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A CONVERSE THEOREM FOR PRACTICAL h-STABILITY OF TIME-VARYING NONLINEAR SYSTEMS

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Abstract. This paper treats the concept of practical uniform h-stability for such perturbed dynamical systems as an extension of practical uniform exponential stability. We present a converse Lyapunov theorem and we give sufficient conditions that guarantee practical uniform h-stability for a time-varying perturbed system using the Gronwall-Bellman inequality and Lyapunov's theory. Some examples are introduced to illustrate the applicability of the main results.

1. Introduction

A dynamical system is a set of objects or phenomena related to them and artificially isolated from the outside world. Its theoretical modeling requires a precise knowledge of the phenomena involved in the system and an aptitude to represent them by mathematical equations. The stability of dynamical systems is the most important criterion in systems design (see [**6, 14, 16, 20**]). The primary objective of a Lyapunov function is to analyze the behavior of trajectories of a dynamical system and how this behavior is preserved after perturbations (see [**10, 15**]). It gives sufficient conditions for stability, asymptotic stability, and so on. There are theorems which establish ([**5, 17, 22**]), at least conceptually, that for many of Lyapunov stability theorems the given conditions are indeed necessary. Such theorems are usually called converse theorems. In [**18, 19**], Pinto introduced a new notion of stability called h-stability (see [**2, 3, 13**]) with the intention of obtaining results about stability for weakly stable differential systems under some perturbations. The various notions of h-stability include several types of known stability properties as uniform stability, exponential asymptotic stability and uniform Lipschitz stability. However, in practice we may only need to stabilize a system into the region of a phase space where the system may oscillate near the state in which the implementation is still acceptable. This concept is called practical stability (see [**1, 4, 7, 12**]) which is very useful for studying the asymptotic behavior of the system in which the origin is not necessarily an equilibrium point. This work introduces a new notion of practical stability called practical h-stability (see [**11**]). The most important and useful tools for investigating stability and h-stability properties behavior of solutions of dynamical equations are Gronwall's inequalities and Lyapunov's techniques. In mathematics, Gronwall's inequalities [**8**] allow one to bound a function that is known to satisfy a certain differential or integral inequality by the solution of the

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corresponding differential or integral equation. They can be use as a technique to prove existence and uniqueness of solutions. The Lyapunov method, which states that if one can find an appropriate Lyapunov function then the system has some stability properties, attracted the attention of many researchers for studying the stability of nonlinear systems. In [**16**], the author defined a converse theorem for exponential stability by requiring a continuously differentiable Lyapunov function and the researchers in [**1**] developed it in the practical case. In addition, in [**21**], a converse theorem of Lyapunov for asymptotic stability of the averaged system is proved in which that the semi-global practical asymptotic stability is implied by the existence of a Lyapunov function whose derivative along the flow of the averaged system is negative definite. They showed that if the averaged system is globally uniformly asymptotically stable, then the origin of the original time-varying system is semi-globally practically asymptotically stable. The main result of this research is to establish a converse theorem when the nonlinear system is semi-globally practically uniformly h-stable by constructing a continuously differentiable Lyapunov function which guarantees certain properties. Then, it is used to obtain necessary and sufficient conditions to ensure the practical h-stability of dynamical systems. The novelty here is to investigate the practical approach of perturbed systems by Lyapunov's technique and the Gronwall-Bellman inequality. This paper is organized as follows. In Section 2, some definitions and notations are summarized and the system description is given. In Section 3, we study the global practical uniform h-stability of certain perturbed systems using a generalization of Gronwall's inequality. In addition, we investigate the global practical uniform h-stability of system (2.1) using Lyapunov's direct method. We state the main of this work in Section 4. Finally, some special cases and examples are provided to illustrate the obtained results in Section 5. Our conclusion is presented in Section 6.

2. Notations and Definitions

Throughout this paper, we deal with $\mathbb{R}_+ = [0, +\infty[$ and $\mathbb{R}^n$ the n-dimensional Euclidean space. Let $\|\cdot\|$ be the corresponding Euclidean norm. Let $\mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ be the space of all continuous functions from $\mathbb{R}_+ \times \mathbb{R}^n$ to $\mathbb{R}^n$. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices.

In this paper, we are interested to show necessary and sufficient conditions for practical uniform h-stability of solution for the following non-autonomous differential system

$$\dot{x}(t) = f(t, x), \qquad x(t_0) = x_0, \qquad t \geq t_0 \geq 0, \tag{2.1}$$

where $t \in \mathbb{R}_+$ is the time, $x \in \mathbb{R}^n$ is the state and $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t. Let $x(t) = x(t, t_0, x_0)$ denote by the unique solution of system (2.1) throughout $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$.

First, we start by presenting the notion of global uniform h-stability for system (2.1) which is introduced by Pinto in [**18**]. Let $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a positive, continuous and bounded function.

Definition 2.1. The system (2.1) is said to be globally uniformly h-stable if there exists $c \geq 1$, such that for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$

$$\|x(t)\| \leq \frac{c\|x_0\|h(t)}{h(t_0)}.$$

Now, we give the definition of practical h-stability which will be used in subsequent main results.

Definition 2.2. Given $\sigma > \rho > 0$, the origin of system (2.1) is said to be $(\sigma \to \rho)h$-stable if

(1) for each $r \in (0, \sigma)$ there exists a finite number $v(r) > 0$, such that for all $t_0 \in \mathbb{R}_+$

$$\|x_0\| \le r \;\Rightarrow\; \|x(t)\| \le v(r), \qquad \forall\, t \ge t_0;$$

and

(2) there exist $c \ge 1$ and $r > 0$, such that for all $t_0 \in \mathbb{R}_+$

$$\|x(t)\| \le \frac{c\|x_0\|h(t)}{h(t_0)} + \rho, \qquad \forall\, t \ge t_0, \quad \forall\, \|x_0\| \le r.$$

The system (2.1) is said to be semi-globally practically uniformly h-stable if for all numbers σ and ρ with $\infty > \sigma > \rho > 0$, the system (2.1) is $(\sigma \to \rho)h$-stable.

Definition 2.3. The origin of system (2.1) is said to be globally practically uniformly h-stable if there exist $c \ge 1$ and $\rho > 0$, such that for all $t \ge t_0$ and $x_0 \in \mathbb{R}^n$, we have

$$\|x(t)\| \le \frac{c\|x_0\|h(t)}{h(t_0)} + \rho. \tag{2.2}$$

Remark 2.4.

(1) The practical uniform exponential stability is a particular case of practical uniform h-stability, by taking $h(t) = e^{-\lambda t}$ with $\lambda > 0$.

(2) The inequality (2.2) implies that $x(t)$ will be globally uniformly bounded by a small bound $\rho > 0$, that is, $\|x(t)\|$ will be small for sufficient large t.

In this article, we will also investigate the global practical uniform h-stability of time-varying nonlinear perturbed systems of the form

$$\dot{x}(t) = B(t)x(t) + \psi(t, x), \qquad x(t_0) = x_0, \qquad t \ge t_0 \ge 0, \tag{2.3}$$

where $B(\cdot)$ is an $n \times n$ matrix whose entries are all real-valued continuous functions of $t \in \mathbb{R}_+$ and $\psi \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$ is locally Lipschitz in x, uniformly in t.
This system is seen as a perturbation of the nominal system

$$\dot{x}(t) = B(t)x(t), \qquad x(t_0) = x_0. \tag{2.4}$$

The stability behavior of the origin as an equilibrium point for the linear time-varying system (2.4) can be completely characterized in terms of the state transition matrix $R(t, t_0)$ associated to $B(\cdot)$ as follows

$$x(t) = R(t, t_0)x_0, \quad x_0 \in \mathbb{R}^n, \qquad t \ge t_0 \ge 0.$$

The following lemma presents the global uniform h-stability of system (2.4) in terms of $R(t, t_0)$.

Lemma 2.5 ([**19**]). *The system* (2.4) *is globally uniformly h-stable if and only if there exist $c \ge 1$ and a positive continuous bounded function h on $\mathbb{R}_+$, such that for all $t_0 \in \mathbb{R}_+$*

$$\|R(t, t_0)\| \le \frac{ch(t)}{h(t_0)}, \quad \forall\, t \ge t_0.$$

To study the practical approach behavior of solutions of certain perturbed systems, we need the following technical lemmas.

Lemma 2.6 (Gronwall-Bellman Inequality).
Let $\theta, \gamma : \mathbb{R}_+ \to \mathbb{R}$ be continuous functions and $\vartheta : \mathbb{R}_+ \to \mathbb{R}_+$ a function, such that

$$\dot{\vartheta}(t) \leq \theta(t)\vartheta(t) + \gamma(t), \quad \forall\, t \geq t_0. \tag{2.5}$$

Then, for all $t_0 \geq 0$, we have

$$\vartheta(t) \leq \vartheta(t_0) \exp\Big(\int_{t_0}^{t} \theta(\tau)d\tau\Big) + \int_{t_0}^{t} \exp\Big(\int_{s}^{t} \theta(\tau)d\tau\Big)\gamma(s)ds, \quad \forall\, t \geq t_0.$$

Proof. See appendix. □

Lemma 2.7 (Generalization of Gronwall's inequality).
Let θ be a non-negative function on $\mathbb{R}_+$, that satisfies the following integral inequality

$$\theta(t) \leq b + \int_{t_0}^{t} \varpi(s)\theta^{\alpha}(s)ds, \quad b \geq 0, \quad \alpha \geq 0,$$

where ϖ is a non-negative continuous function for $t \geq t_0 \geq 0$. For $0 \leq \alpha < 1$, we have

$$\theta(t) \leq \Big[b^{1-\alpha} + (1-\alpha)\int_{t_0}^{t} \varpi(s)ds\Big]^{\frac{1}{1-\alpha}}.$$

Proof. See appendix. □

Lemma 2.8. *Let $a,\ b \geq 0$ and $p \geq 1$. Then,*

$$(a+b)^p \leq 2^{p-1}(a^p + b^p).$$

3. Global Practical Uniform h-stability

In this section, we investigate the global practical uniform h-stability of such perturbed systems under different conditions on the perturbed term using the generalization of Gronwall's inequality.

Let $h : \mathbb{R}_+ \to \mathbb{R}_+^*$ be a positive, continuous and bounded function.

Theorem 3.1. *Consider the perturbed system* (2.3). *The perturbation term ψ satisfies the following condition:*

$$\|\psi(t,x)\| \leq \chi(t)\|x\|^{\alpha} + \varrho(t), \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, t \geq 0, \quad 0 \leq \alpha < 1, \tag{3.1}$$

where χ and ϱ are non-negative continuous functions on $\mathbb{R}_+$ satisfying

$$\int_0^t \frac{\varrho(s)}{h(s)}ds \leq M_1, \qquad \int_0^t \frac{\chi(s)}{h(s)}ds \leq M_2, \qquad M_1,\ M_2 > 0, \qquad \forall\, t \geq 0. \tag{3.2}$$

Suppose that the system (2.4) *is globally uniformly h-stable, then the system* (2.3) *is globally practically uniformly h-stable.*

Proof. Let $x(t)$ be the solution of system (2.3), then

$$x(t) = R(t,t_0)x_0 + \int_{t_0}^{t} R(t,s)\psi(s,x(s))ds,$$

where $R(t,t_0)$ is the state transition matrix of the linear system (2.4). Thus, from the global uniform h-stability of system (2.4) and the condition (3.1), we obtain

$$\|x(t)\| \leq \frac{c\|x_0\|h(t)}{h(t_0)} + ch(t)\int_{t_0}^{t} \frac{\chi(s)\|x(s)\|^{\alpha} + \varrho(s)}{h(s)}ds.$$

Then,

$$\frac{\|x(t)\|}{h(t)} \leq c\left(\frac{\|x_0\|}{h(t_0)} + M_1\right) + c\int_{t_0}^{t} \frac{\chi(s)\|x(s)\|^{\alpha}}{h(s)}ds.$$

Put, $\mu(t) = \dfrac{\|x(t)\|}{h(t)}$ to get

$$\mu(t) \leq c\Big(\mu(t_0) + M_1\Big) + c\int_{t_0}^{t} \frac{\chi(s)\mu(s)^{\alpha}}{h(s)^{1-\alpha}}ds.$$

By applying Lemma 2.7, we obtain for all $t \geq t_0$

$$\mu(t) \leq \left[\Big(c\mu(t_0) + cM_1\Big)^{1-\alpha} + c(1-\alpha)\|h\|_{\infty}^{\alpha}\int_{t_0}^{t} \frac{\chi(s)}{h(s)}ds\right]^{\frac{1}{1-\alpha}},$$

with $\|h\|_{\infty} = \sup_{t\geq 0}\{h(t)\}$. Hence, by using Lemma 2.8, we get

$$\mu(t) \leq 2^{\frac{\alpha}{1-\alpha}}c\mu(t_0) + 2^{\frac{\alpha}{1-\alpha}}\Big(c(1-\alpha)M_2\Big)^{\frac{1}{1-\alpha}}\|h\|_{\infty}^{\frac{\alpha}{1-\alpha}} + 2^{\frac{\alpha}{1-\alpha}}cM_1.$$

We deduce that, for all $t \geq t_0$ and all $x_0 \in \mathbb{R}^n$ the solution of system (2.3) satisfies

$$\|x(t)\| \leq \frac{b\|x_0\|h(t)}{h(t_0)} + \rho,$$

where $b = 2^{\frac{\alpha}{1-\alpha}}c$ and $\rho = 2^{\frac{\alpha}{1-\alpha}}\Big(c(1-\alpha)M_2\|h\|_{\infty}\Big)^{\frac{1}{1-\alpha}} + 2^{\frac{\alpha}{1-\alpha}}cM_1\|h\|_{\infty}$.
This finishes the proof. □

Specializing the previous theorem, when $\chi(t) = 0$ we obtain the following corollary.

Corollary 3.2. *Suppose that the system* (2.4) *is globally uniformly h-stable where h is decreasing on $\mathbb{R}_+$, then the system* (2.3) *is globally practically uniformly h-stable under the condition on the perturbation term ψ is as follows:*

$$\|\psi(t,x)\| \leq \varrho(t), \qquad \forall\, x \in \mathbb{R}^n, \qquad \forall\, t \geq 0,$$

where ϱ is a non-negative continuous integrable function on $\mathbb{R}_+$.

Next, we introduce the direct Lyapunov method in the following theorem, which allows us to determine the global practical uniform h-stability of system (2.1) by requiring the existence of Lyapunov functions that satisfy certain conditions.

Theorem 3.3. *Assume that h is a positive, bounded, continuously differentiable and decreasing function on $\mathbb{R}_+$. In addition, there exist constants numbers $c_1,\ c_2,\ p > 0,\ a, \eta \geq 0$ and $M > 0$, such that*

$$\int_0^t \frac{h(t)}{h(s)}ds \leq M, \quad \forall\, t \geq 0. \tag{3.3}$$

Suppose that there exists a function $W : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ continuously differentiable satisfying the following properties.

(1) $c_1\|x\|^p \le W(t,x) \le c_2\|x\|^p + a, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n,$
(2) $\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x) f(t,x) \le \frac{h'(t)}{h(t)} W(t,x) + \eta, \qquad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n.$

Then, the system (2.1) *is globally practically uniformly* $h^{\frac{1}{p}}$*-stable.*

Proof. Let $x(t) = x(t,t_0,x_0)$ be the solution of system (2.1) through $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. One has,

$$\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x) f(t,x) \le \frac{h'(t)}{h(t)} W(t,x) + \eta.$$

Using Lemma 2.6, we obtain for all $t \ge t_0$

$$\begin{aligned} W(t,x) &\le W(t_0,x_0) \exp\left(\int_{t_0}^t \frac{h'(s)}{h(s)} ds\right) + \eta \int_{t_0}^t \exp\left(\int_s^t \frac{h'(\tau)}{h(\tau)} d\tau\right) ds \\ &\le W(t_0,x_0) \frac{h(t)}{h(t_0)} + M\eta. \end{aligned}$$

It follows that, for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$ the solution of system (2.1) satisfies

$$\|x(t)\| \le \left(\frac{c_2}{c_1}\|x_0\|^p \frac{h(t)}{h(t_0)} + \frac{M\eta + a}{c_1}\right)^{\frac{1}{p}}.$$

We discriminate two cases:

(1) If $p > 1$, then for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$, we get

$$\|x(t)\| \le \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} \|x_0\| \left(\frac{h(t)}{h(t_0)}\right)^{\frac{1}{p}} + \left(\frac{M\eta + a}{c_1}\right)^{\frac{1}{p}}.$$

(2) If $p \le 1$, from Lemma 2.8, we obtain for all $t \ge t_0$ and all $x_0 \in \mathbb{R}^n$

$$\|x(t)\| \le 2^{\frac{1-p}{p}} \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} \|x_0\| \left(\frac{h(t)}{h(t_0)}\right)^{\frac{1}{p}} + 2^{\frac{1-p}{p}} \left(\frac{M\eta + a}{c_1}\right)^{\frac{1}{p}}.$$

The proof is completed. □

4. Converse Theorem

The purpose of this section is to establish a converse theorem in the case when the origin is not an equilibrium point of the nonlinear system (2.1), but it is assumed that there exists a non-negative constant f_0, such that $\|f(t,0)\| \le f_0$, for all $t \ge 0$. In Theorem 4.2 that follows, we show under this assumption that there exists a Lyapunov function W that satisfies conditions similar but not the same as those in Theorem 3.3. We first state the following lemma which will be used later.

Lemma 4.1 ([**9**]). *Consider the nonlinear system* (2.1). *Let* $\phi(\tau;t,x)$ *be a solution of the system that starts at* (t,x), *and let* $\phi_x(\tau;t,x) = \frac{\partial}{\partial x}\phi(\tau;t,x)$ *and suppose* $\left\|\frac{\partial f}{\partial x}(t,x)\right\| \le L$, *where* L *is a positive constant. Then,*

$$\|\phi_x(\tau;t,x)\| \le e^{L(\tau - t)}. \tag{4.1}$$

Theorem 4.2. *Consider the nonlinear system* (2.1) *with* f *is differentiable and suppose the Jacobian matrix* $\left[\frac{\partial f}{\partial x}\right]$ *is bounded on* D, *where* $D \subset \mathbb{R}^n$ *is an open connected set that contains the origin. Let* $\sigma > \rho > 0$ *be given and suppose that the trajectory of the system is* $(\sigma \to \rho)h$*-stable where* h *is a decreasing function and* h' *exists and is continuous on* $\mathbb{R}_+$. *Assume that there exists* $\delta > 0$, *such that*

$$\frac{h(t)}{h(t+\delta)} \leq \zeta, \qquad \zeta > 0, \qquad \forall\ t \geq 0. \tag{4.2}$$

Then, there exist a natural number $p \geq 2$ *and a Lyapunov function* $W : \mathbb{R}_+ \times D \to \mathbb{R}$ *continuously differentiable that satisfy the following properties.*

(*i*) $c_1\|x\|^p \leq W(t,x) \leq c_2\|x\|^p + a, \qquad (t,x) \in \mathbb{R}_+ \times D,$

(*ii*) $\left\|\frac{\partial W}{\partial x}(t,x)\right\| \leq c_3\|x\|^{p-1} + b, \qquad (t,x) \in \mathbb{R}_+ \times D,$

(*iii*) $\frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x)f(t,x) \leq \frac{h'(t)}{h(t)}W(t,x) + d, \qquad (t,x) \in \mathbb{R}_+ \times D,$

for some positive constants $c_1,\ c_2,\ c_3,\ a,\ b$ *and* d.

Proof. Let $\phi(\tau;t,x)$ denote the solution of system (2.1) that starts at (t,x), that is, $\phi(t;t,x) = x$ and let $L > 0$ denote the bound of the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$. We have,

$$\begin{aligned}
\left|\frac{d}{d\tau}\phi^T(\tau;t,x)\phi(\tau;t,x)\right| &= \left|2\phi^T(\tau;t,x)f(\tau,\phi(\tau;t,x))\right| \\
&\leq 2\|\phi(\tau;t,x)\|\|f(\tau,\phi(\tau;t,x))\| \\
&= 2\|\phi(\tau;t,x)\|\|f(\tau,\phi(\tau;t,x)) - f(\tau,0) + f(\tau,0)\| \\
&\leq 2L\|\phi(\tau;t,x)\|^2 + 2f_0\|\phi(\tau;t,x)\|,
\end{aligned}$$

which implies that

$$-2L\|\phi(\tau;t,x)\|^2 - 2f_0\|\phi(\tau;t,x)\| \leq \frac{d}{d\tau}\phi^T(\tau;t,x)\phi(\tau;t,x). \tag{4.3}$$

Let, $\varphi(\tau) = -\|\phi(\tau;t,x)\|$. By using (4.3), we obtain (as in [**16**], Example 3.9, pp. 103–104) that

$$D^+\varphi(\tau) \leq -L\varphi(\tau) + f_0,$$

such that

$$D^+\varphi(t) = \limsup_{T\to 0^+}\frac{1}{T}\Big(\varphi(t+T) - \varphi(t)\Big).$$

By applying the comparison lemma (see [**16**], pp. 102–103), we conclude that

$$\Big(\|x\| + \frac{f_0}{L}\Big)e^{-L(\tau-t)} \leq \|\phi(\tau;t,x)\| + \frac{f_0}{L}\cdot$$

Now, using the fact that $(\lambda_1+\lambda_2)^n \leq 2^n(\lambda_1^n+\lambda_2^n)$, for all $n \in \mathbb{N}^*$ and $\lambda_1,\ \lambda_2 \geq 0$, we have

$$\begin{aligned}\left[\left(\|x\|+\frac{f_0}{L}\right)e^{-L(\tau-t)}\right]^p &= \left(\|x\|+\frac{f_0}{L}\right)^p e^{-pL(\tau-t)}\\ &\leq \left(\|\phi(\tau;t,x)\|+\frac{f_0}{L}\right)^p\\ &\leq 2^p\|\phi(\tau;t,x)\|^p+2^p\left(\frac{f_0}{L}\right)^p.\end{aligned}$$

We deduce that,

$$\|\phi(\tau;t,x)\|^p+\left(\frac{f_0}{L}\right)^p \geq \frac{1}{2^p}\left(\|x\|+\frac{f_0}{L}\right)^p e^{-pL(\tau-t)}. \tag{4.4}$$

Define the function $W:\mathbb{R}_+\times D\to\mathbb{R}_+$ by

$$W(t,x)=h(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\left(\phi^T(\tau;t,x)\phi(\tau;t,x)\right)^{\frac{p}{2}}+\left(\frac{f_0}{L}\right)^p\right)d\tau,$$

where δ is a positive constant. On one side, we have

$$\begin{aligned}W(t,x) &= h(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\|\phi(\tau;t,x)\|^p+\left(\frac{f_0}{L}\right)^p\right)d\tau\\ &\leq h(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\left(\frac{c\|x\|h(\tau)}{h(t)}+\rho\right)^p+\left(\frac{f_0}{L}\right)^p\right)d\tau.\end{aligned}$$

From Lemma 2.8 and the condition (4.2), we obtain

$$W(t,x)\leq c_2\|x\|^p+a,$$

with $c_2=2^pc^p\delta$ and $a=\left(2^p\rho^p+\left(\frac{f_0}{L}\right)^p\right)\zeta$.

On the other side, using the fact (4.4), we obtain

$$\begin{aligned}W(t,x) &\geq \frac{1}{2^p}h(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\|x\|+\frac{f_0}{L}\right)^p e^{-pL(\tau-t)}d\tau\\ &\geq \frac{1}{2^p}\int_0^{\delta}e^{-pLs}ds\|x\|^p.\end{aligned}$$

Hence,

$$W(t,x)\geq c_1\|x\|^p,$$

with $c_1=\frac{1}{2^prL}(1-e^{-2rL\delta})$. This proves the property (i).

We define now the functions $\phi_t(\tau;t,x)$ and $\phi_x(\tau;t,x)$ as follows:

$$\phi_t(\tau;t,x)=\frac{\partial\phi}{\partial t}(\tau;t,x),\quad \phi_x(\tau;t,x)=\frac{\partial\phi}{\partial x}(\tau;t,x).$$

To prove (ii), let

$$\frac{\partial W}{\partial x}(t,x)=ph(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\phi^T(\tau;t,x)^{p-1}\phi_x(\tau;t,x)d\tau.$$

It follows from Lemma 4.1 that

$$\begin{aligned}\left\|\frac{\partial W}{\partial x}(t,x)\right\| &\leq ph(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\frac{c\|x\|h(\tau)}{h(t)}+\rho\right)^{p-1}e^{L(\tau-t)}d\tau \\ &\leq 2^{p-1}ph(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\left(\frac{c\|x\|h(\tau)}{h(t)}\right)^{p-1}+\rho^{p-1}\right)e^{L(\tau-t)}d\tau \\ &= \frac{2^{p-1}c^{p-1}p}{L}(e^{L\delta}-1)\|x\|^{p-1}+\frac{2^{p-1}\rho^{p-1}p\zeta}{L}(e^{L\delta}-1).\end{aligned}$$

Hence, for all $t \geq t_0$, we get

$$\left\|\frac{\partial W}{\partial x}(t,x)\right\| \leq c_3\|x\|^{p-1}+b,$$

with $c_3 = \dfrac{2^{p-1}c^{p-1}p}{L}(e^{L\delta}-1)$ and $b = \dfrac{2^{p-1}\rho^{p-1}p\zeta}{L}(e^{L\delta}-1)$. Therefore, (ii) is satisfied.

The derivative of W along the trajectories of system (2.1) is given by

$$\begin{aligned}&\frac{\partial W}{\partial t}(t,x)+\frac{\partial W}{\partial x}(t,x)f(t,x) \\ &= h'(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\|\phi(\tau;t,x)\|^p+\left(\frac{f_0}{L}\right)^p\right)d\tau \\ &\quad +\frac{h(t)}{h(t+\delta)}\|\phi^T(t+\delta;t,x)\|^p+\left(\frac{f_0}{L}\right)^p\frac{h(t)}{h(t+\delta)}-\|x\|^p-\left(\frac{f_0}{L}\right)^p \\ &\quad +ph(t)\int_t^{t+\delta}\frac{1}{h(\tau)}\left(\phi^T(\tau;t,x)\right)^{p-1}\Big[\phi_t(\tau;t,x)+\phi_x(\tau;t,x)f(t,x)\Big]d\tau.\end{aligned}$$

One has, from [**16**]

$$\phi_t(\tau;t,x)+\phi_x(\tau;t,x)f(t,x)\equiv 0, \qquad \forall\, \tau\geq t.$$

Then,

$$\begin{aligned}&\frac{\partial W}{\partial t}(t,x)+\frac{\partial W}{\partial x}(t,x)f(t,x) \\ &\quad\leq \frac{h'(t)}{h(t)}W(t,x)+\left(2^pc^p\left(\frac{h(t+\delta)}{h(t)}\right)^{p-1}\|x\|^p+\left(2^p\rho^p+\left(\frac{f_0}{L}\right)^p\right)\zeta.\right.\end{aligned}$$

For each $r\in(0,\sigma)$, there exists a finite number $v(r)>0$, such that $\|x_0\|\leq r$ implies that $x(t)$ is defined and $\|x(t)\|\leq v(r)$. We deduce that,

$$\frac{\partial W}{\partial t}(t,x)+\frac{\partial W}{\partial x}(t,x)f(t,x)\leq\frac{h'(t)}{h(t)}W(t,x)+d,$$

with $d=(2^pc^p-1)v(r)^p+\left(2^p\rho^p+\left(\dfrac{f_0}{L}\right)^p\right)\zeta$, which proves (iii). The proof is completed. □

Now, we apply the converse theorem to perturbed systems. We shall be interested in the relation between the solution of the unperturbed system (2.1) and the solution of the perturbed system

$$\dot{x}=f(t,x)+\psi(t,x), \qquad x(t_0)=x_0, \qquad t\geq t_0\geq 0, \tag{4.5}$$

where $f,\ \psi \in \mathcal{C}(\mathbb{R}_+ \times D, D)$ are locally Lipschitz in x, uniformly in t. Precisely, we give sufficient conditions to study that if the nominal system (2.1) is $(\sigma \to \rho)h$-stable then the perturbed system (4.5) is semi-globally practically uniformly $h^{\frac{1}{p}}$-stable. Before proposing our theorem, we introduce the following assumption:
$(\mathcal{A})$ The perturbation term ψ satisfies:

$$\|\psi(t,x)\| \le \chi(t)\|x\| + \varrho(t), \qquad \forall\, x \in D, \qquad \forall\, t \in \mathbb{R}_+,$$

where χ and ϱ are non-negative continuous integrable functions on $\mathbb{R}_+$.

Theorem 4.3. *Assume that assumption $(\mathcal{A})$ holds. We consider the perturbed system (4.5), where the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded on D, where $D \subset \mathbb{R}^n$ is an open connected set that contains the origin. Let $\sigma > \rho > 0$ be given and suppose that the system (2.1) is $(\sigma \to \rho)h$-stable where h is a decreasing function and h' exists and is continuous on $\mathbb{R}_+$, the condition (4.2) is satisfied and there exists a positive constant M, such that*

$$\int_0^t \frac{h(t)}{h(s)} ds \le M, \qquad \forall\, t \ge 0. \tag{4.6}$$

Then, the system (4.5) is semi-globally practically uniformly $h^{\frac{1}{p}}$-stable, where $p \ge 2$ is a natural number.

Proof. By Theorem 4.2, there exist functions W and h satisfying these three properties $(i) - (iii)$. The derivative of W along the trajectories of system (4.5) satisfies

$$\begin{aligned}
& \frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x)\Big(f(t,x) + \psi(t,x)\Big) \\
\le\ & \frac{h'(t)}{h(t)} W(t,x) + d + \left\|\frac{\partial W}{\partial x}(t,x)\right\| \|\psi(t,x)\| \\
\le\ & \Big[\frac{h'(t)}{h(t)} + \frac{c_3}{c_1}\chi(t)\Big] W(t,x) + c_3\varrho(t)\|x\|^{p-1} + b\chi(t)\|x\| + b\varrho(t) + d.
\end{aligned}$$

For each $r \in (0,\sigma)$, there exists a finite number $v(r) > 0$, such that $\|x_0\| \le r$ implies that $x(t)$ is defined and $\|x(t)\| \le v(r)$. Then,

$$\begin{aligned}
& \frac{\partial W}{\partial t}(t,x) + \frac{\partial W}{\partial x}(t,x)\Big(f(t,x) + \psi(t,x)\Big) \\
\le\ & \Big[\frac{h'(t)}{h(t)} + \frac{c_3}{c_1}\chi(t)\Big] W(t,x) + c_3\varrho(t)v(r)^{p-1} + b\chi(t)v(r) + b\varrho(t) + d \\
=\ & \Big[\frac{h'(t)}{h(t)} + \frac{c_3}{c_1}\chi(t)\Big] W(t,x) + \big(A+b\big)\varrho(t) + B\chi(t) + d,
\end{aligned}$$

where $A = c_3 v(r)^{p-1}$ and $B = b + v(r)$. Hence, by Lemma 2.6, we get for all $t \ge t_0$

$$\begin{aligned}
W(t,x) \ \le\ & W(t_0,x_0)\frac{h(t)}{h(t_0)} \exp\left(\frac{c_3}{c_1}\int_{t_0}^t \chi(s)ds\right) + \int_{t_0}^t \frac{h(t)}{h(s)} \exp\left(\frac{c_3}{c_1}\int_s^t \chi(\tau)d\tau\right) \\
& \times \Big(\big(A+b\big)\varrho(s) + B\chi(s) + d\Big) ds \\
\le\ & W(t_0,x_0)\frac{h(t)}{h(t_0)} e^{\frac{c_3}{c_1}M_1} + e^{\frac{c_3}{c_1}M_1}\big(A+b\big)M_2 + e^{\frac{c_3}{c_1}M_1} BM_1 + e^{\frac{c_3}{c_1}M_1} dM,
\end{aligned}$$

with $M_1 = \int_0^\infty \chi(s)ds$ and $M_2 = \int_0^\infty \varrho(s)ds$. Therefore, for all $t \geq t_0$ and all $x_0 \in D$ the solution of system (2.3) satisfies the inequality

$$\begin{aligned} \|x(t)\| \quad \leq \quad & \left(\frac{c_2}{c_1}\right)^{\frac{1}{p}} e^{\frac{c_3}{pc_1}M_1}\|x_0\| \left(\frac{h(t)}{h(t_0)}\right)^{\frac{1}{p}} + e^{\frac{c_3}{pc_1}M_1}(A+b)^{\frac{1}{p}}\left(\frac{M_2}{c_1}\right)^{\frac{1}{p}} \\ & + e^{\frac{c_3}{pc_1}M_1}\left(\frac{BM_1}{c_1}\right)^{\frac{1}{p}} + e^{\frac{c_3}{pc_1}M_1}\left(\frac{dM+a_1}{c_1}\right)^{\frac{1}{p}}, \end{aligned}$$

which implies that the system (4.5) is semi-globally practically uniformly $h^{\frac{1}{p}}$-stable. This ends the proof. □

5. Examples

The purpose of this section is to illustrate the main results by given some numerical examples and simulations.

Example 5.1. Consider the following system:

$$\begin{cases} \dot{x}_1 = -2tx_1 + \gamma x_2 + \dfrac{e^{-t^2}}{(1+t^2)^2}(x_1^2+x_2^2)^{\frac{1}{8}} + \dfrac{e^{-t^2}}{(1+t^2)\sqrt{1+x_1^2}} \\ \dot{x}_2 = -\gamma x_1 - 2tx_2 + \dfrac{\sqrt{2}te^{-t^2}}{(1+t^2)^2}(x_1^2+x_2^2)^{\frac{1}{8}}, \end{cases} \tag{5.1}$$

where $x = (x_1, x_2)^T \in \mathbb{R}^2$, $\gamma > 0$ and $t \in \mathbb{R}_+$. The above mentioned example is exactly the system (2.3), with

$$B(t) = \begin{pmatrix} -2t & \gamma \\ -\gamma & -2t \end{pmatrix}, \qquad \psi(t,x) = \begin{pmatrix} \dfrac{e^{-t^2}}{(1+t^2)^2}(x_1^2+x_2^2)^{\frac{1}{8}} + \dfrac{e^{-t^2}}{(1+t^2)\sqrt{1+x_1^2}} \\ \dfrac{\sqrt{2}te^{-t^2}}{(1+t^2)^2}(x_1^2+x_2^2)^{\frac{1}{8}} \end{pmatrix}.$$

The state transition matrix of the linear system is given by

$$R(t,t_0) = e^{-(t^2-t_0^2)}\varphi(t-t_0), \qquad \forall\, t \geq 0,$$

with

$$\varphi(t) = \begin{pmatrix} \cos\gamma t & \sin\gamma t \\ -\sin\gamma t & \cos\gamma t \end{pmatrix}.$$

Therefore, the linear system $\dot{x} = B(t)x$ is globally uniformly h-stable with $c = 1$ and $h(t) = e^{-t^2}$ is a positive continuous bounded function on $\mathbb{R}_+$.
On the other side,

$$\|\psi(t,x)\| \leq \frac{2e^{-t^2}}{(1+t^2)^{\frac{3}{2}}}\|x\|^{\frac{1}{4}} + \frac{\sqrt{2}e^{-t^2}}{1+t^2}, \qquad \forall\, x \in \mathbb{R}^2, \qquad \forall\, t \in \mathbb{R}_+.$$

Put, $\chi(t) = \dfrac{2e^{-t^2}}{(1+t^2)^{\frac{3}{2}}}$ and $\varrho(t) = \dfrac{\sqrt{2}e^{-t^2}}{1+t^2}$, which are non-negative continuous functions on $\mathbb{R}_+$ with the conditions (3.1) and (3.2) are satisfied. Then, all hypotheses of Theorem 3.1 are fulfilled and the perturbed system (5.1) is globally practically uniformly h-stable with $\alpha = \dfrac{1}{4}\cdot$

The trajectory of system (5.1) with respect the initial state $\Big(x_1(0), x_2(0)\Big) = (1,1)$ and $\gamma = 1$, is depicted in Figure 1.

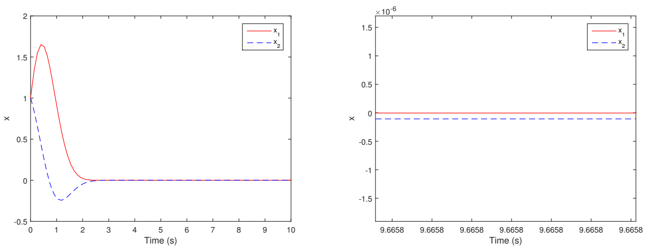

FIGURE 1. *The trajectory of the state* $x(t) = (x_1, x_2)$ *of system* (5.1).

Example 5.2. We consider the scalar equation:

$$\dot{x} = -\frac{2+t}{5(1+t)}x + \frac{\arctan(t)e^{-t}}{1+x^4}, \qquad x \in \mathbb{R}, \qquad t \geq 0. \tag{5.2}$$

Put, $h(t) = \dfrac{e^{-t}}{1+t}$ is a positive, continuous, bounded and decreasing function on $\mathbb{R}_+$ which satisfies (3.3). Let, $W(t,x) = x^5$, that is continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}$. Therefore, all hypotheses of Theorem 3.3 are fulfilled with $c_1 = c_2 = 1,\ a = 0,\ \eta = \dfrac{5\pi}{2}$ and $p = 5$. Consequently, the system (5.2) is globally practically uniformly $h^{\frac{1}{5}}$-stable.
For simulation of system (5.2) we select the initial state $x(0) = 2$. The result is depicted in Figure 2.

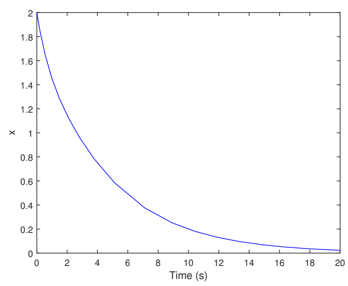

FIGURE 2. *The trajectory of the state* $x(t)$ *of system* (5.2).

Example 5.3. We consider the scalar system

$$\dot{x} = -\frac{2+t}{1+t}x + \frac{e^{-t}}{1+|x|} + \frac{te^{-2t}}{1+x^2}x + \frac{1}{1+t^2}, \qquad x \in \mathbb{R}, \qquad t \geq 0. \tag{5.3}$$

The above example is exactly the system (4.5) with

$$f(t,x) = -\frac{2+t}{1+t}x + \frac{e^{-t}}{1+|x|}, \qquad \psi(t,x) = \frac{te^{-2t}}{1+x^2}x + \frac{1}{1+t^2},$$

where f is differentiable and the Jacobian matrix $\left[\frac{\partial f}{\partial x}\right]$ is bounded on $\mathbb{R}$. The nominal system $\dot{x} = f(t,x)$ is globally practically uniformly h-stable with $c = 1$ and $h(t) = \frac{e^{-t}}{1+t}$ is a positive continuous bounded function on $\mathbb{R}_+$ and verifies the conditions (4.2) and (4.6). Moreover, it is easy to check the assumption $(\mathcal{A})$ with $\chi(t) = te^{-2t}$ and $\varrho(t) = \frac{1}{1+t^2}$ that are non-negative continuous integrable functions on $\mathbb{R}_+$. Consequently, from Theorem 4.3 we conclude the semi-global practical uniform h-stability of system (5.3).
For simulation of system (5.3) we select the initial state $x(0) = 0$. The result is depicted in Figure 3.

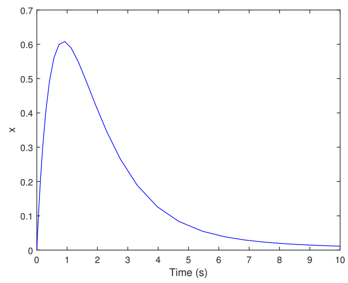

FIGURE 3. *The trajectory of the state $x(t)$ of system* (5.3).

6. Conclusion

We have investigated some new conditions for global practical uniform h-stability of perturbed differential equations. A converse Lyapunov theorem for semi-global practical uniform h-stability of nonlinear time-varying systems has been established. We have illustrated this use in the practical h-stability of perturbed systems. Finally, some illustrative examples were given to demonstrate the validity of the results.

Appendix

Proof of Lemma 2.6. We write (2.5) as

$$\dot{\vartheta}(t) - \theta(t)\vartheta(t) \leq \gamma(t), \qquad \forall\, t \geq t_0.$$

One has,

$$\frac{d}{ds}\left(\vartheta(s)\exp\left(-\int_{t_0}^{s}\theta(\tau)d\tau\right)\right) \leq \gamma(s)\exp\left(-\int_{t_0}^{s}\theta(\tau)d\tau\right), \quad s \geq t_0.$$

Then, for all $t_0 \in \mathbb{R}_+$

$$\int_{t_0}^{t}\frac{d}{ds}\left(\vartheta(s)\exp\left(-\int_{t_0}^{s}\theta(\tau)d\tau\right)\right) \leq \int_{t_0}^{t}\gamma(s)\exp\left(-\int_{t_0}^{s}\theta(\tau)d\tau\right)ds, \qquad \forall\, t \geq t_0,$$

and it follows that

$$\vartheta(t)\exp\left(-\int_{t_0}^{t}\theta(\tau)d\tau\right) - \vartheta(t_0) \leq \int_{t_0}^{t}\gamma(s)\exp\left(-\int_{t_0}^{s}\theta(\tau)d\tau\right)ds.$$

Therefore,

$$\vartheta(t) \leq \vartheta(t_0)\exp\left(\int_{t_0}^{t}\theta(\tau)d\tau\right) + \int_{t_0}^{t}\gamma(s)\exp\left(\int_{s}^{t}\theta(\tau)d\tau\right)ds, \qquad \forall\, t \geq t_0.$$

□

Proof of Lemma 2.7. Let,

$$\varphi(t) = \int_{t_0}^{t}\varpi(s)x^{\alpha}(s)ds, \qquad 0 \leq \alpha < 1, \qquad t \geq t_0.$$

Then,

$$\dot{\varphi}(t) \leq \varpi(t)\Big(b + \varphi(t)\Big)^{\alpha}, \qquad \forall\, t \geq t_0.$$

Therefore, for all $t \geq t_0$

$$x(t) \leq b + \varphi(t) \leq \left(b^{1-\alpha} + (1-\alpha)\int_{t_0}^{t}\varpi(s)ds\right)^{\frac{1}{1-\alpha}}.$$

□

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