,T]$, which is an obvious extension of the previous results.

1. Introduction

This paper is devoted to study the smoothness of a weak solutions to the Cauchy problem for three-dimensional micropolar fluid system (see [**4**]) describing the flow of a viscous incompressible fluid, namely,

$$\begin{cases} \partial_t u-(\mu+\varkappa)\Delta u-\varkappa\nabla\times\omega+(u\cdot\nabla)u+\nabla\pi=0, \\ \partial_t\omega-\gamma\Delta\omega-\kappa\nabla\nabla\cdot\omega+2\varkappa\omega+(u\cdot\nabla)\omega-\varkappa\nabla\times u=0, \\ \nabla\cdot u=0, \\ u\left(x,0\right)=u_0\left(x\right), \qquad \omega\left(x,0\right)=\omega_0\left(x\right). \end{cases} \tag{1.1}$$

Here $u=u(x,t)\in\mathbb{R}^3$ denotes the velocity of the fluid, $\omega=\omega(x,t)\in\mathbb{R}^3$ denotes the micro-rotational vector field, π denotes the total pressure field of the fluid motion, while u_0 and ω_0 are given initial velocity and initial micro-rotational fields with $\nabla\cdot u_0=0$ in the sense of distribution. The constants $\mu,\varkappa,\gamma$ and κ are positive numbers associated to the properties of the material : μ is the kinematic viscosity, $\varkappa$ is the vortex viscosity, γ and κ are spin viscosities.

The existences of weak and strong solutions for micropolar fluid equations were treated by Galdi and Rionero [**9**] and Yamaguchi [**17**], respectively. However, as same as the 3D Navier-Stokes equations, the problem on the regularity or finite time singularity for the weak solution still remains unsolved. Regularity can only been derived when certain growth conditions are satisfied. For more details, we can refer to [**1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 18**] and the references cited therein.

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Very recently, Tran and Yu [**16**] derived the following regularity criterion for weak solutions of Navier-Stokes equations ($\omega = 0$) as long

$$\int_0^T \frac{\|\pi(t)\|_{L^r}^{\frac{2r}{2r-3}}}{(1+\|u(\cdot,t)\|_{L^3})^{\alpha}} dt < \infty \quad \text{with} \quad \alpha = \begin{cases} \frac{6}{2r-3}, & r \geq 3, \\ \frac{2r}{2r-3}, & \frac{9}{4} \leq r \leq 3, \\ 3, & \frac{3}{2} < r \leq \frac{9}{4} \end{cases}.$$

In this paper, inspired by the paper of Tran and Yu [**16**], we will show that a weak solution (u, ω) is smooth under the assumption that

$$\int_0^T \frac{\|\nabla\pi(t)\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u(\cdot,t)\|_{L^3}^{\alpha} + \|\omega(\cdot,t)\|_{L^3}^{\alpha}} dt < \infty \quad \text{with} \quad \alpha = \begin{cases} 3, & 1 < r \leq \frac{9}{7}, \\ \frac{2r}{3(r-1)}, & \frac{9}{7} < r < 3. \end{cases}$$

Our result improves the classical $L^p(0,T;L^q(\mathbb{R}^3))$ regularity criteria for the pressure by factors of certain negative powers of the scaling invariant norms.

Now we present our main result as follows.

Theorem 1.1. *Let $T > 0$ be a given time and $(u_0, \omega_0) \in H^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$. Suppose that (u, ω) is a weak solution of system (1.1) in the time interval $[0, T)$ for some $0 < T < \infty$. If the pressure π satisfies the following condition*

$$\int_0^T \frac{\|\nabla\pi(t)\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u(\cdot,t)\|_{L^3}^{\alpha} + \|\omega(\cdot,t)\|_{L^3}^{\alpha}} dt < \infty \quad \textit{with} \quad \alpha = \begin{cases} 3, & 1 < r \leq \frac{9}{7}, \\ \frac{2r}{3(r-1)}, & \frac{9}{7} < r < 3, \end{cases} \tag{1.2}$$

then (u, ω) is a regular solution on $\mathbb{R}^3 \times [0, T]$.

Remark 1.2. It should be noted that (1.2) presents a new type of regularity criterion via the pressure by factors of certain negative powers of the scaling invariant norms $\|u\|_{L^3}$ and $\|\omega\|_{L^3}$. The regularity criterion (1.2) still true for the 3D incompressible Navier-Stokes equations.

2. Proof of Theorem 1.1

Now we are in a position to give the proof of Theorem 1.1.

Proof: Since the initial data $(u_0, \omega_0) \in H^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$ with $\nabla \cdot u_0 = 0$, there exists a unique local strong solution (u, ω) of the 3D micropolar equations on $(0, T)$ (see [**9, 13**]). Moreover, the strong solution can be proved to exist on a maximal time interval using the standard local solution extension technique. Thus the proof of Theorem 1.1 is reduced to establishing regular estimates uniformly on $(0, T)$, and then the local strong solution (u, ω) can be continuously extended to the time $t = T$ by standard continuation process. Therefore, in what follows, we may as well assume that the solution (u, ω) is sufficiently smooth on $(0, T)$.

Now, we first multiply both sides of the equation (1.1) by $u\,|u|$, and integrate with respect to x over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned} & \frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^3\, dx + (\mu+\varkappa)\int_{\mathbb{R}^3} |\nabla u|^2\,|u|\, dx + \frac{4}{9}(\mu+\varkappa)\int_{\mathbb{R}^3} \left|\nabla\,|u|^{\frac{3}{2}}\right|^2 dx \\ = & -\int_{\mathbb{R}^3} \nabla\pi\cdot u\,|u|\, dx + \varkappa\int_{\mathbb{R}^3} \nabla\times\omega\cdot u\,|u|\, dx, \end{aligned} \tag{2.1}$$

where we have used the following identities due to the divergence free property:

$$\int_{\mathbb{R}^3} (u\cdot\nabla u)\cdot u\,|u|\, dx = \frac{1}{3}\int_{\mathbb{R}^3} u\cdot\nabla\,|u|^3\, dx = 0,$$

$$\begin{aligned}\int_{\mathbb{R}^3}\Delta u\cdot u\,|u|\,dx &= -\int_{\mathbb{R}^3}|\nabla u|^2\,|u|\,dx-\int_{\mathbb{R}^3}|\nabla\,|u|\,|^2\,|u|\,dx\\ &= -\int_{\mathbb{R}^3}|\nabla u|^2\,|u|\,dx-\frac{4}{9}\int_{\mathbb{R}^3}\left|\nabla\,|u|^{\frac{3}{2}}\right|^2dx.\end{aligned}$$

Similarly, multiply both sides of the equation (1.1) by $\omega\,|\omega|$, and integrate with respect to x over $\mathbb{R}^3$. After suitable integration by parts, we obtain

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}|\omega|^3\,dx+(\gamma+\frac{\kappa}{2})\int_{\mathbb{R}^3}|\nabla\omega|^2\,|\omega|\,dx+\frac{4}{9}(\gamma+\kappa)\int_{\mathbb{R}^3}\left|\nabla\,|\omega|^{\frac{3}{2}}\right|^2dx\\ &\qquad+\frac{\kappa}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2\,|\omega|\,dx+2\varkappa\int_{\mathbb{R}^3}|\omega|^3\,dx\\ \leq\;&\varkappa\int_{\mathbb{R}^3}\nabla\times u\cdot\omega\,|\omega|\,dx,\end{aligned}\tag{2.2}$$

where we have used the fact that $\nabla(\nabla\cdot\omega)=\nabla\times(\nabla\times\omega)+\Delta\omega$ yields

$$\begin{aligned}&-\int_{\mathbb{R}^3}\nabla(\nabla\cdot\omega)\cdot\omega\,|\omega|\,dx\\ =\;&-\int_{\mathbb{R}^3}(\nabla\times(\nabla\times\omega)+\Delta\omega)\cdot\omega\,|\omega|\,dx\\ =\;&\int_{\mathbb{R}^3}|\nabla\times\omega|^2\,|\omega|\,dx+\int_{\mathbb{R}^3}\nabla\times\omega\cdot\nabla\,|\omega|\times\omega dx\\ &\qquad+\int_{\mathbb{R}^3}|\nabla\omega|^2\,|\omega|\,dx+\frac{2}{3}\int_{\mathbb{R}^3}\left|\nabla\,|\omega|^{\frac{3}{2}}\right|^2dx\\ \geq\;&\int_{\mathbb{R}^3}|\nabla\times\omega|^2\,|\omega|\,dx-\frac{1}{2}\int_{\mathbb{R}^3}(|\nabla\times\omega|^2\,|\omega|+|\nabla\,|\omega|\,|^2\,|\omega|)dx\\ &\qquad+\int_{\mathbb{R}^3}|\nabla\omega|^2\,|\omega|\,dx+\frac{4}{9}\int_{\mathbb{R}^3}\left|\nabla\,|\omega|^{\frac{3}{2}}\right|^2dx\\ =\;&\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2\,|\omega|\,dx+\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\omega|^2\,|\omega|\,dx+\frac{4}{9}\int_{\mathbb{R}^3}\left|\nabla\,|\omega|^{\frac{3}{2}}\right|^2dx.\end{aligned}$$

Combining (2.1) and (2.2), it follows that

$$\begin{aligned}&\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^3+|\omega|^3)dx+\frac{4}{9}(\mu+\varkappa)\int_{\mathbb{R}^3}\left|\nabla\,|u|^{\frac{3}{2}}\right|^2dx+\frac{4}{9}(\gamma+\kappa)\int_{\mathbb{R}^3}\left|\nabla\,|\omega|^{\frac{3}{2}}\right|^2dx\\ &\qquad+(\mu+\varkappa)\int_{\mathbb{R}^3}|\nabla u|^2\,|u|\,dx+(\gamma+\frac{\kappa}{2})\int_{\mathbb{R}^3}|\nabla\omega|^2\,|\omega|\,dx\\ &\qquad+\frac{\kappa}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2\,|\omega|\,dx+2\varkappa\int_{\mathbb{R}^3}|\omega|^3\,dx\\ \leq\;&\left|\int_{\mathbb{R}^3}\nabla\pi\cdot u\,|u|\,dx\right|+\varkappa\int_{\mathbb{R}^3}|\omega|\,|u|\,|\nabla u|\,dx+\varkappa\int_{\mathbb{R}^3}|u|\,|\omega|\,|\nabla\omega|\,dx\\ =\;&I+J+K,\end{aligned}\tag{2.3}$$

where we have use the following identities

$$\begin{aligned}\int_{\mathbb{R}^3}\nabla\times\omega\cdot u\,|u|\,dx &= -\int_{\mathbb{R}^3}|u|\,\omega\cdot\nabla\times udx-\int_{\mathbb{R}^3}\omega\cdot\nabla\,|u|\times udx,\\ \int_{\mathbb{R}^3}\nabla\times u\cdot\omega\,|\omega|\,dx &= -\int_{\mathbb{R}^3}|\omega|\,u\cdot\nabla\times\omega dx-\int_{\mathbb{R}^3}u\cdot\nabla\,|\omega|\times\omega dx,\end{aligned}$$

and the facts that

$$|\nabla \times u| \leq |\nabla u| \quad \text{and} \quad |\nabla |u|| \leq |\nabla u|.$$

Then we shall estimate the above terms I, J and K one by one. For J, by using the Hölder inequality and the Young inequality

$$\begin{aligned}
J &\leq \varkappa \left\| |\omega| |u|^{\frac{1}{2}} \right\|_{L^2} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2} \\
&\leq \frac{(\mu+\varkappa)}{4} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2 + C \left\| |\omega| |u|^{\frac{1}{2}} \right\|_{L^2}^2 \\
&\leq \frac{(\mu+\varkappa)}{4} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2 + C \|\omega\|_{L^3}^2 \left\| |u|^{\frac{1}{2}} \right\|_{L^6}^2 \\
&\leq \frac{(\mu+\varkappa)}{4} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2 + C \|\omega\|_{L^3}^2 \|u\|_{L^3} \\
&\leq \frac{(\mu+\varkappa)}{4} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2 + C(\frac{2}{3} \|\omega\|_{L^3}^3 + \frac{1}{3} \|u\|_{L^3}^3). \qquad (2.4)
\end{aligned}$$

Similarly, we can estimate K as

$$\begin{aligned}
K &\leq \varkappa \left\| |u| |\omega|^{\frac{1}{2}} \right\|_{L^2} \left\| |\omega|^{\frac{1}{2}} |\nabla \omega| \right\|_{L^2} \\
&\leq \frac{\gamma}{2} \left\| |\omega|^{\frac{1}{2}} |\nabla \omega| \right\|_{L^2}^2 + C(\frac{2}{3} \|u\|_{L^3}^3 + \frac{1}{3} \|\omega\|_{L^3}^3). \qquad (2.5)
\end{aligned}$$

To estimate the term involving $\nabla\pi$, Taking $\nabla\nabla\cdot$ on both sides of the equation (1.1), it follows that $\pi = (-\Delta)^{-1}\nabla \cdot (u \cdot \nabla u)$, where we have used the facts $\nabla \cdot u = 0$ and $\nabla \cdot (\nabla \times \omega) = 0$. Then the Calderón-Zygmund inequality implies that

$$\|\nabla \pi\|_{L^s} = \left\| \nabla(-\Delta)^{-1}\nabla \cdot (u \cdot \nabla u) \right\|_{L^s} \leq C \|u \cdot \nabla u\|_{L^s}, \quad \text{for all } 1 < s < \infty.$$

Case 1. If $1 < r \leq \frac{9}{7}$, Hölder's inequality and Young's inequality gives

$$\begin{aligned}
I &\leq \int_{\mathbb{R}^3} |\nabla \pi| |u|^2 dx = \int_{\mathbb{R}^3} |\nabla \pi|^{\frac{2r}{9-5r}} |\nabla \pi|^{\frac{9-7r}{9-5r}} |u|^2 dx \\
&\leq \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \|\nabla \pi\|_{L^{\frac{9}{5}}}^{\frac{9-7r}{9-5r}} \|u\|_{L^9}^2 \\
&\leq C \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \|u \cdot \nabla u\|_{L^{\frac{9}{5}}}^{\frac{9-7r}{9-5r}} \left\| |u|^{\frac{3}{2}} \right\|_{L^6}^{\frac{4}{3}} \\
&\leq C \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \left\| |u|^{\frac{1}{2}} \left(|u|^{\frac{1}{2}} |\nabla u| \right) \right\|_{L^{\frac{9}{5}}}^{\frac{9-7r}{9-5r}} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^{\frac{4}{3}} \\
&\leq C \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \left\| |u|^{\frac{1}{2}} \right\|_{L^{18}}^{\frac{9-7r}{9-5r}} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^{\frac{9-7r}{9-5r}} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^{\frac{4}{3}} \\
&= C \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^{\frac{9-7r}{3(9-5r)}} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^{\frac{9-7r}{9-5r}} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^{\frac{4}{3}} \\
&= C \|\nabla \pi\|_{L^r}^{\frac{2r}{9-5r}} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^{\frac{15-9r}{9-5r}} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^{\frac{9-7r}{9-5r}} \\
&\leq \left(C \|\nabla \pi\|_{L^r}^{\frac{2r}{3(r-1)}} \right)^{\frac{3(r-1)}{9-5r}} \left(\left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^2 \right)^{\frac{15-9r}{2(9-5r)}} \left(\left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2 \right)^{\frac{9-7r}{2(9-5r)}} \\
&\leq C \|\nabla \pi\|_{L^r}^{\frac{2r}{3(r-1)}} + \frac{(\mu+\varkappa)}{9} \left\| \nabla |u|^{\frac{3}{2}} \right\|_{L^2}^2 + \frac{(\mu+\varkappa)}{4} \left\| |u|^{\frac{1}{2}} |\nabla u| \right\|_{L^2}^2. \qquad (2.6)
\end{aligned}$$

Now we can insert (2.4), (2.5) and (2.6) into (2.3), it follows that

$$\begin{aligned} &\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^3+|\omega|^3)dx+\frac{(\mu+\varkappa)}{3}\int_{\mathbb{R}^3}\left|\nabla|u|^{\frac{3}{2}}\right|^2dx+\frac{4}{9}(\gamma+\kappa)\int_{\mathbb{R}^3}\left|\nabla|\omega|^{\frac{3}{2}}\right|^2dx \\ &\qquad+\frac{(\mu+\varkappa)}{2}\int_{\mathbb{R}^3}|\nabla u|^2|u|\,dx+\frac{(\gamma+\kappa)}{2}\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|\,dx \\ &\qquad+\frac{\kappa}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|\,dx+2\varkappa\int_{\mathbb{R}^3}|\omega|^3\,dx \\ \le\;& C\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3), \end{aligned}$$

which implies that

$$\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^3+|\omega|^3)dx\le C\left(\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^3+\|\omega\|_{L^3}^3}+1\right)\left(\|u\|_{L^3}^3+\|\omega\|_{L^3}^3\right).$$

Then the bounds for L^3-norms of u and ω follow from standard Gronwall's inequality. By standard arguments of continuation of local solutions, we conclude that the solutions $(u(x,t),\omega(x,t))$ can be extended beyond $t=T$.

Case 2. If $\frac{9}{7}<r<3$, we start from (2.3) and I can be estimated as

$$\begin{aligned} I &\le \int_{\mathbb{R}^3}|\nabla\pi|\,|u|^2\,dx \\ &\le \|\nabla\pi\|_{L^r}\left\||u|^2\right\|_{L^{\frac{r}{r-1}}}=\|\nabla\pi\|_{L^r}\left\||u|^{\frac{3}{2}}\right\|_{L^{\frac{4r}{3(r-1)}}}^{\frac{4}{3}} \\ &\le C\|\nabla\pi\|_{L^r}\left(\left\||u|^{\frac{3}{2}}\right\|_{L^2}^{\frac{7r-9}{4r}}\left\||u|^{\frac{3}{2}}\right\|_{L^6}^{\frac{9-3r}{4r}}\right)^{\frac{4}{3}} \\ &\le C\|\nabla\pi\|_{L^r}\|u\|_{L^3}^{\frac{7r-9}{2r}}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^2}^{\frac{3-r}{r}} \\ &= \left(C\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}\right)^{1-\frac{3-r}{2r}}\left(\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^2}^2\right)^{\frac{3-r}{2r}} \\ &\le C\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}+\frac{(\mu+\varkappa)}{9}\left\|\nabla|u|^{\frac{3}{2}}\right\|_{L^2}^2. \end{aligned} \tag{2.7}$$

Inserting (2.7), (2.5) and (2.6) into (2.3) yields

$$\begin{aligned} &\frac{1}{3}\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^3+|\omega|^3)dx+\frac{(\mu+\varkappa)}{3}\int_{\mathbb{R}^3}\left|\nabla|u|^{\frac{3}{2}}\right|^2dx+\frac{4}{9}(\gamma+\kappa)\int_{\mathbb{R}^3}\left|\nabla|\omega|^{\frac{3}{2}}\right|^2dx+2\varkappa\int_{\mathbb{R}^3}|\omega|^3\,dx \\ &\quad+\frac{3}{4}(\mu+\varkappa)\int_{\mathbb{R}^3}|\nabla u|^2|u|\,dx+\frac{(\gamma+\kappa)}{2}\int_{\mathbb{R}^3}|\nabla\omega|^2|\omega|\,dx+\frac{\kappa}{2}\int_{\mathbb{R}^3}|\nabla\times\omega|^2|\omega|\,dx \\ \le\;& C\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3) \\ \le\;& C\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}\left(\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{7r-9}{3(r-1)}}\right)+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3) \end{aligned}$$

which implies that

$$
\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^3}(|u|^3+|\omega|^3)dx\\
\leq\ & C\,\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}\left(\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{7r-9}{3(r-1)}}\right)+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3)\\
=\ & C\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}}\left(\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}\right)\left(\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{7r-9}{3(r-1)}}\right)\\
&\qquad+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3)\\
=\ & C\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}}\\
&\quad\times\left(\|u\|_{L^3}^3+\|\omega\|_{L^3}^3+\|u\|_{L^3}^{\frac{2r}{3(r-1)}}\|\omega\|_{L^3}^{\frac{7r-9}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}\|u\|_{L^3}^{\frac{7r-9}{3(r-1)}}\right)\\
&\qquad+\ C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3)\\
=\ & C\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}}\\
&\times\left(\|u\|_{L^3}^3+\|\omega\|_{L^3}^3+\left(\|u\|_{L^3}^3\right)^{\frac{2r}{9(r-1)}}\left(\|\omega\|_{L^3}^3\right)^{\frac{7r-9}{9(r-1)}}+\left(\|\omega\|_{L^3}^3\right)^{\frac{2r}{9(r-1)}}\left(\|u\|_{L^3}^3\right)^{\frac{7r-9}{9(r-1)}}\right)\\
&\qquad+\ C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3)\\
\leq\ & C\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}}\left(3\,\|u\|_{L^3}^3+3\,\|\omega\|_{L^3}^3\right)+C(\|\omega\|_{L^3}^3+\|u\|_{L^3}^3)\\
=\ & C\left(\frac{\|\nabla\pi\|_{L^r}^{\frac{2r}{3(r-1)}}}{\|u\|_{L^3}^{\frac{2r}{3(r-1)}}+\|\omega\|_{L^3}^{\frac{2r}{3(r-1)}}}+1\right)(\|u\|_{L^3}^3+\|\omega\|_{L^3}^3)
\end{aligned}
$$

which implies the deserved estimate by Gronwall's inequality. Then, by using the standard arguments of the continuation of local solutions, it is easy to conclude that the solution $(u(x,t),\omega(x,t))$ can be smoothly extended beyond T . This completes the proof of Theorem 1.1. □

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