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EVALUATION OF CONVOLUTION SUMS ENTAILING MIXED DIVISOR FUNCTIONS FOR A CLASS OF LEVELS

Ebénézer Ntienjem

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Abstract. Let $0 < n, \alpha, \beta \in \mathbb{N}$ be such that $\gcd(\alpha, \beta) = 1$. We carry out the evaluation of the convolution sums $\sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \alpha k+\beta l=n}} \sigma(k)\sigma_3(l)$ and $\sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \alpha k+\beta l=n}} \sigma_3(k)\sigma(l)$ for all levels $\alpha\beta \in \mathbb{N}$, by using in particular modular forms. We next apply convolution sums belonging to this class of levels to determine formulae for the number of representations of a positive integer n by the quadratic forms in twelve variables $\sum_{i=1}^{12} x_i^2$ when the level $\alpha\beta \equiv 0 \pmod 4$, and $\sum_{i=1}^{6} (\, x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)$ when the level $\alpha\beta \equiv 0 \pmod 3$. Our approach is then illustrated by explicitly evaluating the convolution sum for $\alpha\beta = 3$, 6, 7, 8, 9, 12, 14, 15, 16, 18, 20, 21, 27, 32. These convolution sums are then applied to determine explicit formulae for the number of representations of a positive integer n by quadratic forms in twelve variables.

1. Introduction

Convolution functions are defined in many areas in natural science. Some of these areas are signal processing and mathematics. In a nutshell, a convolution function in mathematical context is the definition of a third function in term of two, not necessary different, functions. Therefore, a convolution sum is a special case of a convolution function.

In number theory, the problem began with the evaluation of the following summation:

$$\sigma_r(1)\sigma_s(n-1) + \sigma_r(2)\sigma_s(n-2) + \cdots + \sigma_r(n-2)\sigma_s(2) + \sigma_r(n-1)\sigma_s(1), \quad (1.1)$$

wherein $0 < n, r, s, k \in \mathbb{N}$ and $\sigma_k(n)$ denotes the sum of the k^{th} powers of the positive divisors of n, see J. Liouville [**14**]. Hence, research on convolution sums and the number of representation of a natural number by quadratic forms can be traced back to Liouville [**14**] as noted by K. S. Williams [**27**, chap. 12]. Remark that Liouville's pseudonym was Besge or Besgue. It is worth mentioning that Liouville did state the results on convolution sums and did provide the proofs by examples. Results on convolution sums (1.1) were also later obtained by Glaisher [**4**], Lahiri

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[**11, 12**] and Ramanujan [**23**] for the pairs (r,s) such that r and s are odd positive integers and $r+s<14$.

We denote in the following by $\mathbb{N}$ the set of natural numbers, $\mathbb{N}^*$ the set of natural numbers without zero, i.e., $\mathbb{N}\setminus\{0\}$, $\mathbb{Z}$ the set of integers, $\mathbb{Q}$ the set of rational numbers, $\mathbb{R}$ the set of real numbers. and $\mathbb{C}$ the set of complex numbers.

Suppose that $n\in\mathbb{N}$ and $k\in\mathbb{N}$. Then formally define the sum $\sigma_k(n)$ of the k^{th} powers of the positive divisors of n by

$$\sigma_k(n)=\sum_{0<\delta|n}\delta^k. \tag{1.2}$$

It is obvious from the definition that $\sigma_k(m)=0$ for all $m\notin\mathbb{N}$. We write $d(n)$ and $\sigma(n)$ as a shorthand for $\sigma_0(n)$ and $\sigma_1(n)$, respectively.

Assume that the positive integers $\alpha\le\beta$ are given and that $i,j\in\mathbb{N}^*$. Then we now formally define the convolution sums as follows

$$W_{(\alpha,\beta)}^{2i-1,2j-1}(n)=\sum_{\substack{(k,l)\in\mathbb{N}^2\\ \alpha\,k+\beta\,l=n}}\sigma_{2i-1}(k)\sigma_{2j-1}(l). \tag{1.3}$$

We set $W_{(\alpha,\beta)}^{2i-1,2j-1}(n)=0$ if for all $(k,l)\in\mathbb{N}^2$ it holds that $\alpha\,k+\beta\,l\neq n$.

The convolution sum as defined by (1.3) above is obviously a generalization of that defined by (1.1). Trivially one obtains (1.1) from (1.3) when one sets $(\alpha,\beta)=(1,1)$. In 1997, G. Melfi [**16, 17**] considered the generalized definition by specializing it to the case $(\alpha,\beta)=(2,1)$ and $i+j\le 3$. Approximately ten years later, in 2007, E. Royer [**24**] proposed a method to evaluate convolution sums involving the divisor functions (1.3), which used quasi-modular forms. Nowadays, research on the evaluation of convolutions sums tends to utilize the approach based on modular forms and theta functions, especially when the weight is greater than 4.

In this paper, we evaluate the convolution sums

$$W_{(\alpha,\beta)}^{2i-1,2j-1}(n),\quad W_{(\alpha,\beta)}^{2j-1,2i-1}(n),\quad W_{(\beta,\alpha)}^{2i-1,2j-1}(n)\quad\text{and}\quad W_{(\beta,\alpha)}^{2j-1,2i-1}(n)$$

for the level $\alpha\beta\in\mathbb{N}$ whenever $i,j\in\mathbb{N}^*$ are such that $i+j=3$. We let

$$\mathfrak{N}=\{\,2^{\nu}\mho\,|\,\nu\in\{0,1,2,3\}\text{ and }\mho\text{ is a finite product of distinct odd primes}\,\}.$$

Then these convolution sums are evaluated for the levels $\alpha\beta\in\mathfrak{N}$ and $\alpha\beta\in\mathbb{N}^*\setminus\mathfrak{N}$.

So far known convolution sums whose evaluation involved mixed divisor functions $\sigma_{2i-1}(n)$ and $\sigma_{2j-1}(n)$, where $i,j\in\mathbb{N}^*$ are such that $i+j=3$, are displayed in the following table.

Level $\alpha\beta$	$\mathbf{i+j}$	**Authors**	**References**
1	$1+2, 2+1$	J. Liouville, D. B. Lahiri, S. Ramanujan	[**14, 11, 12, 23**]
2	$1+2$	Ş. Alaca & F. Uygul & K. S. Williams, J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams, N. Cheng & K. S. Williams	[**2, 3, 5**]
2	$2+1$	N. Cheng & K. S. Williams, J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams	[**3, 5**]

3	$1+2$	O. X. M. Yao & E. X. W. Xia	[**28**]
4	$1+2, 2+1$	N. Cheng & K. S. Williams	[**3**]
6	$1+2$	B. Köklüce	[**9**]
12	$1+2$	B. Köklüce & H. Eser	[**10**]

Table 1: Known convolution sums $W_{(\alpha,\beta)}^{2i-1,2j-1}(n)$ of level $\alpha\beta$

The evaluation of these convolution sums, especially for a class of levels is new. We then apply the result for this class of levels to determine the convolution sums for

- $\alpha\beta = 7, 8, 9, 14, 15, 16, 18, 20, 21, 27, 32$ when $(i,j) = (1,2)$; and
- $\alpha\beta = 3, 6, 7, 8, 9, 12, 14, 15, 16, 18, 20, 21, 27, 32$ when $(i,j) = (2,1)$.

Again, these explicit convolution sums have not been evaluated as yet.

These convolution sums are applied to establish explicit formulae for the number of representations of a positive integer n by the quadratic forms in twelve variables

$$a\sum_{i=1}^{4} x_i^2 + b\sum_{i=5}^{12} x_i^2, \tag{1.4}$$

$$a_1\sum_{i=1}^{8} x_i^2 + b_1\sum_{i=9}^{12} x_i^2, \tag{1.5}$$

$$c\sum_{j=1}^{2}(\,x_{2j-1}^2 \;+\; x_{2j-1}x_{2j} \;+\; x_{2j}^2\,) \;+\; d\sum_{j=3}^{6}(\,x_{2j-1}^2 \;+\; x_{2j-1}x_{2j} \;+\; x_{2j}^2\,) \tag{1.6}$$

and

$$c_1\sum_{j=1}^{4}(\,x_{2j-1}^2 \;+\; x_{2j-1}x_{2j} \;+\; x_{2j}^2\,) \;+\; d_1\sum_{j=5}^{6}(\,x_{2j-1}^2 \;+\; x_{2j-1}x_{2j} \;+\; x_{2j}^2\,) \tag{1.7}$$

where $(a,b), (a_1,b_1), (c,d), (c_1,d_1) \in \mathbb{N}^* \times \mathbb{N}^*$.

Known formulae for the number of representations of a positive integer n by the quadratic forms

- (1.4) and (1.5) are given by B. Köklüce & H. Eser [**10**] and by A. Alaca & Ş. Alaca & Z. Selcuk Aygin [**1**] when $(a,b) = (a_1,b_1) = (1,3)$.
- (1.6) are given by O. X. M. Yao & E. X. W. Xia [**28**] when $(c,d) = (1,1)$.

Based on this structure of the level $\alpha\beta$, we provide a method to determine all pairs $(a,b) \in \mathbb{N}^* \times \mathbb{N}^*$ that are necessary for the determination of the formulae for the number of representations of a positive integer by the quadratic forms (1.4). Then we determine explicit formulae for the number of representations of a positive integer n by the quadratic forms (1.4)– (1.7) whenever $\alpha\beta$ has the above form and is such that $\alpha\beta \equiv 0 \pmod 4$ or $\alpha\beta \equiv 0 \pmod 3$.

We have obtained the results displayed in this paper by using Software for symbolic scientific computation which is composed of the open source software packages *GiNaC*, *Maxima*, *REDUCE*, *SAGE* and the commercial software package *MAPLE*.

2. Essential Background Knowledge

2.1. Modular forms. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Im}(z) > 0\}$, be the upper half-plane and let $\Gamma = G = \mathrm{SL}_2(\mathbb{R}) = \{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mid a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \}$ be the group of 2×2-matrices. Let $N \in \mathbb{N}^*$. Then

$$\Gamma(N) \quad = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \pmod N \right\}$$

is a subgroup of G and is called the *principal congruence subgroup of level N*. A subgroup H of G is called a *congruence subgroup of level N* if it contains $\Gamma(N)$.

For our purposes the following congruence subgroup is relevant:

$$\Gamma_0(N) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \mid c \equiv 0 \pmod N \right\}.$$

Let $k, N \in \mathbb{N}$ and let $\Gamma' \subseteq \Gamma$ be a congruence subgroup of level N. Let $k \in \mathbb{Z}, \gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $f : \mathbb{H} \cup \mathbb{Q} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$. We denote by $f^{[\gamma]_k}$ the function whose value at z is $(cz + d)^{-k} f(\gamma(z))$, i.e., $f^{[\gamma]_k}(z) = (cz + d)^{-k} f(\gamma(z))$. We use the definition of modular forms according to N. Koblitz [**7**, p. 108].

Let us denote by $M_{2k}(\Gamma')$ the set of modular forms of weight $2k$ for Γ', by $S_{2k}(\Gamma')$ the set of cusp forms of weight $2k$ for Γ' and by $E_{2k}(\Gamma')$ the set of Eisenstein forms. The sets $M_{2k}(\Gamma')$, $S_{2k}(\Gamma')$ and $E_{2k}(\Gamma')$ are vector spaces over $\mathbb{C}$. Therefore, $M_{2k}(\Gamma_0(N))$ is the space of modular forms of weight $2k$ for $\Gamma_0(N)$, $S_{2k}(\Gamma_0(N))$ is the space of cusp forms of weight $2k$ for $\Gamma_0(N)$, and $E_{2k}(\Gamma_0(N))$ is the space of Eisenstein forms. The decomposition of the space of modular forms as a direct sum of the space generated by the Eisenstein series and the space of cusp forms, i.e., $M_{2k}(\Gamma_0(N)) = E_{2k}(\Gamma_0(N)) \oplus S_{2k}(\Gamma_0(N))$, is well-known; see for example W. A. Stein 's book (on line version) [**25**, p. 81].

We assume in this paper that $k \in \mathbb{N}^*$ and that χ and ψ are primitive Dirichlet characters with conductors L and R, respectively. W. A. Stein [**25**, p. 86] has noted that

$$E_{2k,\chi,\psi}(q) = C_0 + \sum_{n=1}^{\infty} \Big(\sum_{d|n} \psi(d)\chi(\frac{n}{d})\, d^{2k-1} \Big) q^n, \tag{2.1}$$

where

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{2k,\chi}}{4k} & \text{if } L = 1 \end{cases}$$

and $B_{2k,\chi}$ are the generalized Bernoulli numbers. Theorems 5.8 and 5.9 in Section 5.3 of W. A. Stein [**25**, p. 86] are then applicable.

If the primitive Dirichlet characters χ and ψ are trivial, then their conductors L and R are one, respectively. Therefore, (2.1) can be normalized and then given as follows: $E_{2k}(q) = 1 - \frac{4k}{B_{2k}} \sum\limits_{n=1}^{\infty} \sigma_{2k-1}(n)\, q^n$. This will be the case whenever the level $\alpha\beta$ belongs to $\mathfrak{N}$.

Let $k, m, n \in \mathbb{N}^*$ be such that m is a positive divisor of n. We then use T. Miyake [**18**, Lemma 2.1.3, p. 41] to conclude that

$$M_{2k}(\Gamma_0(m)) \subset M_{2k}(\Gamma_0(n)). \tag{2.2}$$

It also follows that the same inclusion relation holds for the bases, the space of Eisenstein forms of weight $2k$ and the spaces of cusp forms of weight $2k$.

2.2. Eta quotients. On the upper half-plane $\mathbb{H}$, the Dedekind η-function, $\eta(z)$, is defined by $\eta(z) = e^{\frac{2\pi i z}{24}} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})$. Let us set $q = e^{2\pi i z}$. Then it follows that

$$\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) = q^{\frac{1}{24}} F(q), \quad \text{where } F(q) = \prod_{n=1}^{\infty} (1 - q^n).$$

Let $j, N \in \mathbb{N}^*$ and $e_j \in \mathbb{Z}$. Following G. Köhler[**8**, p. 31] an *eta product* or *eta quotient*, $f(z)$, is a finite product of eta functions, which is of the form $\prod_{j|N} \eta(jz)^{e_j}$, wherein N is the *level* of the eta product. If $2k = \frac{1}{2} \sum_{j=1}^{\kappa} e_j$, then an eta quotient $f(z)$ behaves like a modular form of weight $2k$ on $\Gamma_0(N)$ with some multiplier system.

Note that eta function, eta quotient and eta product will use interchangeably as synonyms.

L. J. P. Kilford [**6**, p. 99] and G. Köhler [**8**, Cor. 2.3, p. 37] have given a formulation of the following theorem which is a result of the work of M. Newman [**19, 20**] and G. Ligozat [**13**]. This theorem will be effectively used to exhaustively determine η-quotients, $f(z)$, which belong to $M_{2k}(\Gamma_0(N))$, and especially those η-quotients which are in $S_{2k}(\Gamma_0(N))$.

Theorem 2.1 (M. Newman and G. Ligozat)**.** *Let $N \in \mathbb{N}^*$, $D(N)$ be the set of all positive divisors of N, $\delta \in D(N)$ and $r_\delta \in \mathbb{Z}$. Let furthermore $f(z) = \prod_{\delta \in D(N)} \eta^{r_\delta}(\delta z)$ be an eta quotient. If the following four conditions are satisfied*

(i) $\sum_{\delta \in D(N)} \delta\, r_\delta \equiv 0 \pmod{24}$,

(ii) $\sum_{\delta \in D(N)} \frac{N}{\delta}\, r_\delta \equiv 0 \pmod{24}$,

(iii) $\prod_{\delta \in D(N)} \delta^{r_\delta}$ *is a square in* $\mathbb{Q}$,

(iv) $\forall d \in D(N)$ *it holds that* $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta}\, r_\delta \geq 0$,

then $f(z) \in M_{2k}(\Gamma_0(N))$, where $2k = \frac{1}{2} \sum_{\delta \in D(N)} r_\delta$. Moreover, the η-quotient $f(z)$ is an element of $S_{2k}(\Gamma_0(N))$ if (iv) is replaced by (iv'): $\forall d \in D(N)$ it holds that $\sum_{\delta \in D(N)} \frac{\gcd(\delta, d)^2}{\delta} r_\delta > 0$.

2.3. Convolution sums $\mathbf{W}^{\mathbf{2i-1,2j-1}}_{(\boldsymbol{\alpha},\boldsymbol{\beta})}(\mathbf{n})$. Given $i, j, \alpha, \beta \in \mathbb{N}^*$ such that $i + j = 3$, and $\alpha \leq \beta$, let the convolution sums be defined by (1.3). Suppose that $\gcd(\alpha, \beta) = \delta > 1$ for some $\delta \in \mathbb{N}^*$. Then there exist $\alpha_1, \beta_1 \in \mathbb{N}^*$ such that $\gcd(\alpha_1, \beta_1) =$

$1,\ \alpha = \delta\,\alpha_1$ and $\beta = \delta\,\beta_1$. Hence,

$$\begin{aligned} W_{(\alpha,\beta)}^{2i-1,2j-1}(n) &= \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \alpha\,k+\beta\,l=n}} \sigma_{2i-1}(k)\sigma_{2j-1}(l) \\ &= \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ \delta\,\alpha_1\,k+\delta\,\beta_1\,l=n}} \sigma_{2i-1}(k)\sigma_{2j-1}(l) \\ &= W_{(\alpha_1,\beta_1)}^{2i-1,2j-1}(\frac{n}{\delta}). \end{aligned} \tag{2.3}$$

Therefore, we may simply assume that $\gcd(\alpha,\beta) = 1$. As an immediate consequence of the definition, we note that

- $W_{(\alpha,\beta)}^{2i-1,2j-1}(n) = W_{(\beta,\alpha)}^{2i-1,2j-1}(n)$ whenever $i = j$ holds; and
- $W_{(\alpha,\beta)}^{2i-1,2j-1}(n) = W_{(\beta,\alpha)}^{2j-1,2i-1}(n)$ whenever $\alpha = \beta$ holds. By (2.3) it is sufficient to consider only the case $\alpha = \beta = 1$. We then apply the commutativity of the multiplication, namely, $\sigma(k)\sigma_3(n-k) = \sigma_3(n-k)\sigma(k)$, to yield

$$W_{(1,1)}^{1,3}(n) = \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k+l=n}} \sigma(k)\sigma_3(l) = \sum_{k=0}^{n}\sigma(k)\sigma_3(n-k) = \sum_{\substack{(k,l)\in\mathbb{N}^2 \\ k+l=n}} \sigma_3(k)\sigma(l) = W_{(1,1)}^{3,1}(n). \tag{2.4}$$

We assume that the primitive Dirichlet characters χ and ψ

(1) are trivial whenever $\alpha\beta \in \mathfrak{N}$ holds.
(2) are such that $\chi = \psi$ and that χ is a Legendre-Jacobi-Kronecker symbol otherwise.

Then the following Eisenstein series hold:

$$E_2(q) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n, \tag{2.5}$$

$$E_4(q) = 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n, \tag{2.6}$$

$$E_6(q) = 1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \tag{2.7}$$

$$\begin{aligned} E_{2k,\chi}(q^\lambda) &= E_{2k}(q^\lambda) \otimes \chi(\lambda) \\ &= \chi(\lambda)\left(C_0 + \sum_{n=1}^{\infty}\chi(n)\,\sigma_{2k-1}(n)\,q^{\lambda\,n}\right) \\ &= \chi(\lambda)\,C_0 + \sum_{n=1}^{\infty}\chi(\lambda\,n)\,\sigma_{2k-1}(n)\,q^{\lambda\,n}, \\ &= \chi(\lambda)\,C_0 + \sum_{n=1}^{\infty}\chi(n)\,\sigma_{2k-1}(\frac{n}{\lambda})\,q^{n}, \end{aligned} \tag{2.8}$$

where $\lambda \in \mathbb{N}^*$,

$$C_0 = \begin{cases} 0 & \text{if } L > 1 \\ -\frac{B_{2k,\chi}}{4k} & \text{if } L = 1 \end{cases}$$

and $B_{2k,\chi}$ are the specially generalized Bernoulli numbers.

Note that $E_4(q)$ and $E_6(q)$ are special cases of (2.1) or (2.8) and hold if $\alpha\beta \in \mathfrak{N}$. We state two relevant results for the sequel of this work.

Lemma 2.2. *Let $\alpha, \beta \in \mathbb{N}^*$. Then*

$$(\,\alpha\, E_2(q^\alpha) - E_2(q)\,)\, E_4(q^\beta) \in M_6(\Gamma_0(\alpha\beta))$$

and

$$E_4(q^\alpha)\,(\,E_2(q) - \beta\, E_2(q^\beta)\,) \in M_6(\Gamma_0(\alpha\beta)).$$

Proof. We will only prove the first part since the second part can be shown similarly.

If $\alpha > 0$, then trivially $0 = (\,\alpha\, E_k(q^\alpha) - \alpha\, E_k(q^\alpha)\,) \in M_k(\Gamma_0(\alpha))$ and there is nothing to prove. Therefore, we may suppose that $\alpha \neq 1$ in the sequel. We apply the result proved by W. A. Stein [**25**, Thrms 5.8, 5.9, p. 86] to deduce that $(\,\alpha\, E_2(q^\alpha) - E_2(q)\,) \in M_2(\Gamma_0(\alpha)) \subseteq M_2(\Gamma_0(\alpha\beta)) \subseteq M_6(\Gamma_0(\alpha\beta))$ and $E_4(q^\beta) \in M_4(\Gamma_0(\beta)) \subseteq M_4(\Gamma_0(\alpha\beta))$. Therefore, we obtain $(\,\alpha\, E_2(q^\alpha) - E_2(q)\,)\, E_4(q^\beta) \in M_6(\Gamma_0(\alpha\beta))$. □

The following identity is proven for $\alpha = 1$ by D. B. Lahiri[**11**, p. 149]. For all $\alpha \in \mathbb{N}^*$, it holds that

$$E_2(q^\alpha)\, E_4(q^\alpha) = 1 + 720 \sum_{n\geq 1} \frac{n}{\alpha}\, \sigma_3(\frac{n}{\alpha})\, q^n - 504 \sum_{n\geq 1} \sigma_5(\frac{n}{\alpha})\, q^n. \tag{2.9}$$

The following identity is shown by S. Ramanujan[**23**, Table IV, p. 168]. For all $n \in \mathbb{N}^*$, it holds that

$$W^{1,3}_{(1,1)}(n) = \frac{21}{240}\, \sigma_5(n) + \frac{1}{24}\,(1 - 3n)\, \sigma_3(n) - \frac{1}{240}\, \sigma(n). \tag{2.10}$$

We then apply (2.3) and (2.4) to deduce that for all $\alpha, n \in \mathbb{N}^*$

$$W^{3,1}_{(\alpha,\alpha)}(n) = W^{1,3}_{(\alpha,\alpha)}(n) = \frac{21}{240}\, \sigma_5(\frac{n}{\alpha}) + \frac{1}{24}\,(1 - 3\,\frac{n}{\alpha})\, \sigma_3(\frac{n}{\alpha}) - \frac{1}{240}\, \sigma(\frac{n}{\alpha}). \tag{2.11}$$

When we make use of the just proven identity, we obviously infer that for all $\alpha \in \mathbb{N}^*$

$$E_4(q^\alpha)\, E_2(q^\alpha) = E_2(q^\alpha)\, E_4(q^\alpha) = 1 + 720 \sum_{n\geq 1} \frac{n}{\alpha}\, \sigma_3(\frac{n}{\alpha})\, q^n - 504 \sum_{n\geq 1} \sigma_5(\frac{n}{\alpha})\, q^n. \tag{2.12}$$

Theorem 2.3. *Let $\alpha, \beta \in \mathbb{N}^*$ be such that* $\gcd(\alpha, \beta) = 1$. *Then*

$$\begin{aligned}(\,\alpha\, E_2(q^\alpha) - E_2(q)\,)\, E_4(q^\beta) = \alpha - 1 + \sum_{n=1}^{\infty} \Bigg(24\, \sigma(n) - 24\, \alpha\, \sigma(\frac{n}{\alpha}) \\ + 240\, \alpha\, \sigma_3(\frac{n}{\beta}) - 240\, \sigma_3(\frac{n}{\beta}) + 24 \times 240\, W^{1,3}_{(1,\beta)}(n) - 24 \times 240\, \alpha\, W^{1,3}_{(\alpha,\beta)}(n) \Bigg) q^n\end{aligned} \tag{2.13}$$

and

$$E_4(q^\alpha)\,(\,E_2(q)-\beta\,E_2(q^\beta)\,)=1-\beta+\sum_{n=1}^{\infty}\Big(\,24\,\beta\,\sigma(\frac{n}{\beta})-24\,\sigma(n)$$
$$+\,240\,\sigma_3(\frac{n}{\alpha})-240\,\beta\,\sigma_3(\frac{n}{\alpha})-24\times 240\,W^{3,1}_{(\alpha,1)}(n)+24\times 240\,\beta\,W^{3,1}_{(\alpha,\beta)}(n)\,\Big)q^n. \tag{2.14}$$

Proof. We observe that

$$(\,\alpha\,E_2(q^\alpha)-E_2(q)\,)\,E_4(q^\beta)=\alpha\,E_2(q^\alpha)\,E_4(q^\beta)-E_2(q)\,E_4(q^\beta) \tag{2.15}$$

and

$$E_4(q^\alpha)\,(\,E_2(q)-\beta\,E_2(q^\beta)\,)=E_4(q^\alpha)\,E_2(q)-\beta\,E_4(q^\alpha)\,E_2(q^\beta). \tag{2.16}$$

For the sake of simplicity, we only prove (2.14) since the other can be proven similarly.

$$E_4(q^\alpha)\,E_2(q)=1-24\sum_{n=1}^{\infty}\sigma(n)\,q^n+240\sum_{n=1}^{\infty}\sigma_3(\frac{n}{\alpha})\,q^n-24\times 240\sum_{n=1}^{\infty}W^{3,1}_{(\alpha,1)}(n)\,q^n \tag{2.17}$$

and

$$\beta\,E_4(q^\alpha)\,E_2(q^\beta)=\beta-24\,\beta\sum_{n=1}^{\infty}\sigma(\frac{n}{\beta})\,q^n$$
$$+\,240\,\beta\sum_{n=1}^{\infty}\sigma_3(\frac{n}{\alpha})\,q^n-24\times 240\,\beta\sum_{n=1}^{\infty}W^{3,1}_{(\alpha,\beta)}(n)\,q^n. \tag{2.18}$$

We then put (2.17) and (2.18) together to obtain

$$E_4(q^\alpha)\,(\,E_2(q)-\beta\,E_2(q^\beta)\,)=1-\beta+\sum_{n=1}^{\infty}\Big(\,24\,\beta\,\sigma(\frac{n}{\beta})-24\,\sigma(n)$$
$$+\,240\,\sigma_3(\frac{n}{\alpha})-240\,\beta\,\sigma_3(\frac{n}{\alpha})-24\times 240\,W^{3,1}_{(\alpha,1)}(n)+24\times 240\,\beta\,W^{3,1}_{(\alpha,\beta)}(n)\,\Big)q^n$$

as asserted. □

The following corollary discusses special cases of Theorem 2.3, which are essential for the determination of $W^{2i-1,2j-1}_{(\alpha,\beta)}(n)$.

Corollary 2.4. *Let* $\alpha,\beta\in\mathbb{N}^*$ *be such that* $\gcd(\alpha,\beta)=1$.
If $\alpha\neq 1$ **and** $\beta=1$**, then:**

$$(\,\alpha\,E_2(q^\alpha)-E_2(q)\,)\,E_4(q)=\alpha-1+\sum_{n=1}^{\infty}\Big(\,-24\,\alpha\,\sigma(\frac{n}{\alpha})+240\,\alpha\,\sigma_3(n)$$
$$-\,720\,n\,\sigma_3(n)+504\,\sigma_5(n)-24\times 240\,\alpha\,W^{1,3}_{(\alpha,1)}(n)\,\Big)q^n. \tag{2.19}$$

If $\beta \neq 1$ and $\alpha = 1$, then:

$$E_4(q)\,(\,E_2(q) - \beta\,E_2(q^{\beta})\,) = 1 - \beta + \sum_{n=1}^{\infty}\Big(\,24\,\beta\,\sigma(\frac{n}{\beta}) - 240\,\beta\,\sigma_3(n) + 720\,n\,\sigma_3(n) - 504\,\sigma_5(n) + 24\times 240\,\beta\,W^{3,1}_{(1,\beta)}(n)\,\Big)q^n. \quad (2.20)$$

If $\beta = \alpha \neq 1$, then:

$$(\,\beta\,E_2(q^{\beta}) - E_2(q)\,)\,E_4(q^{\beta}) = \beta - 1 + \sum_{n=1}^{\infty}\Big(\,24\,\sigma(n) - 240\,\sigma_3(\frac{n}{\beta}) + 720\,n\,\sigma_3(\frac{n}{\beta}) - 504\,\beta\,\sigma_5(\frac{n}{\beta}) + 24\times 240\,W^{1,3}_{(1,\beta)}(n)\,\Big)q^n \quad (2.21)$$

and

$$E_4(q^{\alpha})\,(\,E_2(q) - \alpha\,E_2(q^{\alpha})\,) = 1 - \alpha + \sum_{n=1}^{\infty}\Big(\,-24\,\sigma(n) + 240\,\sigma_3(\frac{n}{\alpha}) - 720\,n\,\sigma_3(\frac{n}{\alpha}) + 504\,\alpha\,\sigma_5(\frac{n}{\alpha}) - 24\times 240\,W^{3,1}_{(\alpha,1)}(n)\,\Big)q^n. \quad (2.22)$$

Proof. We only prove the following cases as the others can be proved similarly. By Theorem 2.3

if $\alpha \neq 1$ and $\beta = 1$: then

$$(\,\alpha\,E_2(q^{\alpha}) - E_2(q)\,)\,E_4(q) = \alpha - 1 + \sum_{n=1}^{\infty}\Big(\,-24\,\alpha\,\sigma(\frac{n}{\alpha}) + 240\,\alpha\,\sigma_3(n) - 720\,n\,\sigma_3(n) + 504\,\sigma_5(n) - 24\times 240\,\alpha\,W^{1,3}_{(\alpha,1)}(n)\,\Big)q^n$$

when we also make use of either (2.9 or (2.11).

if $\beta = \alpha \neq 1$: then

$$E_4(q^{\alpha})\,(\,E_2(q) - \alpha\,E_2(q^{\alpha})\,) = 1 - \alpha + \sum_{n=1}^{\infty}\Big(\,-24\,\sigma(n) + 240\,\sigma_3(\frac{n}{\alpha}) - 720\,n\,\sigma_3(\frac{n}{\alpha}) + 504\,\alpha\,\sigma_5(\frac{n}{\alpha}) - 24\times 240\,W^{3,1}_{(\alpha,1)}(n)\,\Big)q^n$$

when we additionally apply (2.11).

□

From (2.19), (2.20), (2.21), and (2.22) it immediately follows that

Lemma 2.5. *Let $1 < \alpha \in \mathbb{N}^*$. Then*

$$(\,\alpha\,E_2(q^{\alpha}) - E_2(q)\,)\,E_4(q) = -\,E_4(q)\,(\,E_2(q) - \alpha\,E_2(q^{\alpha})\,)$$

and

$$(\,\alpha\,E_2(q^{\alpha}) - E_2(q)\,)\,E_4(q^{\alpha}) = -\,E_4(q^{\alpha})\,(\,E_2(q) - \alpha\,E_2(q^{\alpha})\,).$$

3. Evaluating $W^{1,3}_{(\alpha,\beta)}(n)$ and $W^{3,1}_{(\alpha,\beta)}(n)$, where $\alpha\beta \in \mathbb{N}^*$

We carry out an explicit formula for the convolution sums $W^{1,3}_{(\alpha,\beta)}(n)$ and $W^{3,1}_{(\alpha,\beta)}(n)$ wherein $\alpha\beta \in \mathbb{N}^*$.

3.1. Bases of $E_6(\boldsymbol{\Gamma}_0(\boldsymbol{\alpha\beta}))$ and $S_6(\boldsymbol{\Gamma}_0(\boldsymbol{\alpha\beta}))$. Let $\mathcal{D}(\alpha\beta)$ denote the set of all positive divisors of $\alpha\beta$.

In terms of theta series, A. Pizer [**22**] has discussed the existence of a basis of the space of cusp forms of weight $2k \in \mathbb{N}^*$ for $\Gamma_0(\alpha\beta)$ when $\alpha\beta$ is not a perfect square. We suppose in the sequel that the weight $2k \in \mathbb{N}^*$ and we then apply the dimension formulae in T. Miyake's book [**18**, Thrm 2.5.2, p. 60] or W. A. Stein's book [**25**, Prop. 6.1, p. 91] to conclude that

- for the space of Eisenstein forms,
$$\dim(E_{2k}(\Gamma_0(\alpha\beta))) = \sum_{d|\alpha\beta} \varphi(\gcd(d, \frac{\alpha\beta}{d})) = m_E, \tag{3.1}$$
where $m_E \in \mathbb{N}^*$ and φ is the Euler's totient function.
We observe that, if $\alpha\beta \in \mathfrak{N}$, then
$$\dim(E_{2k}(\Gamma_0(\alpha\beta))) = \sum_{\delta|\alpha\beta} \varphi(\gcd(\delta, \frac{\alpha\beta}{\delta})) = \sum_{\delta|\alpha\beta} 1 = \sigma_0(\alpha\beta) = d(\alpha\beta). \tag{3.2}$$
- for the space of cusp forms, $\dim(S_{2k}(\Gamma_0(\alpha\beta))) = m_S$, where $m_S \in \mathbb{N}$.

We use Theorem 2.1 $(i) - (iv')$ to exhaustively determine as many elements of the space $S_{2k}(\Gamma_0(\alpha\beta))$ as possible. From these elements of the space $S_{2k}(\Gamma_0(\alpha\beta))$ we select relevant ones for the purpose of the determination of a basis of this space. The proof of the following theorem provides a method to effectively determine such a basis.

The so-determined basis of the vector space of cusp forms is in general not unique. However, due to the change of basis which is an automorphism, it is sufficient to only consider this basis for our purpose.

Let $\mathcal{C}$ denote the set of primitive Dirichlet characters $\chi(n) = \left(\frac{m}{n}\right)$ as assumed in (2.8), where $m, n \in \mathbb{Z}$ and $\left(\frac{m}{n}\right)$ is the Legendre-Jacobi-Kronecker symbol. Let $D_\chi(\alpha\beta)$ denote the subset of $\mathcal{D}(\alpha\beta)$ associated with the character χ.

Let i, κ be natural numbers. The expression of a natural number in the form $\prod\limits_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime and e_i is in $\mathbb{N}^*$, modulo a permutation of the primes p_i, is standard. In the following we will use this form to express a level $\alpha\beta \in \mathbb{N}^* \setminus \mathfrak{N}$.

We observe that the selection of an applicable primitive Dirichlet character is not obvious. Let us assume that the levels are $2\cdot 3^2$ and 2^4. Then the primitive Dirichlet characters $\left(\frac{-3}{n}\right)$ and $\left(\frac{-4}{n}\right)$ will not be good candidates for the determination of a basis of $E_{2k}(\Gamma_0(2\cdot 3^2))$ and $E_{2k}(\Gamma_0(2^4))$, respectively.

Definition 3.1. Let $i, \kappa \in \mathbb{N}^*$ and $n \in \mathbb{N}$. Let $C \in \mathbb{Z}$ be fixed. Suppose that the level $\alpha\beta \in \mathbb{N}^* \setminus \mathfrak{N}$ is fixed and of the form $\prod\limits_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}^*$. We say that the primitive Dirichlet character $\chi(n) = \left(\frac{C}{n}\right)$ *annihilates* $E_6(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_6(\Gamma_0(\alpha\beta))$ if for some $1 \leq j \leq \kappa$ we have $1 < p_j^{e_j} \in \mathbb{N}^* \setminus \mathfrak{N}$ and $M_\chi(q^\delta)$ vanishes for all $1 < \delta$ positive divisor of $p_j^{e_j}$.

A set $\mathcal{C}$ of primitive Dirichlet characters *annihilates* $E_6(\Gamma_0(\alpha\beta))$ or is an *annihilator* of $E_6(\Gamma_0(\alpha\beta))$ if each $\chi(n) \in \mathcal{C}$ is an annihilator of $E_6(\Gamma_0(\alpha\beta))$.

To illustrate the above definition, suppose that $\alpha\beta = 36$ and the primitive Dirichlet characters is $\chi(n) = \left(\frac{-3}{n}\right)$. Then $C = -3$ so that $|C|$is a positive divisor of $9 = 3^2$. Since it holds that

$$\chi(n) = \left(\frac{-3}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 2 \pmod 3, \\ 0 & \text{if } \gcd(3, n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 3 \end{cases}$$

and

$$\sigma_3(\frac{n}{\delta}) = \begin{cases} 0 & \text{if } \frac{n}{\delta} \notin \mathbb{N}_0, \\ \text{non-zero} & \gcd(3, n) \neq 1, \end{cases}$$

for all $1 < \delta \in \mathcal{D}(9)$, we apply (2.8) to deduce that

$$E_{6,\chi}(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\, \sigma_5(\frac{n}{\delta})\, q^n = 0.$$

Therefore, the primitive Dirichlet characters $\chi(n) = \left(\frac{-3}{n}\right)$ is an annihilator of $E_6(\Gamma_0(36))$.

The following theorem provides a strong criteria when selecting a primitive Dirichlet character for a given level $\alpha\beta \in \mathbb{N}^* \setminus \mathfrak{N}$.

Theorem 3.2. *Let i, κ be in $\mathbb{N}^*$. Let $C \in \mathbb{Z}$ be fixed. Let χ be a primitive Dirichlet character with conductor $|C| > 1$ and let the level $\alpha\beta \in \mathbb{N}^* \setminus \mathfrak{N}$ be fixed and of the form $\prod\limits_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}^*$. Suppose furthermore that $p_j^{e_j} \in \mathbb{N}^* \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$ for some $1 \leq j \leq \kappa$. If the conductor $|C|$ is a positive divisor of p_j and hence of the level $\alpha\beta$, then $\chi(n) = \left(\frac{C}{n}\right)$, for all $n \in \mathbb{N}$, is an annihilator of $E_6(\Gamma_0(\alpha\beta))$.*

Proof. Suppose that $\alpha\beta \in \mathbb{N}^* \setminus \mathfrak{N}$ is fixed and of the form $\prod\limits_{i=1}^{\kappa} p_i^{e_i}$, where p_i is a prime number and e_i is in $\mathbb{N}^*$. As an immediate consequence of the structure of $\alpha\beta$ there exists $1 \leq j \leq \kappa$ such that $p_j^{e_j} \in \mathbb{N}^* \setminus \mathfrak{N}$ is a positive divisor of $\alpha\beta$. Since the conductor $|C|$ is a positive divisor of the level p_j, the existence of $1 < \delta \in \mathcal{D}(p_j^{e_j})$ is given. It is well-known that for each $1 \leq f \leq e_j$ it holds that p_j^f is a positive divisor of $p_j^{e_j}$. On the other hand, it holds that

$$\left(\frac{C}{n}\right) = \begin{cases} 0 & \text{if } \gcd(|C|, n) \neq 1, \\ \text{non-zero} & \text{otherwise.} \end{cases} \tag{3.3}$$

For each $1 < \delta \in \mathcal{D}(p_j^{e_j})$ it holds that $\gcd(|C|, \delta) \neq 1$. Since the conductor of χ is greater than one, we have $C_0 = 0$. Now we apply (2.8) to infer that

$$E_{6,\chi}(q^\delta) = \sum_{n=1}^{\infty} \chi(n)\, \sigma_5(\frac{n}{\delta})\, q^n.$$

Since it also holds that

$$\sigma_5(\frac{n}{\delta}) = \begin{cases} 0 & \text{if } \frac{n}{\delta} \notin \mathbb{N}^*, \\ \text{non-zero} & \text{otherwise,} \end{cases} \tag{3.4}$$

we obtain the stated result by simply putting altogether; that is $E_{6,\chi}(q^\delta) = 0$ for all $1 < \delta \in \mathcal{D}(p_j^{e_j})$. $\square$

If $\alpha\beta \in \mathfrak{N}$ holds, then the primitive Dirichlet characters are trivial. Therefore, the set $\mathcal{C}$ is empty. Hence, the case where $\alpha\beta \in \mathfrak{N}$ holds is a special case of the following theorem.

Theorem 3.3. *The following statements holds:*

(a) Let $\mathcal{C}$ be a set of primitive Dirichlet characters which is not an annihilator of $E_6(\Gamma_0(\alpha\beta))$. Then the set $\mathcal{B}_E = \{ E_6(q^t) \mid t \in \mathcal{D}(\alpha\beta) \} \cup \bigcup_{\chi \in \mathcal{C}} \{ E_{6,\chi}(q^t) \mid t \in D_\chi(\alpha\beta) \}$ is a basis of $E_6(\Gamma_0(\alpha\beta))$.

(b) Let $1 \leq i \leq m_S$ be positive integers, $\delta \in D(\alpha\beta)$ and $(r(i,\delta))_{i,\delta}$ be a table of the powers of $\eta(\delta z)$. Let furthermore $\mathfrak{B}_{\alpha\beta,i}(q) = \prod_{\delta \mid \alpha\beta} \eta^{r(i,\delta)}(\delta z)$ be selected elements of $S_6(\Gamma_0(\alpha\beta))$. Then the set $\mathcal{B}_S = \{ \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_S \}$ is a basis of $S_6(\Gamma_0(\alpha\beta))$.

(c) The set $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ constitutes a basis of $M_6(\Gamma_0(\alpha\beta))$.

Remark 3.4. (r1) Each $\mathfrak{B}_{\alpha\beta,i}(q)$ can be expressed in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{\alpha\beta,i}(n)q^n$, where $1 \leq i \leq m_S$ and for each $n \geq 1$ the coefficient $\mathfrak{b}_{\alpha\beta,i}(n)$ is an integer.

(r2) If we divide the sum that results from Theorem 2.1 (iv'), when $d = N$, by 24, then we obtain the smallest positive degree of q in $\mathfrak{B}_{\alpha\beta,i}(q)$.

Since the existence of a basis of the space of cusp forms in terms of theta series has been proved for all square-free levels by A. Pizer [**22**] and since M. Newman and G. Ligozat, see Theorem 2.1 $(i) - (iv')$, have determined a method to find as many theta series belonging to the space of cusp forms as possible, the proof of this theorem is essentially restricted to show that the selected elements of the space of modular forms of the given level are linearly independent.

Proof. (a) W. A. Stein [**25**, Thrms 5.8 and 5.9, p. 86] has shown that for each t positive divisor of $\alpha\beta$ it holds that $E_6(q^t)$ is in $M_6(\Gamma_0(t))$.

Let ι be the absolute difference between $\dim(E_6(\Gamma_0(\alpha\beta)))$ and the cardinality of $D(\alpha\beta)$.

If $\iota = 0$, then $\alpha\beta$ is in $\mathfrak{N}$ and hence $\mathcal{C} = \emptyset$ and $\mathcal{B}_E = \{ E_6(q^t) \mid t \in \mathcal{D}(\alpha\beta) \}$ is linearly independent. Since the dimension of $E_6(\Gamma_0(\alpha\beta))$ is finite, it follows that it is a basis of $E_6(\Gamma_0(\alpha\beta))$.

Suppose now that $\iota > 0$. Since $E_6(\Gamma_0(t))$ is a vector space and the set $\mathcal{C}$ of primitive Dirichlet characters does not annihilate $E_6(\Gamma_0(\alpha\beta))$, it then holds for each Legendre-Jacobi-Kronecker symbol $\chi \in \mathcal{C}$ and $t \in D_\chi(\alpha\beta)$ that $0 \neq E_{6,\chi}(q^t)$ is in $E_6(\Gamma_0(t))$. Since the dimension of $E_6(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that $\mathcal{B}_E$ is linearly independent. Suppose that for each $\chi \in \mathcal{C}, s \in$

$D_\chi(\alpha\beta)$ we have $z(\chi)_s \in \mathbb{C}$ and that for each $t|\alpha\beta$ we have $x_t \in \mathbb{C}$. Then

$$\sum_{t|\alpha\beta} x_t\, E_6(q^t) + \sum_{\chi\in\mathcal{C}}\Bigg(\sum_{s\in D_\chi(\alpha\beta)} z(\chi)_s E_{6,\chi}(q^s)\Bigg) = 0.$$

We recall that χ is a Legendre-Jacobi-Kronecker symbol; therefore, for all $0 \neq a \in \mathbb{Z}$ it holds that $\left(\frac{a}{0}\right) = 0$. Since the primitive Dirichlet characters χ is not trivial and has a conductor L which we may assume greater than one, we can deduce that $C_0 = 0$ in (2.8). Then we equate the coefficients of q^n for $n \in D(\alpha\beta) \cup \bigcup_{\chi\in\mathcal{C}} \{\, s | s \in D_\chi(\alpha\beta)\,\}$ to obtain the homogeneous system of linear equations in m_E unknowns:

$$\sum_{u|\alpha\beta} \sigma_5(\frac{t}{u}) x_u + \sum_{\chi\in\mathcal{C}} \sum_{v\in D_\chi(\alpha\beta)} \chi(t)\sigma_5(\frac{t}{v}) Z(\chi)_v = 0, \qquad t \in D(\alpha\beta).$$

The determinant of the matrix of this homogeneous system of linear equations is not zero. Hence, the unique solution is $x_t = z(\chi)_s = 0$ for all $t \in D(\alpha\beta)$ and for all $\chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$. So, the set $\mathcal{B}_E$ is linearly independent and hence is a basis of $E_6(\Gamma_0(\alpha\beta))$.

(b) We show that each $\mathfrak{B}_{\alpha\beta,i}(q)$, where $1 \leq i \leq m_S$, is in the space $S_6(\Gamma_0(\alpha\beta))$. This is obviously the case since $\mathfrak{B}_{\alpha\beta,i}(q), 1 \leq i \leq m_S$, are obtained using an exhaustive search which applies items (i)–(iv′) in Theorem 2.1.
Since the dimension of $S_6(\Gamma_0(\alpha\beta))$ is finite, it suffices to show that the set $\mathcal{B}_S$ is linearly independent. Suppose that $x_i \in \mathbb{C}$ and $\sum_{i=1}^{m_S} x_i\, \mathfrak{B}_{\alpha\beta,i}(q) = 0$. Then $\sum_{i=1}^{m_S} x_i\, \mathfrak{B}_{\alpha\beta,i}(q) = \sum_{n=1}^{\infty} (\sum_{i=1}^{m_S} x_i\, \mathfrak{b}_{\alpha\beta,i}(n)\,)q^n = 0$ which gives the homogeneous system of m_S linear equations in m_S unknowns:

$$\sum_{i=1}^{m_S} \mathfrak{b}_{\alpha\beta,i}(n)\, x_i = 0, \qquad 1 \leq n \leq m_S. \tag{3.5}$$

Two cases arise:

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ is i for each $1 \leq i \leq m_S$: Then the square matrix which corresponds to this homogeneous system of m_S linear equations is triangular with 1's on the diagonal. Hence, the determinant of that matrix is 1 and so the unique solution is $x_i = 0$ for all $1 \leq i \leq m_S$.

The smallest degree of $\mathfrak{B}_{\alpha\beta,i}(q)$ is i for $1 \leq i < m_S$: Let n' be the largest positive integer such that $1 \leq i \leq n' < m_S$. Let $\mathcal{B}'_S = \{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq n'\,\}$ and $\mathcal{B}''_S = \{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid n' < i \leq m_S\,\}$. Then $\mathcal{B}_S = \mathcal{B}'_S \cup \mathcal{B}''_S$ and we may consider $\mathcal{B}_S$ as an ordered set. By the case above, the set $\mathcal{B}'_S$ is linearly independent. Hence, the linear independence of the set $\mathcal{B}_S$ depends on that of the set $\mathcal{B}''_S$. Let $A = (\mathfrak{b}_{\alpha\beta,i}(n))$ be the $m_S \times m_S$ matrix in (3.5). If $\det(A) \neq 0$, then $x_i = 0$ for all $1 \leq i \leq m_S$ and we are done. Suppose that $\det(A) = 0$. Then for some $n' < l \leq m_S$ there exists $\mathfrak{B}_{\alpha\beta,l}(q)$ which is causing the system of equations to be inconsistent. We substitute $\mathfrak{B}_{\alpha\beta,l}(q)$ with, say $\mathfrak{B}'_{\alpha\beta,l}(q)$, which does not occur in $\mathcal{B}_S$ and compute the determinant of the new matrix A. Since there are finitely many $\mathfrak{B}_{\alpha\beta,l}(q)$ with $n' < l \leq m_S$ that may cause the system of linear equations to be inconsistent and finitely many elements of $S_6(\Gamma_0(\alpha\beta)) \setminus \mathcal{B}_S$, the procedure will terminate with a consistent system of linear equations. Hence, we will find a set of elements of $S_6(\Gamma_0(\alpha\beta))$ which is linearly independent.

Therefore, the set $\{\, \mathfrak{B}_{\alpha\beta,i}(q) \mid 1 \leq i \leq m_S \,\}$ is linearly independent and hence is a basis of $S_6(\Gamma_0(\alpha\beta))$.

(c) Since $M_6(\Gamma_0(\alpha\beta)) = E_6(\Gamma_0(\alpha\beta)) \oplus S_6(\Gamma_0(\alpha\beta))$, the result follows from (a) and (b).

□

If the level $\alpha\beta$ belongs to the class $\mathfrak{N}$, then Theorem 3.3 (a) is provable by induction on the set of positive divisors of $\alpha\beta$; see for example E. Ntienjem [**21**]. Note that each positive divisor of $\alpha\beta$ is in the class $\mathfrak{N}$ whenever the level $\alpha\beta$ belongs to the class $\mathfrak{N}$. This nice property does not hold in general if the level $\alpha\beta$ belongs to the class $\mathbb{N} \setminus \mathfrak{N}$. For example 18 is an element of the class $\mathbb{N} \setminus \mathfrak{N}$; however, 6 which is a positive divisor of 18 does not belong to $\mathbb{N} \setminus \mathfrak{N}$.

The proof of Theorem 3.3(b) provides us with an effective method to determine a basis of the space of cusp forms of weight $0 < 2k \in \mathbb{N}$ and of level $\alpha\beta$ whenever $\alpha\beta$ belongs to $\mathbb{N}^*$.

3.2. Evaluating the convolution sums $\mathbf{W}^{1,3}_{(\alpha,\beta)}(\mathbf{n})$ and $\mathbf{W}^{3,1}_{(\alpha,\beta)}(\mathbf{n})$. We recall that it is sufficient to assume that the primitive Dirichlet character χ is not trivial since the case χ trivial can be concluded as an immediate corollary.

Lemma 3.5. *Let $\alpha, \beta \in \mathbb{N}$ be such that $gcd(\alpha, \beta) = 1$. Let furthermore $\mathcal{B}_M = \mathcal{B}_E \cup \mathcal{B}_S$ be a basis of $M_6(\Gamma_0(\alpha\beta))$. Then there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}, where 1 \leq j \leq m_S, \chi \in \mathcal{C}, s \in D_\chi(\alpha\beta)$ and $\delta|\alpha\beta$, such that*

$$(\,\alpha\, E_2(q^\alpha) - E_2(q)\,)\, E_4(q^\beta) = \sum_{\delta|\alpha\beta} X_\delta + \sum_{n=1}^{\infty}\Big(-504 \sum_{\delta|\alpha\beta} \sigma_5(\frac{n}{\delta}) X_\delta + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} \sigma_5(\frac{n}{s})\, Z(\chi)_s + \sum_{j=1}^{m_S} \mathfrak{b}_{\alpha\beta,j}(n)\, Y_j \Big)\, q^n. \quad (3.6)$$

and

$$E_4(q^\alpha)\,(\, E_2(q) - \beta\, E_2(q^\beta)\,) = \sum_{\delta|\alpha\beta} X_\delta + \sum_{n=1}^{\infty}\Big(-504 \sum_{\delta|\alpha\beta} \sigma_5(\frac{n}{\delta}) X_\delta + \sum_{\chi\in\mathcal{C}} \sum_{s\in D_\chi(\alpha\beta)} \sigma_5(\frac{n}{s})\, Z(\chi)_s + \sum_{j=1}^{m_S} \mathfrak{b}_{\alpha\beta,j}(n)\, Y_j \Big)\, q^n. \quad (3.7)$$

These also hold for the special cases $\beta = 1 \wedge \alpha \neq 1$, $\alpha = 1 \wedge \beta \neq 1$ and $\beta = \alpha \neq 1$.

Proof. For the sake of simplicity, we only give the proof for (3.7). The other cases can be proven in a similar way.

That $E_4(q^\alpha)\,(\, E_2(q) - \beta\, E_2(q^\beta)\,) \in M_6(\Gamma_0(\alpha\beta))$ follows from Lemma 2.2. Hence, by Theorem 3.3 (c), there exist $X_\delta, Z(\chi)_s, Y_j \in \mathbb{C}, 1 \leq j \leq m_S, \chi \in \mathcal{C}, s \in$

$D(\chi)$ and δ is a divisor of $\alpha\beta$, such that

$$E_4(q^\alpha)\,(\,E_2(q)-\beta\,E_2(q^\beta)\,)=\sum_{\delta|\alpha\beta}X_\delta\,E_6(q^\delta)+\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}Z(\chi)_s\,E_{6,\chi}(q^s)$$
$$+\sum_{j=1}^{m_S}Y_j\,\mathfrak{B}_{\alpha\beta,j}(q)=\sum_{\delta|\alpha\beta}X_\delta+\sum_{n=1}^{\infty}\Bigg(-504\sum_{\delta|\alpha\beta}\sigma_5(\frac{n}{\delta})\,X_\delta$$
$$+\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)}\chi(n)\,\sigma_5(\frac{n}{s})\,Z(\chi)_s+\sum_{j=1}^{m_S}\mathfrak{b}_{\alpha\beta,j}(n)\,Y_j\Bigg)\,q^n.$$

We now consider the special cases $\beta=1\wedge\alpha\neq 1$, $\alpha=1\wedge\beta\neq 1$ and $\beta=\alpha\neq 1$. We give the proof for just one case. We equate the right hand side of (3.7) with that of (2.22) to obtain

$$-504\sum_{\delta|\alpha\beta}X_\delta\,\sigma_5(\frac{n}{\delta})+\sum_{\chi\in\mathcal{C}}\Bigg(\sum_{s\in D_\chi(\alpha\beta)}\chi(n)\,\sigma_5(\frac{n}{s})\,Z(\chi)_s\Bigg)+\sum_{j=1}^{m_S}Y_j\,\mathfrak{b}_{\alpha\beta,j}(n)$$
$$=24\,\sigma(n)-240\,\sigma_3(\frac{n}{\beta})$$
$$+720\,n\,\sigma_3(\frac{n}{\beta})-504\,\beta\,\sigma_5(\frac{n}{\beta})+24\times 240\,W^{1,3}_{(1,\beta)}(n).$$

We then take the coefficients of q^n such that n is in $D(\alpha\beta)$ and $1\leq n\leq m_S$, but as many as the unknown, $X_1,\ldots,X_{\alpha\beta}$, $Z(\chi)_s$ for all $\chi\in\mathcal{C}, s\in D(\chi)$, and $Y_1,\ldots,Y_{m_S}$, to obtain a system of m_E+m_S linear equations whose unique solution determines the values of the unknowns. Hence, we obtain the result. □

For the following theorem, let for the sake of simplicity $X_\delta, Z(\chi)_s$ and Y_j stand for their values obtained in the previous lemma.

Theorem 3.6. *Let n be a positive integer. Then*

Case $\beta=1$ and $\alpha\neq 1$:

$$W^{1,3}_{(\alpha,1)}(n)=\frac{7}{80\,\alpha}\,(1+X_1)\,\sigma_5(n)+\frac{1}{24}\,\sigma_3(n)-\frac{1}{8\,\alpha}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{\alpha})$$
$$+\frac{7}{80\,\alpha}\sum_{\substack{\delta|\alpha\\ \delta\neq 1}}X_\delta\,\sigma_5(\frac{n}{\delta})-\frac{1}{5760\,\alpha}\sum_{j=1}^{m_S}Y_j\,\mathfrak{b}_{\alpha,j}(n)$$
$$-\frac{1}{5760\,\alpha}\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha)}Z(\chi)_s\,\sigma_5(\frac{n}{s})\qquad(3.8)$$

Case $\beta\neq 1$ and $\alpha=1$:

$$W^{3,1}_{(1,\beta)}(n)=\frac{7}{80\,\beta}\,(\,1-X_1\,)\,\sigma_5(n)+\frac{1}{24}\,\sigma_3(n)-\frac{1}{8\,\beta}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{\beta})$$
$$-\frac{7}{80\,\beta}\sum_{\substack{\delta|\beta\\ \delta\neq 1}}X_\delta\,\sigma_5(\frac{n}{\delta})+\frac{1}{5760\,\beta}\sum_{j=1}^{m_S}Y_j\,\mathfrak{b}_{\beta,j}(n)$$
$$+\frac{1}{5760\,\beta}\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\beta)}Z(\chi)_s\,\sigma_5(\frac{n}{s})\qquad(3.9)$$

Case $\beta = \alpha \neq 1$:

$$W^{1,3}_{(1,\beta)}(n) = \frac{7}{80}\,(\,\beta - X_\beta\,)\,\sigma_5(\frac{n}{\beta}) + \frac{1}{24}\,\sigma_3(\frac{n}{\beta}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{\beta}) - \frac{1}{240}\,\sigma(n)$$
$$- \frac{7}{80}\sum_{\substack{\delta|\beta\\ \delta\neq\beta}} X_\delta\,\sigma_5(\frac{n}{\delta}) + \frac{1}{5760}\sum_{j=1}^{m_S} Y_j\,\mathfrak{b}_{\beta,j}(n)$$
$$+ \frac{1}{5760}\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\beta)} Z(\chi)_s\,\sigma_5(\frac{n}{s}) \tag{3.10}$$

and

$$W^{3,1}_{(\alpha,1)}(n) = \frac{7}{80}\,(\,\alpha + X_\alpha\,)\,\sigma_5(\frac{n}{\alpha}) + \frac{1}{24}\,\sigma_3(\frac{n}{\alpha}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{\alpha}) - \frac{1}{240}\,\sigma(n)$$
$$+ \frac{7}{80}\sum_{\substack{\delta|\beta\\ \delta\neq\alpha}} X_\delta\,\sigma_5(\frac{n}{\delta}) - \frac{1}{5760}\sum_{j=1}^{m_S} Y_j\,\mathfrak{b}_{\beta,j}(n)$$
$$- \frac{1}{5760}\sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\beta)} Z(\chi)_s\,\sigma_5(\frac{n}{s}) \tag{3.11}$$

Proof. The proof is straightforward. We only prove the case $\alpha = \beta \neq 1$ and in particular the identity (3.11).

We equate the right hand side of (3.7) with that of (2.22) to yield

$$1 - \alpha + \sum_{n=1}^{\infty}\Bigg(-24\,\sigma(n) + 240\,\sigma_3(\frac{n}{\alpha})$$
$$- 720\,n\,\sigma_3(\frac{n}{\alpha}) + 504\,\alpha\,\sigma_5(\frac{n}{\alpha}) - 24\times 240\,W^{3,1}_{(\alpha,1)}(n)\Bigg)q^n$$
$$= \sum_{\delta|\alpha} X_\delta + \sum_{n=1}^{\infty}\Bigg(-504\sum_{\delta|\alpha} X_\delta\,\sigma_5(\frac{n}{\delta})$$
$$+ \sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha)} Z(\chi)_s\,\sigma_5(\frac{n}{s}) + \sum_{j=1}^{m_S} Y_j\,\mathfrak{b}_{\alpha,j}(n)\Bigg)\,q^n.$$

We then obtain

$$24\times 240\,W^{3,1}_{(\alpha,1)}(n) = -24\,\sigma(n) + 240\,\sigma_3(\frac{n}{\alpha}) - 720\,n\,\sigma_3(\frac{n}{\alpha}) + 504\,\alpha\,\sigma_5(\frac{n}{\alpha})$$
$$+ 504\sum_{\delta|\beta} X_\delta\,\sigma_5(\frac{n}{\delta}) - \sum_{\chi\in\mathcal{C}}\sum_{s\in D_\chi(\alpha\beta)} Z(\chi)_s\,\sigma_5(\frac{n}{s}) - \sum_{j=1}^{m_S} Y_j\,\mathfrak{b}_{\beta,j}(n).$$

We then solve for $W^{3,1}_{(1,\beta)}(n)$ to obtain the stated result. □

Remark 3.7. We immediately observe that in the identity

- (3.8) the part $\frac{1}{24}\,\sigma_3(n) - \frac{1}{8\,\alpha}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{\alpha})$
- (3.9) the part $\frac{1}{24}\,\sigma_3(n) - \frac{1}{8\,\beta}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{\beta})$
- (3.10) the part $\frac{1}{24}\,\sigma_3(\frac{n}{\beta}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{\beta}) - \frac{1}{240}\,\sigma(n)$
- (3.11) the part $\frac{1}{24}\,\sigma_3(\frac{n}{\alpha}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{\alpha}) - \frac{1}{240}\,\sigma(n)$

relies each only on n, α and β. The basis of the modular space $M_6(\Gamma_0(\alpha\beta))$ is not involved.

By Lemma 2.5 and by Lemma 3.5 we obtain the following

Lemma 3.8. *Let* $1 < \alpha \in \mathbb{N}^*$. *Then*

$$W^{1,3}_{(\alpha,1)}(n) = W^{3,1}_{(1,\alpha)}(n) \quad \textit{and} \quad W^{1,3}_{(1,\alpha)}(n) = W^{3,1}_{(\alpha,1)}(n).$$

Theorem 3.9. *Let* $\alpha, \beta \in \mathbb{N}^*$ *be such that* $\gcd(\alpha,\beta) = 1$. *Then*

$$W^{1,3}_{(\alpha,\beta)}(n) = W^{3,1}_{(\beta,\alpha)}(n) \quad \textit{and} \quad W^{1,3}_{(\beta,\alpha)}(n) = W^{3,1}_{(\alpha,\beta)}(n).$$

Proof. Let $\alpha, \beta \in \mathbb{N}^*$ be such that $\gcd(\alpha,\beta) = 1$. Then we have

$$\begin{aligned}
(\,\alpha\, E_2(q^{\alpha}) - E_2(q)\,)\, E_4(q^{\beta}) &= \alpha - 1 + \sum_{n=1}^{\infty}\Big(24\,\sigma(n) - 24\,\alpha\,\sigma(\frac{n}{\alpha}) \\
&+ 240\,\alpha\,\sigma_3(\frac{n}{\beta}) - 240\,\sigma_3(\frac{n}{\beta}) + 24\times 240\, W^{1,3}_{(1,\beta)}(n) - 24\times 240\,\alpha\, W^{1,3}_{(\alpha,\beta)}(n)\Big)q^n \\
&= 1 - \alpha + \sum_{n=1}^{\infty}\Big(-24\,\sigma(n) + 24\,\alpha\,\sigma(\frac{n}{\alpha}) \\
&- 240\,\alpha\,\sigma_3(\frac{n}{\beta}) + 240\,\sigma_3(\frac{n}{\beta}) - 24\times 240\, W^{1,3}_{(1,\beta)}(n) + 24\times 240\,\alpha\, W^{1,3}_{(\alpha,\beta)}(n)\Big)q^n \\
&= E_4(q^{\beta})\,(\,E_2(q) - \,\alpha\, E_2(q^{\alpha})\,)
\end{aligned}$$

and

$$\begin{aligned}
E_4(q^{\alpha})\,(\,E_2(q) - \beta\, E_2(q^{\beta})\,) &= 1 - \beta + \sum_{n=1}^{\infty}\Big(24\,\beta\,\sigma(\frac{n}{\beta}) - 24\,\sigma(n) \\
&+ 240\,\sigma_3(\frac{n}{\alpha}) - 240\,\beta\,\sigma_3(\frac{n}{\alpha}) - 24\times 240\, W^{3,1}_{(\alpha,1)}(n) + 24\times 240\,\beta\, W^{3,1}_{(\alpha,\beta)}(n)\Big)q^n \\
&= \beta - 1 + \sum_{n=1}^{\infty}\Big(-24\,\beta\,\sigma(\frac{n}{\beta}) + 24\,\sigma(n) \\
&- 240\,\sigma_3(\frac{n}{\alpha}) + 240\,\beta\,\sigma_3(\frac{n}{\alpha}) + 24\times 240\, W^{3,1}_{(\alpha,1)}(n) - 24\times 240\,\beta\, W^{3,1}_{(\alpha,\beta)}(n)\Big)q^n \\
&= (\,\beta\, E_2(q^{\beta}) - E_2(q)\,)\, E_4(q^{\alpha})
\end{aligned}$$

Now we consider $W^{1,3}_{(1,\beta)}(n)$ and $W^{3,1}_{(\alpha,1)}(n)$; then apply Corollary 3.8. When we then compare both side in each case, we obtain the stated result. □

We now have the prerequisite to determine a formula for the number of representations of a positive integer n by a quadratic form.

4. Number of Representations of a Positive Integer for this Class of Levels

We discuss in this section the determination of formulae for the number of representations of a positive integer by the quadratic forms (1.4) – (1.7).

4.1. Representations of a positive integer by the quadratic form (1.4) and (1.5). We determine formulae for the number of representations of a positive integer by the quadratic forms (1.4) and (1.5).

4.1.1. *Formulae for the Number of Representations by (1.4) and (1.5).* Let $n \in \mathbb{N}$ and let the number of representations of n by the quadratic form $\sum_{i=1}^{12} x_i^2$ be the combination of

$$r_4(n) = \operatorname{card}(\{(x_1, x_2, x_3, x_4) \in \mathbb{Z}^4 \mid n = \sum_{i=1}^{4} x_i^2 \})$$

and

$$r_8(n) = \operatorname{card}(\{(x_1, x_2, \ldots, x_7, x_8) \in \mathbb{Z}^8 \mid n = \sum_{i=1}^{8} x_i^2 \}).$$

It follows from the definitions that $r_4(0) = r_8(0) = 1$. For all $n \in \mathbb{N}^*$ the Jacobi's identities $r_4(n)$ and $r_8(n)$ are

$$r_4(n) = 8\,\sigma(n) - 32\,\sigma(\frac{n}{4}) \tag{4.1}$$

and

$$r_8(n) = 16\,\sigma_3(n) - 32\,\sigma_3(\frac{n}{2}) + 256\,\sigma_3(\frac{n}{4}), \tag{4.2}$$

respectively, and an arithmetic proof of the identity (4.1) is given by K. S. Williams [**26**] and that of the identity (4.2) is provided by G. A. Lomadze [**15**].

Now, let the number of representations of n by the quadratic form (1.4) or (1.5) be

$$N^{4i,4j}_{(a_1,b_1)}(n) = \operatorname{card}(\{(x_1, x_2, \ldots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid n = a_1 \sum_{i=1}^{4} x_i^2 + b_1 \sum_{i=5}^{12} x_i^2\}),$$

where $a_1, b_1, i, j \in \mathbb{N}^*$ such that $i + j = 3$ indicates the number of variables used in the representation of n. In particular we have

$$N^{4,8}_{(a,b)}(n) = \operatorname{card}(\{(x_1, x_2, \ldots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid n = a \sum_{i=1}^{4} x_i^2 + b \sum_{i=5}^{12} x_i^2\}),$$

and

$$N^{8,4}_{(a_1,b_1)}(n) = \operatorname{card}(\{(x_1, x_2, \ldots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid n = a_1 \sum_{i=1}^{8} x_i^2 + b_1 \sum_{i=9}^{12} x_i^2\}),$$

where $a, b, a_1, b_1 \in \mathbb{N}^*$.

It immediately follows from the definition of $N^{4,8}_{(a,b)}(n)$ and $N^{8,4}_{(a_1,b_1)}(n)$ that if $a, b, a_1, b_1 \in \mathbb{N}^*$ are such that $\gcd(a, b) = d > 1$ and $\gcd(a_1, b_1) = d_1 > 1$ for some $d, d_1 \in \mathbb{N}^*$, then $N^{4,8}_{(a,b)}(n) = N^{1,3}_{(\frac{a}{d},\frac{b}{d})}(\frac{n}{d})$ and $N^{8,4}_{(a_1,b_1)}(n) = N^{8,4}_{(\frac{a_1}{d_1},\frac{b_1}{d_1})}(\frac{n}{d_1})$. Therefore, one can simply assume that $a, b, a_1, b_1 \in \mathbb{N}^*$ are relatively prime.

We then derive the following result:

Theorem 4.1. *Let $n \in \mathbb{N}$ and let $a, b, a_1, b_1 \in \mathbb{N}^*$ be relatively prime. Then*

$$N^{4,8}_{(a,b)}(n) = 8\,\sigma(\frac{n}{a}) - 32\,\sigma(\frac{n}{4a}) + 16\,\sigma_3(\frac{n}{b}) - 32\,\sigma_3(\frac{n}{2b}) + 256\sigma_3(\frac{n}{4b}) + 128\,W^{1,3}_{(a,b)}(n) - 256\,W^{1,3}_{(a,2b)}(n) + 2048\,W^{1,3}_{(a,4b)}(n) - 512\,W^{1,3}_{(4a,b)}(n) + 1024\,W^{1,3}_{(2a,b)}(\frac{n}{2}) - 8192\,W^{1,3}_{(a,b)}(\frac{n}{4})$$

and

$$N^{8,4}_{(a_1,b_1)}(n) = 8\,\sigma(\frac{n}{b_1}) - 32\,\sigma(\frac{n}{4b_1}) + 16\,\sigma_3(\frac{n}{a_1}) - 32\,\sigma_3(\frac{n}{2a_1}) + 256\sigma_3(\frac{n}{4a_1}) + 128\,W^{3,1}_{(a_1,b_1)}(n) - 512\,W^{3,1}_{(a_1,4b_1)}(n) - 256\,W^{3,1}_{(2a_1,b_1)}(n) + 1024\,W^{3,1}_{(a_1,2b_1)}(\frac{n}{2}) + 2048\,W^{3,1}_{(4a_1,b_1)}(n) - 8192\,W^{3,1}_{(a_1,b_1)}(\frac{n}{4}).$$

Proof. It suffices to prove the identity $N^{4,8}_{(a,b)}(n)$ since the other can be proved similarly or by using Theorem 3.9.

We have

$$N^{4,8}_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2\\ a\,l+b\,m=n}} r_4(l)r_8(m) = r_4(\frac{n}{a})r_8(0)+r_4(0)r_8(\frac{n}{b})+\sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^*\\ a\,l+b\,m=n}} r_4(l)r_8(m).$$

We make use of (4.1) to obtain

$$N^{4,8}_{(a,b)}(n) = 8\,\sigma(\frac{n}{a}) - 32\,\sigma(\frac{n}{4a}) + 16\,\sigma_3(\frac{n}{b}) - 32\,\sigma_3(\frac{n}{2b}) + 256\sigma_3(\frac{n}{4b}) + \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^*\\ al+bm=n}} (8\,\sigma(l) - 32\,\sigma(\frac{l}{4}))(16\,\sigma_3(m) - 32\,\sigma_3(\frac{m}{2}) + 256\,\sigma_3(\frac{m}{4})).$$

We know that

$$(8\,\sigma(l) - 32\,\sigma(\frac{l}{4}))(16\,\sigma_3(m) - 32\,\sigma_3(\frac{m}{2}) + 256\,\sigma_3(\frac{m}{4})) = 128\,\sigma(l)\sigma_3(m) - 256\,\sigma(l)\sigma_3(\frac{m}{2}) + 2048\,\sigma(l)\sigma_3(\frac{m}{4}) - 512\,\sigma(\frac{l}{4})\sigma_3(m) + 1024\,\sigma(\frac{l}{4})\sigma_3(\frac{m}{2}) - 8192\,\sigma_3(\frac{l}{4})\sigma_3(\frac{m}{4})$$

In the sequel of this proof, we assume that the evaluation of

$$W^{1,3}_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^*\\ al+bm=n}} \sigma(l)\sigma_3(m),$$

$W^{1,3}_{(a,2b)}(n)$, $W^{1,3}_{(a,4b)}(n)$, $W^{1,3}_{(4a,b)}(n)$, $W^{1,3}_{(4a,2b)}(n)$ and $W^{1,3}_{(4a,4b)}(n)$ are known.

Let $u, v \in \mathbb{N}^*$ be given and $f, g : \mathbb{N} \mapsto \mathbb{N}$ be injective functions such that $f(n) = u \cdot n$ and $g(n) = v \cdot n$ for each $n \in \mathbb{N}$.

When we simultaneously apply the functions f and g with l and m as argument, respectively, we derive

$$W^{1,3}_{(a,b)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^*\\ al+bm=n}} \sigma(\frac{l}{u})\sigma_3(\frac{m}{v}) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^*\\ ua\,l+vb\,m=n}} \sigma(l)\sigma_3(m) = W^{1,3}_{(ua,vb)}(n)$$

We set $(u,v) = (1,1),(1,2),(1,4),(4,1),(4,2),(4,4)$ accordingly and then finally put all these evaluations together to obtain the stated result for $N^{4,8}_{(a,b)}(n)$. □

From this proof, one immediately observe that a formula for the number of representations of a positive integer n by the quadratic forms (1.4) and (1.5) depends on the evaluated convolution sums for some given levels $ab, 2ab$ and $4ab$ with $a,b \in \mathbb{N}^*$ relatively prime.

Based on this observation, we only take into consideration those levels $\alpha\beta$ which are multiple of 4; that is $\alpha\beta \equiv 0 \pmod 4$.

4.1.2. *Determination of all relevant* $(\mathbf{a},\mathbf{b}) \in \mathbb{N}^* \times \mathbb{N}^*$ *for* $N_{(a,b)}(n)$ *for a given* $\alpha\beta \in \mathbb{N}^*$. We carry out a method to determine all pairs $(a,b) \in \mathbb{N}^* \times \mathbb{N}^*$ which are necessary for the determination of $N_{(a,b)}(n)$ for a given level $\alpha\beta \in \mathbb{N}^*$ such that $\alpha\beta \equiv 0 \pmod 4$ holds.

Let $\Lambda = \frac{\alpha\beta}{4} = 2^{\nu-2}\mho$, $P_4 = \{p_0 = 2^{\nu-2}\} \cup \bigcup_{j>1} \{p_j \mid p_j \text{ is a prime divisor of } \mho\}$ and $\mathcal{P}(P_4)$ be the power set of P_4. Then for each $Q \in \mathcal{P}(P_4)$ we define $\mu(Q) = \prod_{p\in Q} p$. We set $\mu(Q) = 1$ if Q is an empty set. Let now

$$\Omega_4 = \{(\mu(Q_1),\mu(Q_2)) \mid \text{ there exist } Q_1,Q_2 \in \mathcal{P}(P_4) \text{ such that } \gcd(\mu(Q_1),\mu(Q_2)) = 1 \text{ and } \mu(Q_1)\,\mu(Q_2) = \Lambda\,\}.$$

Observe that $\Omega_4 \neq \emptyset$ since $(1,\Lambda) \in \Omega_4$.

To illustrate our method, suppose that $\alpha\beta = 2^3 \cdot 3 \cdot 5$. Then $\Lambda = 2 \cdot 3 \cdot 5$, $P_4 = \{2,3,5\}$ and $\Omega_4 = \{(1,30),(2,15),(3,10),(5,6)\}$.

Proposition 4.2. *Suppose that the level* $\alpha\beta \in \mathbb{N}^*$ *and* $\alpha\beta \equiv 0 \pmod 4$. *Furthermore, suppose that* Ω_4 *is defined as above. Then for all* $n \in \mathbb{N}$ *the set* Ω_4 *contains all pairs* $(a,b) \in \mathbb{N}^* \times \mathbb{N}^*$ *such that* $N_{(a,b)}(n)$ *can be obtained by applying* $W_{(\alpha,\beta)}(n)$ *and some other evaluated convolution sums.*

Proof. We prove this by induction on the structure of the level $\alpha\,\beta$.

Suppose that $\alpha\beta = 2^\nu p_2$, where $\nu \in \{2,3\}$ and p_2 is an odd prime. Then by the above definitions we have $\Lambda = 2^{\nu-2}p_2$, $P_4 = \{\,2^{\nu-2}, p_2\,\}$,

$$\mathcal{P}(P_4) = \{\,\emptyset, \{2^{\nu-2}\}, \{p_2\}, \{\,2^{\nu-2}, p_2\}\,\},$$

and $\Omega_4 = \{\,(1, 2^{\nu-2}p_2), (2^{\nu-2}, p_2)\,\}$.

Following the observation made at the end of the proof of Theorem 4.1, we note that $\alpha\beta = 4ab = 2^\nu p_2$. Hence, $ab = 2^{\nu-2}p_2$ which leads immediately to $N_{(a,b)}(n)$.

We show that Ω_4 is the largest such set. Assume now that there exists another set, say Ω'_4, which results from the above definitions. Then there are two cases.

Case $\Omega'_4 \subseteq \Omega_4$**:** There is nothing to show. So, we are done.

Case $\Omega_4 \subset \Omega'_4$**:** Let $(e,f) \in \Omega'_4 \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2$ and $\gcd(e,f) = 1$, we must have either $(e,f) = (1, 2^{\nu-2}p_2)$ or $(e,f) = (2^{\nu-2}, p_2)$. So, $(e,f) \in \Omega_4$. Hence, $\Omega_4 = \Omega'_4$.

Suppose now that $\alpha\beta = 2^\nu p_2 p_3$, where $\nu \in \{2,3\}$ and p_2, p_3 are distinct odd primes. Then by the induction hypothesis and by the above definitions we have essentially

$$\Omega_4 = \{\,(1, 2^{\nu-2}p_2p_3), (2^{\nu-2}, p_2p_3), (2^{\nu-2}p_2, p_3), (2^{\nu-2}p_3, p_2)\,\}.$$

One notes that $\alpha\beta = 4ab = 2^{\nu}p_2p_3$. Hence, $ab = 2^{\nu-2}p_2p_3$ which immediately gives $N_{(a,b)}(n)$.

Again, we show that Ω_4 is the largest such set. Suppose that there exist another set, say Ω_4', which results from the above definitions. Two cases arise.

Case $\Omega_4' \subseteq \Omega_4$: There is nothing to prove. So, we are done.

Case $\Omega_4 \subset \Omega_4'$: Let $(e,f) \in \Omega_4' \setminus \Omega_4$. Since $ef = 2^{\nu-2}p_2p_3$ and $\gcd(e,f) = 1$, we must have $(e,f) = (1, 2^{\nu-2}p_2p_3)$ or $(e,f) = (2^{\nu-2}, p_2p_3)$ or $(e,f) = (2^{\nu-2}p_2, p_3)$ or $(e,f) = (2^{\nu-2}p_3, p_2)$. So, $(e,f) \in \Omega_4$. Hence, $\Omega_4 = \Omega_4'$.

□

We then deduce in conjunction with Theorem 3.9 the following:

Corollary 4.3. *Let $n \in \mathbb{N}$, $\alpha\beta \in \mathbb{N}^*$ with $\alpha\beta \equiv 0 \pmod 4$ and Ω_4 be determined as above. Then for each $(a,b), (a_1,b_1) \in \Omega_4$ we have*

$$\begin{aligned}N^{4,8}_{(a,b)}(n) = N^{8,4}_{(b,a)}(n) = & \, 8\,\sigma(\frac{n}{a}) - 32\,\sigma(\frac{n}{4a}) + 16\,\sigma_3(\frac{n}{b}) - 32\,\sigma_3(\frac{n}{2b}) + 256\sigma_3(\frac{n}{4b}) \\ & + 128\,W^{1,3}_{(a,b)}(n) - 256\,W^{1,3}_{(a,2b)}(n) + 2048\,W^{1,3}_{(a,4b)}(n) \\ & - 512\,W^{1,3}_{(4a,b)}(n) + 1024\,W^{1,3}_{(2a,b)}(\frac{n}{2}) - 8192\,W^{1,3}_{(a,b)}(\frac{n}{4})\end{aligned}$$

and

$$\begin{aligned}N^{8,4}_{(a_1,b_1)}(n) = N^{4,8}_{(b_1,a_1)}(n) = & \, 8\,\sigma(\frac{n}{b_1}) - 32\,\sigma(\frac{n}{4b_1}) + 16\,\sigma_3(\frac{n}{a_1}) - 32\,\sigma_3(\frac{n}{2a_1}) \\ & + 256\sigma_3(\frac{n}{4a_1}) + 128\,W^{3,1}_{(a_1,b_1)}(n) - 512\,W^{3,1}_{(a_1,4b_1)}(n) \\ & - 256\,W^{3,1}_{(2a_1,b_1)}(n) + 1024\,W^{3,1}_{(a_1,2b_1)}(\frac{n}{2}) + 2048\,W^{3,1}_{(4a_1,b_1)}(n) \\ & - 8192\,W^{3,1}_{(a_1,b_1)}(\frac{n}{4})\end{aligned}$$

From this corollary it follows that if $a = b = a_1 = b_1$, then $N^{4,8}_{(a,a)}(n) = N^{8,4}_{(a,a)}(n)$, especially $N^{4,8}_{(1,1)}(n) = N^{8,4}_{(1,1)}(n)$.

4.2. Representations of a positive integer by the quadratic form (1.6) and (1.7). We now determine formulae for the number of representations of a positive integer by the octonary quadratic forms (1.6) and (1.7).

4.2.1. *Formulae for the Number of Representations by (1.6) and (1.7).* Let $n \in \mathbb{N}$ and let $s_4(n)$ and $s_8(n)$denote the number of representations of n by the quaternary quadratic form $\sum_{i=1}^{2} (\, x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)$, that is,

$$s_4(n) = \operatorname{card}(\{(x_1,x_2,x_3,x_4) \in \mathbb{Z}^4 \mid n = \sum_{i=1}^{2} (\, x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)\})$$

and the quadratic form $\sum_{i=1}^{4} (\, x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)$, that means,

$$s_8(n) = \operatorname{card}(\{(x_1,x_2,\dots,x_7,x_8) \in \mathbb{Z}^8 \mid n = \sum_{i=1}^{4} (\, x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)\}),$$

respectively. It is obvious that $s_4(0) = s_8(0) = 1$. J. G. Huard et al. [**5**], G. A. Lomadze [**15**] and K. S. Williams [**27**, Thrm 17.3, p. 225] have proved that for all $n \in \mathbb{N}^*$

$$s_4(n) = 12\,\sigma(n) - 36\,\sigma(\frac{n}{3}). \tag{4.3}$$

G. A. Lomadze [**15**] has proved that for all $n \in \mathbb{N}^*$

$$s_8(n) = 24\,\sigma_3(n) + 216\,\sigma_3(\frac{n}{3}). \tag{4.4}$$

Now, let the number of representations of n by the quadratic form (1.6) be

$$R^{4,8}_{(c,d)}(n) = \text{card}(\{(x_1, x_2, \ldots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid n = c \sum_{i=1}^{2} (\,x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,) + d \sum_{i=3}^{6} (\,x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)\,\}),$$

and that of the quadratic form (1.7) be

$$R^{8,4}_{(c_1,d_1)}(n) = \text{card}(\{(x_1, x_2, \ldots, x_{11}, x_{12}) \in \mathbb{Z}^{12} \mid n = c_1 \sum_{i=1}^{4} (\,x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,) + d_1 \sum_{i=5}^{6} (\,x_{2i-1}^2 + x_{2i-1}x_{2i} + x_{2i}^2\,)\,\}),$$

where $c, d, c_1, d_1 \in \mathbb{N}^*$.

From these definitions suppose that $c, d, c_1, d_1 \in \mathbb{N}^*$ are such that $\gcd(c, d) = e > 1$ and $\gcd(c1, d_1) = e_1 > 1$for some $e, e_1 \in \mathbb{N}^*$. Then $R^{4,8}_{(c,d)}(n) = R^{4,8}_{(\frac{c}{e},\frac{d}{e})}(\frac{n}{e})$ and $R^{8,4}_{(c_1,d_1)}(n) = R^{8,4}_{(\frac{c_1}{e_1},\frac{d_1}{e_1})}(\frac{n}{e_1})$, respectively. Hence, one can simply assume that $c, d, c_1, d_1 \in \mathbb{N}^*$ are relatively prime.

We infer the following

Theorem 4.4. *Let $n \in \mathbb{N}$ and $c, d, c_1, d_1 \in \mathbb{N}^*$ be relatively prime. Then*

$$\begin{aligned} R^{4,8}_{(c,d)}(n) =& 12\,\sigma(\frac{n}{c}) - 36\,\sigma(\frac{n}{3c}) + 24\,\sigma_3(\frac{n}{d}) + 216\,\sigma_3(\frac{n}{3d}) + 288\,W^{1,3}_{(c,d)}(n) \\ & + 2592\,W^{1,3}_{(c,3d)}(n) - 864\,W^{1,3}_{(3c,d)}(n) - 7776\,W^{1,3}_{(c,d)}(\frac{n}{3}) \end{aligned}$$

and

$$\begin{aligned} R^{8,4}_{(c_1,d_1)}(n) =& 24\,\sigma_3(\frac{n}{c_1}) + 216\,\sigma_3(\frac{n}{3c_1}) + 12\,\sigma(\frac{n}{d_1}) - 36\,\sigma(\frac{n}{3d_1}) + 288\,W^{3,1}_{(c_1,d_1)}(n) \\ & - 864\,W^{3,1}_{(c_1,3d_1)}(n) + 2592\,W^{3,1}_{(3c_1,d_1)}(n) - 7776\,W^{3,1}_{(c_1,d_1)}(\frac{n}{3}). \end{aligned}$$

Proof. It suffices to prove the identity $R^{8,4}_{(c_1,d_1)}(n)$ since the other can be proved similarly or by applying Theorem 3.9.

We have

$$R^{8,4}_{(c_1,d_1)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^2 \\ c_1\,l+d_1\,m=n}} s_8(l)s_4(m)$$
$$= s_8(\frac{n}{c_1})s_4(0) + s_8(0)s_4(\frac{n}{d_1}) + \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^* \\ c_1\,l+d_1\,m=n}} s_8(l)s_4(m).$$

We make use of (4.1) and (4.2) to obtain

$$R^{8,4}_{(c_1,d_1)}(n) = 24\,\sigma_3(\frac{n}{c_1}) + 216\,\sigma_3(\frac{n}{3c_1}) + 12\,\sigma(\frac{n}{d_1}) - 36\,\sigma(\frac{n}{3d_1})$$
$$+ \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^* \\ c_1l+d_1m=n}} (24\,\sigma_3(l) + 216\,\sigma_3(\frac{l}{3}))(12\,\sigma(m) - 36\,\sigma(\frac{m}{3})).$$

We know that

$$(24\,\sigma_3(l) + 216\,\sigma_3(\frac{l}{3}))(12\,\sigma(m) - 36\,\sigma(\frac{m}{3})) =$$
$$288\,\sigma_3(l)\sigma(m) - 864\,\sigma_3(l)\sigma(\frac{m}{3}) + 2592\,\sigma_3(\frac{l}{3})\sigma(m) - 7776\,\sigma(\frac{l}{3})\sigma_3(\frac{m}{3})$$

In the sequel of this proof, we assume that the evaluation of

$$W^{3,1}_{(c_1,d_1)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^* \\ c_1l+d_1m=n}} \sigma_3(l)\sigma(m),$$

$W^{3,1}_{(c_1,3d_1)}(n)$, $W^{3,1}_{(3c_1,d_1)}(n)$ and $W^{3,1}_{(3c_1,3d_1)}(n)$ are known.

Let $u, v \in \mathbb{N}^*$ be given and $f, g : \mathbb{N} \mapsto \mathbb{N}$ be injective functions such that $f(n) = u \cdot n$ and $g(n) = v \cdot n$ for each $n \in \mathbb{N}$.

When we simultaneously apply the functions f and g with l and m as argument, respectively, we derive

$$W^{3,1}_{(c_1,d_1)}(n) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^* \\ c_1l+d_1m=n}} \sigma_3(\frac{l}{u})\sigma(\frac{m}{v}) = \sum_{\substack{(l,m)\in\mathbb{N}^*\times\mathbb{N}^* \\ uc_1\,l+vd_1\,m=n}} \sigma_3(l)\sigma(m) = W^{1,3}_{(uc_1,vd_1)}(n)$$

We set $(u, v) = (1, 1), (1, 3), (3, 1), (3, 3)$ accordingly and then finally put all these evaluations together to obtain the stated result for $R^{8,4}_{(c_1,d_1)}(n)$. □

From this proof, we note that a formula for the number of representations of a positive integer n by the quadratic form (1.6) and (1.7) depends on the evaluated convolution sums for some given levels c_1d_1 and $3c_1d_1$ with $c_1, d_1 \in \mathbb{N}^*$ relatively prime.

As a consequence, we do consider only the levels $\alpha\beta$ which are divisible by 3; that is $\alpha\beta \equiv 0 \pmod 3$.

4.2.2. *Determination of all relevant* $(\mathbf{c}, \mathbf{d}) \in \mathbb{N}^* \times \mathbb{N}^*$ *for* $R_{(c,d)}(n)$ *for a given level* $\alpha\beta \in \mathbb{N}^*$. The following method determine all pairs $(c, d) \in \mathbb{N}^* \times \mathbb{N}^*$ necessary for the determination of $R_{(c,d)}(n)$ for a given $\alpha\beta \in \mathbb{N}^*$ belonging to the above class. The following method is quasi similar to the one used in Subsection 4.1.2.

Let $\Delta = \frac{\alpha\beta}{3} = \frac{2^\nu \mho}{3}$. Let $P_3 = \{p_0 = 2^\nu\} \cup \bigcup_{j>2} \{p_j \mid p_j \text{ is a prime divisor of } \mho\}$. Let $\mathcal{P}(P_3)$ be the power set of P_3. Then for each $Q \in \mathcal{P}(P_3)$ we define $\mu(Q) = \prod_{p \in Q} p$. We set $\mu(Q) = 1$ if Q is an empty set. Let now Ω_3 be defined in a similar way as Ω_4 in Subsection 4.1.2; however, with Δ instead of Λ, i.e.,

$$\Omega_3 = \{(\mu(Q_1), \mu(Q_2)) \mid \text{ there exist } Q_1, Q_2 \in \mathcal{P}(P_3) \text{ such that } \gcd(\mu(Q_1), \mu(Q_2)) = 1 \text{ and } \mu(Q_1)\,\mu(Q_2) = \Delta\}.$$

Note that $\Omega_3 \neq \emptyset$ since $(1, \Delta) \in \Omega_3$.

As an example, suppose again that $\alpha\beta = 2^3 \cdot 3 \cdot 5$. Then $\Delta = 2^3 \cdot 5$, $P_3 = \{2^3, 5\}$ and $\Omega_3 = \{(1, 40), (5, 8)\}$.

Proposition 4.5. *Suppose that the level $\alpha\beta \in \mathbb{N}^*$ and $\alpha\beta \equiv 0 \pmod 3$. Suppose in addition that Ω_3 be defined as above. Then for all $n \in \mathbb{N}$ the set Ω_3 contains all pairs $(c, d) \in \mathbb{N}^* \times \mathbb{N}^*$ such that $R_{(c,d)}(n)$ can be obtained by applying $W_{(\alpha,\beta)}(n)$ and some other evaluated convolution sums.*

Proof. Similar to the proof of Proposition 4.2. □

We then infer in collaboration with Theorem 3.9 the following:

Corollary 4.6. *Let $n \in \mathbb{N}$, $\alpha\beta \in \mathbb{N}^*$ with $\alpha\beta \equiv 0 \pmod 3$ and Ω_3 be determined as above. Then for each $(c, d), (c_1, d_1) \in \Omega_3$ we obtain*

$$\begin{aligned} R^{4,8}_{(c,d)}(n) \;=\; R^{8,4}_{(d,c)}(n) \;=\; & 12\,\sigma(\frac{n}{c}) - 36\,\sigma(\frac{n}{3c}) + 24\,\sigma_3(\frac{n}{d}) + 216\,\sigma_3(\frac{n}{3d}) + 288\,W^{1,3}_{(c,d)}(n) \\ & + 2592\,W^{1,3}_{(c,3d)}(n) - 864\,W^{1,3}_{(3c,d)}(n) - 7776\,W^{1,3}_{(c,d)}(\frac{n}{3}) \end{aligned}$$

and

$$\begin{aligned} R^{8,4}_{(c_1,d_1)}(n) \;=\; R^{4,8}_{(d_1,c_1)}(n) \;=\; & 24\,\sigma_3(\frac{n}{c_1}) + 216\,\sigma_3(\frac{n}{3c_1}) + 12\,\sigma(\frac{n}{d_1}) - 36\,\sigma(\frac{n}{3d_1}) \\ & + 288\,W^{3,1}_{(c_1,d_1)}(n) - 864\,W^{3,1}_{(c_1,3d_1)}(n) + 2592\,W^{3,1}_{(3c_1,d_1)}(n) \\ & - 7776\,W^{3,1}_{(c_1,d_1)}(\frac{n}{3}) \end{aligned}$$

It immediately follows that if $c = d = c_1 = d_1$, then $R^{4,8}_{(c,c)}(n) = R^{8,4}_{(c,c)}(n)$, especially $R^{4,8}_{(1,1)}(n) = R^{8,4}_{(1,1)}(n)$.

5. Evaluation of the Convolution Sums for some Levels in $\mathfrak{N}$

In this section, we give explicit formulae for the convolution sums

- $W^{1,3}_{(\alpha,\beta)}(n)$ when $\alpha\beta = 7, 8, 14, 15, 20, 21$; and
- $W^{3,1}_{(\alpha,\beta)}(n)$ when $\alpha\beta = 3, 4, 6, 7, 8, 12, 14, 15, 20, 21$;

When we apply Equation 2.2, we conclude that for example

$$M_6(\Gamma_0(3)) \subset M_6(\Gamma_0(6)) \subset M_6(\Gamma_0(12)) \tag{5.1}$$

and

$$M_6(\Gamma_0(4)) \subset M_6(\Gamma_0(8)) \subset M_6(\Gamma_0(16)) \subset M_6(\Gamma_0(32)). \tag{5.2}$$

This implies the same inclusion relation for the bases, the space of Eisenstein forms of weight 6 and the spaces of cusp forms of weight 6.

5.1. Bases of $E_6(\Gamma_0(\alpha\beta))$ and $S_6(\Gamma_0(\alpha\beta))$ for these values of $\alpha\beta$. We apply the dimension formulae in T. Miyake's book [**18**, Thrm 2.5.2, p. 60] or [**25**, Prop. 6.1, p. 91] and (3.2) (3.2) to deduce that

$$\begin{aligned}
\dim(E_6(\Gamma_0(\alpha\beta))) &= d(\alpha\beta), & \dim(S_6(\Gamma_0(\alpha\beta))) &= 1 \text{ when } 2 < \alpha\beta < 6 \text{ holds},\\
\dim(S_6(\Gamma_0(6))) &= 3, & \dim(S_6(\Gamma_0(7))) &= 3,\\
\dim(S_6(\Gamma_0(8))) &= 3, & \dim(S_6(\Gamma_0(12))) &= 7,\\
\dim(S_6(\Gamma_0(14))) &= 8, & \dim(S_6(\Gamma_0(15))) &= 8,\\
\dim(S_6(\Gamma_0(20))) &= 12, & \text{and } \dim(S_6(\Gamma_0(21))) &= 12.
\end{aligned}$$

We apply Theorem 2.1 as mentioned in Subsection 3.1 to determine as many elements of

$$S_6(\Gamma_0(8)),\quad S_6(\Gamma_0(14)),\quad S_6(\Gamma_0(15)),\quad S_6(\Gamma_0(20)) \quad\text{and}\quad S_6(\Gamma_0(21))$$

as possible. Then we apply Remark 3.4 (r2) when selecting basis elements of a given space of cusp forms as stated in the proof of Theorem 3.3 (b).

Corollary 5.1. *The following statement holds.*

(a) Let $\kappa = 12, 14, 15, 20, 21$ and $\delta \in D(\kappa)$. Then the set

$$\mathcal{B}_{E,\kappa} = \{\, M(q^t) \;\mid\; t|\kappa \,\}$$

is a basis of $E_6(\Gamma_0(\kappa))$.

(b) Let $i \in \mathbb{N}^*$ satisfies $1 \le i \le m_\kappa$, where $m_\kappa = 7, 8, 12$.
Let $\delta_1 \in D(12)$ and $(r(i,\delta_1))_{i,\delta_1}$ be the Table 2 of the powers of $\eta(\delta_1 z)$.
Let $\delta_2 \in D(14)$ and $(r(j,\delta_2))_{j,\delta_2}$ be the Table 3 of the powers of $\eta(\delta_2 z)$.
Let $\delta_3 \in D(15)$ and $(r(k,\delta_3))_{k,\delta_3}$ be the Table 4 of the powers of $\eta(\delta_3 z)$.
Let $\delta_4 \in D(20)$ and $(r(j,\delta_4))_{j,\delta_4}$ be the Table 6 of the powers of $\eta(\delta_4 z)$.
Let $\delta_5 \in D(21)$ and $(r(k,\delta_5))_{k,\delta_5}$ be the Table 7 of the powers of $\eta(\delta_5 z)$.
Let furthermore

$$\mathfrak{B}_{\kappa,i}(q) = \prod_{\delta\,|\,\kappa} \eta^{r(i,\delta)}(\delta\, z)$$

be selected elements of $S_6(\Gamma_0(\kappa))$.
Then the set

$$\mathcal{B}_{S,\kappa} = \{\, \mathfrak{B}_{\kappa,i}(q) \;\mid\; 1 \le i \le m_\kappa \,\}$$

is a basis of $S_6(\Gamma_0(\kappa))$.

(c) The set

$$\mathcal{B}_{M,\kappa} = \mathcal{B}_{E,\kappa} \cup \mathcal{B}_{S,\kappa}$$

constitutes a basis of $M_6(\Gamma_0(\kappa))$ for $\kappa = 12, 14, 15, 20, 21$.

By Remark 3.4 (r1), $\mathfrak{B}_{\kappa,i}(q)$ can be expressed in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{\kappa,i}(n) q^n$, where $\kappa = 12, 14, 15, 20, 21$.

Proof. It follows immediately from Theorem 3.3. □

5.2. Evaluation of $\mathbf{W}_{(\alpha,\beta)}^{2i-1,2j-1}(\mathbf{n})$ when $\alpha\beta = 3, 4, 6, 7, 8, 12, 14, 15, 20, 21$. We consider in the following two corollaries to discuss the case $(\,\alpha\,E_2(q^{\alpha}) - E_2(q)\,)\,E_4(q^{\beta})$ and the case $E_4(q^{\beta})\,(\,E_2(q) - \alpha\,E_2(q^{\alpha})\,)$ for some values of $\alpha\beta$ since the other cases can be handled similarly.

The first corollary deals with the case $(\,\alpha\,E_2(q^{\alpha}) - E_2(q)\,)\,E_4(q^{\beta})$. It is sufficient to consider only the special cases $1 \neq \alpha = \beta$ and $\alpha > 1$ whenever $\beta = 1$.

Corollary 5.2. *We have*

$$(\,7\,E_2(q^7) - E_2(q)\,)\,E_4(q^7) = 6 + \sum_{n=1}^{\infty}\Big(-\frac{1}{2451}\,\sigma_5(n) + \frac{14707}{2451}\,\sigma_5(\frac{n}{7}) + \frac{19440}{817}\,\mathfrak{b}_{21,1}(n) + \frac{5760}{19}\,\mathfrak{b}_{21,2}(n) + \frac{843120}{817}\,\mathfrak{b}_{21,3}(n)\Big)\,q^n, \quad (5.3)$$

$$(\,7\,E_2(q^7) - E_2(q)\,)\,E_4(q) = 6 + \sum_{n=1}^{\infty}\Big(-\frac{2101}{2451}\,\sigma_5(n) + \frac{16807}{2451}\,\sigma_5(\frac{n}{7}) + \frac{843120}{817}\,\mathfrak{b}_{21,1}(n) + \frac{282240}{19}\,\mathfrak{b}_{21,2}(n) + \frac{46675440}{817}\,\mathfrak{b}_{21,3}(n)\Big)\,q^n, \quad (5.4)$$

$$(\,8\,E_2(q^8) - E_2(q)\,)\,E_4(q^8) = 7 + \sum_{n=1}^{\infty}\Big(14 - \frac{1}{5376}\,\sigma_5(n) - \frac{5}{1792}\,\sigma_5(\frac{n}{2}) + \frac{7}{48}\,\sigma_5(\frac{n}{4}) + \frac{20}{7}\,\sigma_5(\frac{n}{8}) + \frac{765}{32}\,\mathfrak{b}_{32,1}(n) + \frac{135}{2}\,\mathfrak{b}_{32,2}(n) + 360\,\mathfrak{b}_{32,3}(n)\Big)\,q^n, \quad (5.5)$$

$$(\,8\,E_2(q^8) - E_2(q)\,)\,E_4(q) = 7 + \sum_{n=1}^{\infty}\Big(-\frac{37}{42}\,\sigma_5(n) + \frac{5}{14}\,\sigma_5(\frac{n}{2}) + \frac{10}{7}\,\sigma_5(\frac{n}{4}) + \frac{128}{21}\,\sigma_5(\frac{n}{8}) + 1260\,\mathfrak{b}_{32,1}(n) + 6480\,\mathfrak{b}_{32,2}(n) + 23040\,\mathfrak{b}_{32,3}(n)\Big)\,q^n, \quad (5.6)$$

$$(\,14\,E_2(q^{14}) - E_2(q)\,)\,E_4(q^{14}) = 13 + \sum_{n=1}^{\infty}\Big(-\frac{1}{51471}\,\sigma_5(n) - \frac{20}{51471}\,\sigma_5(\frac{n}{2}) + \frac{2101}{7353}\,\sigma_5(\frac{n}{7}) + \frac{42020}{7353}\,\sigma_5(\frac{n}{14}) + \frac{19600}{817}\,\mathfrak{b}_{14,1}(n) + \frac{254400}{817}\,\mathfrak{b}_{14,2}(n) + \frac{916880}{817}\,\mathfrak{b}_{14,3}(n) + \frac{255680}{817}\,\mathfrak{b}_{14,4}(n) + \frac{523520}{817}\,\mathfrak{b}_{14,5}(n) + \frac{4369920}{817}\,\mathfrak{b}_{14,6}(n) - \frac{1437600}{817}\,\mathfrak{b}_{14,7}(n) - \frac{1280}{817}\,\mathfrak{b}_{14,8}(n)\Big)\,q^n, \quad (5.7)$$

$$
\begin{aligned}
(\,14\,E_2(q^{14})-E_2(q)\,)\,E_4(q) = 13 + \sum_{n=1}^{\infty}\Big(& -\frac{6853}{7353}\,\sigma_5(n)+\frac{1600}{7353}\,\sigma_5(\frac{n}{2}) \\
& +\frac{24010}{7353}\,\sigma_5(\frac{n}{7})+\frac{76832}{7353}\,\sigma_5(\frac{n}{14})+\frac{2184880}{817}\,\mathfrak{b}_{14,1}(n)+\frac{36980160}{817}\,\mathfrak{b}_{14,2}(n) \\
& +\frac{179693360}{817}\,\mathfrak{b}_{14,3}(n)+\frac{153292160}{817}\,\mathfrak{b}_{14,4}(n)+\frac{102126080}{817}\,\mathfrak{b}_{14,5}(n) \\
& +\frac{1505817600}{817}\,\mathfrak{b}_{14,6}(n)-\frac{282011520}{817}\,\mathfrak{b}_{14,7}(n)+\frac{716800}{817}\,\mathfrak{b}_{14,8}(n)\Big)\,q^n, \quad (5.8)
\end{aligned}
$$

$$
\begin{aligned}
(\,15\,E_2(q^{15})-E_2(q)\,)\,E_4(q^{15}) = 14 + \sum_{n=1}^{\infty}\Big(& \frac{274}{8934471}\,\sigma_5(n)+\frac{5457}{992719}\,\sigma_5(\frac{n}{3}) \\
& -\frac{8090239}{8934471}\,\sigma_5(\frac{n}{5})+\frac{14791494}{992719}\,\sigma_5(\frac{n}{15})+\frac{3405800}{141817}\,\mathfrak{b}_{15,1}(n) \\
& +\frac{1158952}{10909}\,\mathfrak{b}_{15,2}(n)-\frac{35264}{10909}\,\mathfrak{b}_{15,3}(14n)+\frac{2061176}{10909}\,\mathfrak{b}_{15,4}(n) \\
& +\frac{177635528}{141817}\,\mathfrak{b}_{15,5}(n)+\frac{2367936}{10909}\,\mathfrak{b}_{15,6}(n)+\frac{34294464}{10909}\,\mathfrak{b}_{15,7}(n) \\
& +\frac{1203968}{10909}\,\mathfrak{b}_{15,8}(n),\Big)\,q^n, \quad (5.9)
\end{aligned}
$$

$$
\begin{aligned}
(\,15\,E_2(q^{15})-E_2(q)\,)\,E_4(q) = 14 + \sum_{n=1}^{\infty}\Big(& -\frac{917834}{992719}\,\sigma_5(n)-\frac{1460295}{992719}\,\sigma_5(\frac{n}{3}) \\
& -\frac{104862815}{992719}\,\sigma_5(\frac{n}{5})+\frac{121139010}{992719}\,\sigma_5(\frac{n}{15})+\frac{413824680}{141817}\,\mathfrak{b}_{15,1}(n) \\
& +\frac{756726120}{10909}\,\mathfrak{b}_{15,2}(n)-\frac{93504960}{10909}\,\mathfrak{b}_{15,3}(n)+\frac{263947320}{10909}\,\mathfrak{b}_{15,4}(n) \\
& +\frac{90853616520}{141817}\,\mathfrak{b}_{15,5}(n)+\frac{283798080}{10909}\,\mathfrak{b}_{15,6}(n)+\frac{3149876160}{10909}\,\mathfrak{b}_{15,7}(n) \\
& -\frac{339972480}{10909}\,\mathfrak{b}_{15,8}(n)\Big)\,q^n, \quad (5.10)
\end{aligned}
$$

$$(\,20\,E_2(q^{20}) - E_2(q)\,)\,E_4(q^{20}) == 19 + \sum_{n=1}^{\infty}\bigg(\,\frac{1567}{117445608}\,\sigma_5(n)$$
$$-\frac{110791}{117445608}\,\sigma_5(\frac{n}{2}) - \frac{10004}{233027}\,\sigma_5(\frac{n}{4}) - \frac{702215}{234891216}\,\sigma_5(\frac{n}{5})$$
$$-\frac{10264633}{234891216}\,\sigma_5(\frac{n}{10}) + \frac{93420884}{4893567}\,\sigma_5(\frac{n}{20}) + \frac{5594215}{233027}\,\mathfrak{b}_{20,1}(n)$$
$$+\frac{16718864}{233027}\,\mathfrak{b}_{20,2}(n) + \frac{89883520}{233027}\,\mathfrak{b}_{20,3}(n) + \frac{32106736}{233027}\,\mathfrak{b}_{20,4}(n)$$
$$-\frac{527968775}{466054}\,\mathfrak{b}_{20,5}(n) + \frac{86133984}{233027}\,\mathfrak{b}_{20,6}(n) + \frac{1416189312}{233027}\,\mathfrak{b}_{20,7}(n)$$
$$+\frac{4970832}{7517}\,\mathfrak{b}_{20,8}(n) + \frac{976297728}{233027}\,\mathfrak{b}_{20,9}(n) + \frac{618500736}{233027}\,\mathfrak{b}_{20,10}(n)$$
$$-\frac{64669696}{7517}\,\mathfrak{b}_{20,11}(n) - \frac{4227164544}{233027}\,\mathfrak{b}_{20,12}(n)\bigg)\,q^n, \quad (5.11)$$

$$(\,20\,E_2(q^{20}) - E_2(q)\,)\,E_4(q) == 19 + \sum_{n=1}^{\infty}\bigg(-\frac{53619409}{58722804}\,\sigma_5(n)$$
$$-\frac{188714723}{58722804}\,\sigma_5(\frac{n}{2}) + \frac{2103232}{1631189}\,\sigma_5(\frac{n}{4}) + \frac{9257815}{8388972}\,\sigma_5(\frac{n}{5})$$
$$+\frac{398439023}{58722804}\,\sigma_5(\frac{n}{10}) + \frac{68258944}{4893567}\,\sigma_5(\frac{n}{20}) + \frac{960956950}{233027}\,\mathfrak{b}_{20,1}(n)$$
$$+\frac{7006131104}{233027}\,\mathfrak{b}_{20,2}(n) + \frac{31247296000}{233027}\,\mathfrak{b}_{20,3}(n) + \frac{31145902336}{233027}\,\mathfrak{b}_{20,4}(n)$$
$$+\frac{15378042050}{233027}\,\mathfrak{b}_{20,5}(n) + \frac{32291496384}{233027}\,\mathfrak{b}_{20,6}(n) + \frac{542908645632}{233027}\,\mathfrak{b}_{20,7}(n)$$
$$+\frac{309602112}{7517}\,\mathfrak{b}_{20,8}(n) + \frac{653138738688}{233027}\,\mathfrak{b}_{20,9}(n) - \frac{678376410624}{233027}\,\mathfrak{b}_{20,10}(n)$$
$$-\frac{13917769216}{7517}\,\mathfrak{b}_{20,11}(n) - \frac{1138525192704}{233027}\,\mathfrak{b}_{20,12}(n)\bigg)\,q^n, \quad (5.12)$$

$$(\,21\,E_2(q^{21}) - E_2(q)\,)\,E_4(q^{21}) = 20 + \sum_{n=1}^{\infty}\bigg(-\frac{1}{223041}\,\sigma_5(n)14 - \frac{350}{31863}\,\sigma_5(\frac{n}{7})$$
$$+\frac{1821}{91}\,\sigma_5(\frac{n}{21}) + \frac{254880}{10621}\,\mathfrak{b}_{21,1}(n) + \frac{77040}{247}\,\mathfrak{b}_{21,2}(n)$$
$$+\frac{11969280}{10621}\,\mathfrak{b}_{21,3}(n) + \frac{4320}{13}\,\mathfrak{b}_{21,4}(n) + \frac{8640}{13}\,\mathfrak{b}_{21,5}(n)$$
$$+\frac{38880}{13}\,\mathfrak{b}_{21,6}(n) + \frac{13200}{13}\,\mathfrak{b}_{21,7}(n) + \frac{11520}{13}\,\mathfrak{b}_{21,8}(n)$$
$$+\frac{132480}{13}\,\mathfrak{b}_{21,9}(n) - \frac{50400}{13}\,\mathfrak{b}_{21,10}(n) - \frac{89280}{13}\,\mathfrak{b}_{21,11}(n) + \frac{4320}{13}\,\mathfrak{b}_{21,12}(n)\bigg)\,q^n, \quad (5.13)$$

$$(21\, E_2(q^{21}) - E_2(q)\,)\, E_4(q) = 20 + \sum_{n=1}^{\infty}\Big(-\frac{10121}{10621}\,\sigma_5(n) + \frac{24010}{10621}\,\sigma_5(\frac{n}{7}) + \frac{243}{13}\,\sigma_5(\frac{n}{21}) + \frac{46134720}{10621}\,\mathfrak{b}_{21,1}(n) + \frac{18925200}{247}\,\mathfrak{b}_{21,2}14(n) + \frac{4185951840}{10621}\,\mathfrak{b}_{21,3}(n) + \frac{5352480}{13}\,\mathfrak{b}_{21,4}(n) + \frac{4415040}{13}\,\mathfrak{b}_{21,5}(n14) + \frac{19867680}{13}\,\mathfrak{b}_{21,6}(n) + \frac{25900560}{13}\,\mathfrak{b}_{21,7}(n) - \frac{1451520}{13}\,\mathfrak{b}_{21,8}(n) + \frac{82373760}{13}\,\mathfrak{b}_{21,9}(n) - \frac{18416160}{13}\,\mathfrak{b}_{21,10}(n) - \frac{9979200}{13}\,\mathfrak{b}_{21,11}(n) - \frac{4082400}{13}\,\mathfrak{b}_{21,12}(n)\Big)\, q^n, \quad (5.14)$$

$$(5\, E_2(q^{5}) - E_2(q)\,)\, E_4(q^3) == 4 + \sum_{n=1}^{\infty}\Big(-\frac{71366}{8934471}\,\sigma_5(n) - \frac{812655}{992719}\,\sigma_5(\frac{n}{3}) - \frac{107126590}{8934471}\,\sigma_5(\frac{n}{5}) + \frac{16694415}{992719}\,\sigma_5(\frac{n}{15}) + \frac{2832680}{141817}\,\mathfrak{b}_{21,1}(n) + \frac{5125480}{10909}\,\mathfrak{b}_{21,2}(n) - \frac{10139840}{10909}\,\mathfrak{b}_{21,3}(n) - \frac{17525320}{10909}\,\mathfrak{b}_{21,4}(n) - \frac{1113650920}{141817}\,\mathfrak{b}_{21,5}(n) + \frac{31878720}{10909}\,\mathfrak{b}_{21,6}(n) + \frac{740168640}{10909}\,\mathfrak{b}_{21,7}(n) - \frac{4481920}{10909}\,\mathfrak{b}_{21,8}(n)\Big)\, q^n, \quad (5.15)$$

$$(5\, E_2(q^{5}) - E_2(q)\,)\, E_4(q^4) == 4 + \sum_{n=1}^{\infty}\Big(-\frac{292153}{117445608}\,\sigma_5(n) - \frac{4512719}{117445608}\,\sigma_5(\frac{n}{2}) - \frac{10268}{699081}\,\sigma_5(\frac{n}{4}) + \frac{1690315}{117445608}\,\sigma_5(\frac{n}{5}) + \frac{25485149}{117445608}\,\sigma_5(\frac{n}{10}) + \frac{6238012}{1631189}\,\sigma_5(\frac{n}{20}) + \frac{5300495}{233027}\,\mathfrak{b}_{20,1}(n) + \frac{2624176}{233027}\,\mathfrak{b}_{20,2}(n) + \frac{14691200}{233027}\,\mathfrak{b}_{20,3}(n) - \frac{196596016}{233027}\,\mathfrak{b}_{20,4}(n) + \frac{150021475}{233027}\,\mathfrak{b}_{20,5}(n) + \frac{645547296}{233027}\,\mathfrak{b}_{20,6}(n) + \frac{1108142208}{233027}\,\mathfrak{b}_{20,7}(n) - \frac{87462672}{7517}\,\mathfrak{b}_{20,8}(n) - \frac{4875587328}{233027}\,\mathfrak{b}_{20,9}(n) - \frac{1996649856}{233027}\,\mathfrak{b}_{20,10}(n) + \frac{126800896}{7517}\,\mathfrak{b}_{20,11}(n) + \frac{18330282624}{233027}\,\mathfrak{b}_{20,12}(n)\Big)\, q^n, \quad (5.16)$$

$$
\begin{aligned}
(\,7\,E_2(q^7) - E_2(q)\,)\,E_4(q^2) == 6 + \sum_{n=1}^{\infty}\Big(& -\frac{2101}{51471}\,\sigma_5(n) - \frac{42020}{51471}\,\sigma_5(\frac{n}{2}) \\
& + \frac{2401}{7353}\,\sigma_5(\frac{n}{7}) + \frac{48020}{7353}\,\sigma_5(\frac{n}{14}) + \frac{2800}{817}\,\mathfrak{b}_{14,1}(n) \\
& + \frac{372480}{817}\,\mathfrak{b}_{14,2}(n) + \frac{2820080}{817}\,\mathfrak{b}_{14,3}(n) - \frac{1644160}{817}\,\mathfrak{b}_{14,4}(n) \\
& + \frac{8142080}{817}\,\mathfrak{b}_{14,5}(n) - \frac{4753920}{817}\,\mathfrak{b}_{14,6}(n) + \frac{36097440}{817}\,\mathfrak{b}_{14,7}(n) \\
& + \frac{34958080}{817}\,\mathfrak{b}_{14,8}(n)\Big)\,q^n, \qquad (5.17)
\end{aligned}
$$

$$
\begin{aligned}
(\,7\,E_2(q^7) - E_2(q)\,)\,E_4(q^3) == 6 + \sum_{n=1}^{\infty}\Big(& -\frac{2101}{223041}\,\sigma_5(n) - \frac{63030}{74347}\,\sigma_5(\frac{n}{3}) \\
& + \frac{2401}{31863}\,\sigma_5(\frac{n}{7}) + \frac{72030}{10621}\,\sigma_5(\frac{n}{21}) - \frac{4930000}{10621}\,\mathfrak{b}_{21,1}(n) \\
& - \frac{2344560}{817}\,\mathfrak{b}_{21,2}(n) - \frac{10694880}{817}\,\mathfrak{b}_{21,3}(n) + \frac{5131200}{817}\,\mathfrak{b}_{21,4}(n) \\
& + \frac{901920}{43}\,\mathfrak{b}_{21,5}(n) - \frac{48257040}{817}\,\mathfrak{b}_{21,6}(n) + \frac{460703520}{10621}\,\mathfrak{b}_{21,7}(n) \\
& + \frac{137650560}{817}\,\mathfrak{b}_{21,8}(n) - \frac{199923120}{817}\,\mathfrak{b}_{21,9}(n) + \frac{55816800}{817}\,\mathfrak{b}_{21,10}(n) \\
& + \frac{439823520}{817}\,\mathfrak{b}_{21,11}(n) + \frac{394960}{817}\,\mathfrak{b}_{21,12}(n)\Big)\,q^n, \qquad (5.18)
\end{aligned}
$$

Proof. These identities follow immediately when one sets for example $(\alpha,\beta) = (1,7), (1,8), (1,14), (1,20), (1,21)$ in Lemma 3.5. In case of $\alpha\beta = 20$, we take all n belonging to the set $\{\,1,2,\dots,16,17,20\,\}$ to obtain a system of 18 linear equations with unknowns X_δ and Y_j, where $\delta \in D(20)$ and $1 \le i \le 12$. □

The second corollary handles the case $E_4(q^\alpha)(\,E_2(q) - \beta\,E_2(q^\beta)\,)$. We only consider the special cases $1 \neq \alpha = \beta$ and $\alpha > 1$ whenever $\beta = 1$.

Corollary 5.3. *It holds that*

$$
\begin{aligned}
E_4(q)\,(\,E_2(q) - 12\,E_2(q^{12})\,) = -11 + \sum_{n=1}^{\infty}\Big(& \frac{1153}{1274}\,\sigma_5(n) - \frac{279}{1274}\,\sigma_5(\frac{n}{2}) - \frac{789}{1274}\,\sigma_5(\frac{n}{3}) \\
- \frac{640}{637}\,\sigma_5(\frac{n}{4}) - \frac{2451}{1274}\,\sigma_5(\frac{n}{6}) - \frac{5184}{637}\,\sigma_5(\frac{n}{12}) & - \frac{200916}{91}\,\mathfrak{b}_{12,1}(n) - \frac{362664}{13}\,\mathfrak{b}_{12,2}(n) \\
- \frac{85968}{7}\,\mathfrak{b}_{12,3}(n) - \frac{23880384}{91}\,\mathfrak{b}_{12,4}(n) & + 3456\,\mathfrak{b}_{12,5}(n) + \frac{841536}{7}\,\mathfrak{b}_{12,6}(n) \\
& - 290304\,\mathfrak{b}_{12,7}(n)\Big)\,q^n, \qquad (5.19)
\end{aligned}
$$

$$E_4(q^{12})\,(\,E_2(q)-12\,E_2(q^{12})\,)=-11+\sum_{n=1}^{\infty}\biggl(-\frac{1}{61152}\,\sigma_5(n)+\frac{11}{20384}\,\sigma_5(\frac{n}{2})+\frac{61}{20384}\,\sigma_5(\frac{n}{3})+\frac{20}{1911}\,\sigma_5(\frac{n}{4})+\frac{899}{20384}\,\sigma_5(\frac{n}{6})-\frac{7044}{637}\,\sigma_5(\frac{n}{12})-\frac{8739}{364}\,\mathfrak{b}_{12,1}(n)-\frac{39321}{182}\,\mathfrak{b}_{12,2}(n)+\frac{837}{7}\,\mathfrak{b}_{12,3}(n)-\frac{80484}{91}\,\mathfrak{b}_{12,4}(n)-504\,\mathfrak{b}_{12,5}(n)-\frac{324}{7}\,\mathfrak{b}_{12,6}(n)-864\,\mathfrak{b}_{12,7}(n)\biggr)\,q^n, \quad (5.20)$$

$$E_4(q)\,(\,E_2(q)-15\,E_2(q^{15})\,)=-14+\sum_{n=1}^{\infty}\biggl(\frac{917834}{992719}\,\sigma_5(n)+\frac{1460295}{992719}\,\sigma_5(\frac{n}{3})+\frac{104862815}{992719}\,\sigma_5(\frac{n}{5})-\frac{121139010}{992719}\,\sigma_5(\frac{n}{15})-\frac{413824680}{141817}\,\mathfrak{b}_{15,1}(n)-\frac{756726120}{10909}\,\mathfrak{b}_{15,2}(n)+\frac{93504960}{10909}\,\mathfrak{b}_{15,3}(n)-\frac{263947320}{10909}\,\mathfrak{b}_{15,4}(n)-\frac{90853616520}{141817}\,\mathfrak{b}_{15,5}(n)-\frac{283798080}{10909}\,\mathfrak{b}_{15,6}(n)-\frac{3149876160}{10909}\,\mathfrak{b}_{15,7}(n)+\frac{339972480}{10909}\,\mathfrak{b}_{15,8}(n)\biggr)\,q^n, \quad (5.21)$$

$$E_4(q^{15})\,(\,E_2(q)-15\,E_2(q^{15})\,)=-14+\sum_{n=1}^{\infty}\biggl(-\frac{274}{8934471}\,\sigma_5(n)-\frac{5457}{992719}\,\sigma_5(\frac{n}{3})+\frac{8090239}{8934471}\,\sigma_5(\frac{n}{5})-\frac{14791494}{992719}\,\sigma_5(\frac{n}{15})-\frac{3405800}{141817}\,\mathfrak{b}_{15,1}(n)-\frac{1158952}{10909}\,\mathfrak{b}_{15,12}(n)+\frac{35264}{10909}\,\mathfrak{b}_{15,3}(n)-\frac{2061176}{10909}\,\mathfrak{b}_{15,4}(n)-\frac{177635528}{141817}\,\mathfrak{b}_{15,5}(n)-\frac{2367936}{10909}\,\mathfrak{b}_{15,6}(n)-\frac{34294464}{10909}\,\mathfrak{b}_{15,7}(n)-\frac{1203968}{10909}\,\mathfrak{b}_{15,8}(n)\biggr)\,q^n, \quad (5.22)$$

$$E_4(q)\,(\,E_2(q)-20\,E_2(q^{20})\,)=-19+\sum_{n=1}^{\infty}\biggl(\frac{53619409}{58722804}\,\sigma_5(n)+\frac{188714723}{58722804}\,\sigma_5(\frac{n}{2})$$
$$-\frac{2103232}{1631189}\,\sigma_5(\frac{n}{4})-\frac{9257815}{8388972}\,\sigma_5(\frac{n}{5})-\frac{398439023}{58722804}\,\sigma_5(\frac{n}{10})$$
$$-\frac{68258944}{4893567}\,\sigma_5(\frac{n}{20})-\frac{960956950}{233027}\,\mathfrak{b}_{20,1}(n)-\frac{7006131104}{233027}\,\mathfrak{b}_{20,12}(n)$$
$$-\frac{31247296000}{233027}\,\mathfrak{b}_{20,3}(n)-\frac{31145902336}{233027}\,\mathfrak{b}_{20,4}(n)-\frac{15378042050}{233027}\,\mathfrak{b}_{20,5}(n)$$
$$-\frac{32291496384}{233027}\,\mathfrak{b}_{20,6}(n)-\frac{542908645632}{233027}\,\mathfrak{b}_{20,7}(n)-\frac{309602112}{7517}\,\mathfrak{b}_{20,8}(n)$$
$$-\frac{653138738688}{233027}\,\mathfrak{b}_{20,9}(n)+\frac{678376410624}{233027}\,\mathfrak{b}_{20,10}(n)+\frac{13917769216}{7517}\,\mathfrak{b}_{20,11}(n)$$
$$+\frac{1138525192704}{233027}\,\mathfrak{b}_{20,12}(n)\biggr)\,q^n, \quad (5.23)$$

$$E_4(q^{20})\,(\,E_2(q)-20\,E_2(q^{20})\,)=-19+\sum_{n=1}^{\infty}\biggl(-\frac{1567}{117445608}\,\sigma_5(n)+\frac{110791}{117445608}\,\sigma_5(\frac{n}{2})$$
$$+\frac{10004}{233027}\,\sigma_5(\frac{n}{4})+\frac{702215}{234891216}\,\sigma_5(\frac{n}{5})+\frac{10264633}{234891216}\,\sigma_5(\frac{n}{10})$$
$$-\frac{93420884}{4893567}\,\sigma_5(\frac{n}{20})-\frac{5594215}{233027}\,\mathfrak{b}_{20,1}(n)-\frac{16718864}{233027}\,\mathfrak{b}_{20,2}(n)$$
$$-\frac{89883520}{233027}\,\mathfrak{b}_{20,3}(n)-\frac{32106736}{233027}\,\mathfrak{b}_{20,4}(n)+\frac{527968775}{466054}\,\mathfrak{b}_{20,5}(n)$$
$$-\frac{86133984}{233027}\,\mathfrak{b}_{20,6}(n)-\frac{1416189312}{233027}\,\mathfrak{b}_{20,7}(n)-\frac{4970832}{7517}\,\mathfrak{b}_{20,8}(n)$$
$$-\frac{976297728}{233027}\,\mathfrak{b}_{20,9}(n)-\frac{618500736}{233027}\,\mathfrak{b}_{20,10}(n)+\frac{64669696}{7517}\,\mathfrak{b}_{20,11}(n)$$
$$+\frac{4227164544}{233027}\,\mathfrak{b}_{20,12}(n)\biggr)\,q^n, \quad (5.24)$$

$$E_4(q)\,(\,E_2(q)-21\,E_2(q^{21})\,)=-20+\sum_{n=1}^{\infty}\biggl(\frac{10121}{10621}\,\sigma_5(n)-\frac{4050}{10621}\,\sigma_5(\frac{n}{3})$$
$$-\frac{24010}{10621}\,\sigma_5(\frac{n}{7})-\frac{194481}{10621}\,\sigma_5(\frac{n}{21})+\frac{75600}{10621}\,\mathfrak{b}_{21,1}(n)$$
$$-\frac{27075600}{817}\,\mathfrak{b}_{21,2}(n)-\frac{233275680}{817}\,\mathfrak{b}_{21,3}(n)-\frac{400645440}{817}\,\mathfrak{b}_{21,4}(n)$$
$$-\frac{171387360}{817}\,\mathfrak{b}_{21,5}(n)-\frac{997680240}{817}\,\mathfrak{b}_{21,6}(n)-\frac{23349501360}{10621}\,\mathfrak{b}_{21,7}(n)$$
$$+\frac{430513920}{817}\,\mathfrak{b}_{21,8}(n)-\frac{4488019920}{817}\,\mathfrak{b}_{21,9}(n)+\frac{561042720}{817}\,\mathfrak{b}_{21,10}(n)$$
$$-\frac{162254880}{817}\,\mathfrak{b}_{21,11}(n)-\frac{3554640}{817}\,\mathfrak{b}_{21,12}(n)\biggr)\,q^n, \quad (5.25)$$

$$E_4(q^{21})\,(\,E_2(q)-21\,E_2(q^{21})\,) = -20+\sum_{n=1}^{\infty}\Big(\frac{1}{223041}\,\sigma_5(n)+\frac{30}{74347}\,\sigma_5(\frac{n}{3}) + \frac{350}{31863}\,\sigma_5(\frac{n}{37})-\frac{212541}{10621}\,\sigma_5(\frac{n}{21})-\frac{80}{10621}\,\mathfrak{b}_{21,1}(n) - \frac{58800}{817}\,\mathfrak{b}_{21,2}(n)-\frac{430560}{817}\,\mathfrak{b}_{21,3}(n)-\frac{624000}{817}\,\mathfrak{b}_{21,4}(n) + \frac{2400}{43}\,\mathfrak{b}_{21,5}(n)-\frac{1026960}{817}\,\mathfrak{b}_{21,6}(n)-\frac{22742160}{10621}\,\mathfrak{b}_{21,7}(n) + \frac{1159680}{817}\,\mathfrak{b}_{21,8}(n)-\frac{4078800}{817}\,\mathfrak{b}_{21,9}(n)-\frac{126240}{817}\,\mathfrak{b}_{21,10}(n) + \frac{1260000}{817}\,\mathfrak{b}_{21,11}(n)-\frac{19600}{817}\,\mathfrak{b}_{21,12}(n)\Big)\,q^n. \tag{5.26}$$

Proof. Similar to that of Corollary 5.2. □

We now state and prove our main result of this section.

Corollary 5.4. *Let n be a positive integer. Then*

$$W^{3,1}_{(1,3)}(n) = W^{1,3}_{(3,1)}(n) = \frac{1}{104}\,\sigma_5(n)+\frac{81}{1040}\,\sigma_5(\frac{n}{3})+\frac{1}{24}\,\sigma_3(n)-\frac{1}{24}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{3})-\frac{1}{104}\,\mathfrak{b}_{9,1}(n), \tag{5.27}$$

$$W^{3,1}_{(3,1)}(n) = W^{1,3}_{(1,3)}(n) = \frac{1}{1040}\,\sigma_5(n)+\frac{9}{104}\,\sigma_5(\frac{n}{3})+\frac{1}{24}\,\sigma_3(\frac{n}{3})-\frac{1}{8}\,n\,\sigma_3(\frac{n}{3}) - \frac{1}{240}\,\sigma(n)+\frac{1}{312}\,\mathfrak{b}_{9,1}(n), \tag{5.28}$$

$$W^{3,1}_{(1,4)}(n) = W^{1,3}_{(4,1)}(n) = \frac{1}{192}\,\sigma_5(n)+\frac{1}{64}\,\sigma_5(\frac{n}{2})+\frac{1}{15}\,\sigma_5(\frac{n}{4})+\frac{1}{24}\,\sigma_3(n)-\frac{1}{32}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{4})-\frac{1}{64}\,\mathfrak{b}_{32,1}(n), \tag{5.29}$$

$$W^{3,1}_{(4,1)}(n) = W^{1,3}_{(1,4)}(n) = \frac{1}{3840}\,\sigma_5(n)+\frac{1}{256}\,\sigma_5(\frac{n}{2})+\frac{1}{12}\,\sigma_5(\frac{n}{4})+\frac{1}{24}\,\sigma_3(\frac{n}{4})-\frac{1}{8}\,n\,\sigma_3(\frac{n}{4}) - \frac{1}{240}\,\sigma(n)+\frac{1}{256}\,\mathfrak{b}_{32,1}(n), \tag{5.30}$$

$$W^{3,1}_{(1,6)}(n) = W^{1,3}_{(6,1)}(n) = \frac{5}{2184}\,\sigma_5(n)+\frac{2}{273}\,\sigma_5(\frac{n}{2})+\frac{27}{1456}\,\sigma_5(\frac{n}{3})+\frac{27}{455}\,\sigma_5(\frac{n}{6}) + \frac{1}{24}\,\sigma_3(n)-\frac{1}{48}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{6})-\frac{101}{4368}\,\mathfrak{b}_{12,1}(n) - \frac{121}{546}\,\mathfrak{b}_{12,2}(n)+\frac{3}{14}\,\mathfrak{b}_{12,3}(n), \tag{5.31}$$

$$W^{3,1}_{(6,1)}(n) = W^{1,3}_{(1,6)}(n) = \frac{1}{21840}\,\sigma_5(n)+\frac{1}{1092}\,\sigma_5(\frac{n}{2})+\frac{3}{728}\,\sigma_5(\frac{n}{3})+\frac{15}{182}\,\sigma_5(\frac{n}{6}) + \frac{1}{24}\,\sigma_3(\frac{n}{6})-\frac{1}{8}\,n\,\sigma_3(\frac{n}{6})-\frac{1}{240}\,\sigma(n)+\frac{3}{728}\,\mathfrak{b}_{12,1}(n) + \frac{19}{546}\,\mathfrak{b}_{12,2}(n)-\frac{1}{28}\,\mathfrak{b}_{12,3}(n), \tag{5.32}$$

$$W_{(2,3)}^{3,1}(n) = W_{(3,2)}^{1,3}(n) = \frac{1}{2184}\,\sigma_5(n) + \frac{5}{546}\,\sigma_5(\frac{n}{2}) + \frac{27}{7280}\,\sigma_5(\frac{n}{3}) + \frac{27}{364}\,\sigma_5(\frac{n}{6}) + \frac{1}{24}\,\sigma_3(\frac{n}{2}) - \frac{1}{24}\,n\,\sigma_3(\frac{n}{2}) - \frac{1}{240}\,\sigma(\frac{n}{3}) - \frac{1}{2184}\,\mathfrak{b}_{12,1}(n) + \frac{4}{273}\,\mathfrak{b}_{12,2}(n) - \frac{3}{28}\,\mathfrak{b}_{12,3}(n), \tag{5.33}$$

$$W_{(3,2)}^{3,1}(n) = W_{(2,3)}^{1,3}(n) = \frac{1}{4368}\,\sigma_5(n) + \frac{1}{1365}\,\sigma_5(\frac{n}{2}) + \frac{15}{728}\,\sigma_5(\frac{n}{3}) + \frac{6}{91}\,\sigma_5(\frac{n}{6}) + \frac{1}{24}\,\sigma_3(\frac{n}{3}) - \frac{1}{16}\,n\,\sigma_3(\frac{n}{3}) - \frac{1}{240}\,\sigma(\frac{n}{2}) - \frac{1}{4368}\,\mathfrak{b}_{12,1}(n) - \frac{1}{182}\,\mathfrak{b}_{12,2}(n) + \frac{1}{14}\,\mathfrak{b}_{12,3}(n), \tag{5.34}$$

$$W_{(1,12)}^{3,1}(n) = W_{(12,1)}^{1,3}(n) = \frac{121}{174720}\,\sigma_5(n) + \frac{93}{58240}\,\sigma_5(\frac{n}{2}) + \frac{263}{58240}\,\sigma_5(\frac{n}{3}) + \frac{2}{273}\,\sigma_5(\frac{n}{4}) + \frac{817}{58240}\,\sigma_5(\frac{n}{6}) + \frac{27}{455}\,\sigma_5(\frac{n}{12}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{96}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{12}) - \frac{5581}{174720}\,\mathfrak{b}_{12,1}(n) - \frac{1679}{4160}\,\mathfrak{b}_{12,2}(n) - \frac{199}{1120}\,\mathfrak{b}_{12,3}(n) - \frac{41459}{10920}\,\mathfrak{b}_{12,4}(n) + \frac{1}{20}\,\mathfrak{b}_{12,5}(n) + \frac{487}{280}\,\mathfrak{b}_{12,6}(n) - \frac{21}{5}\,\mathfrak{b}_{12,7}(n) \tag{5.35}$$

$$W_{(12,1)}^{3,1}(n) = W_{(1,12)}^{1,3}(n) = -\frac{1}{118560}\,\sigma_5(n) + \frac{3}{55328}\,\sigma_5(\frac{n}{2}) + \frac{17}{63232}\,\sigma_5(\frac{n}{3}) + \frac{1}{1092}\,\sigma_5(\frac{n}{4}) + \frac{1705}{442624}\,\sigma_5(\frac{n}{6}) + \frac{15}{182}\,\sigma_5(\frac{n}{12}) + \frac{1}{24}\,\sigma_3(\frac{n}{12}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{12}) - \frac{1}{240}\,\sigma(n) + \frac{33}{7904}\,\mathfrak{b}_{12,1}(n) + \frac{55}{1456}\,\mathfrak{b}_{12,2}(n) - \frac{93}{4864}\,\mathfrak{b}_{12,3}(n) + \frac{2545}{10374}\,\mathfrak{b}_{12,4}(n) + \frac{1}{38}\,\mathfrak{b}_{12,5}(n) - \frac{3}{133}\,\mathfrak{b}_{12,6}(n) + \frac{15}{38}\,\mathfrak{b}_{12,7}(n), \tag{5.36}$$

$$W_{(3,4)}^{3,1}(n) = W_{(4,3)}^{1,3}(n) = \frac{1}{13440}\,\sigma_5(n) + \frac{9}{58240}\,\sigma_5(\frac{n}{2}) + \frac{23}{4480}\,\sigma_5(\frac{n}{3}) + \frac{1}{1365}\,\sigma_5(\frac{n}{4}) + \frac{901}{58240}\,\sigma_5(\frac{n}{6}) + \frac{6}{91}\,\sigma_5(\frac{n}{12}) + \frac{1}{24}\,\sigma_3(\frac{n}{4}) - \frac{1}{24}\,n\,\sigma_3(\frac{n}{4}) - \frac{1}{240}\,\sigma(\frac{n}{3}) - \frac{1}{13440}\,\mathfrak{b}_{12,1}(n) - \frac{89}{29120}\,\mathfrak{b}_{12,2}(n) + \frac{33}{1120}\,\mathfrak{b}_{12,3}(n) + \frac{71}{3640}\,\mathfrak{b}_{12,4}(n) - \frac{7}{20}\,\mathfrak{b}_{12,5}(n) + \frac{111}{280}\,\mathfrak{b}_{12,6}(n) - \frac{3}{5}\,\mathfrak{b}_{12,7}(n) \tag{5.37}$$

$$W_{(4,3)}^{3,1}(n) = W_{(3,4)}^{1,3}(n) = -\frac{1}{698880}\,\sigma_5(n) + \frac{107}{232960}\,\sigma_5(\frac{n}{2}) + \frac{61}{232960}\,\sigma_5(\frac{n}{3}) + \frac{5}{546}\,\sigma_5(\frac{n}{4}) + \frac{803}{232960}\,\sigma_5(\frac{n}{6}) + \frac{27}{364}\,\sigma_5(\frac{n}{12}) + \frac{1}{24}\,\sigma_3(\frac{n}{3}) - \frac{1}{32}\,n\,\sigma_3(\frac{n}{3}) - \frac{1}{240}\,\sigma(\frac{n}{4}) + \frac{1}{698880}\,\mathfrak{b}_{12,1}(n) - \frac{47}{116480}\,\mathfrak{b}_{12,2}(n) + \frac{19}{4480}\,\mathfrak{b}_{12,3}(n) + \frac{5099}{43680}\,\mathfrak{b}_{12,4}(n) - \frac{1}{80}\,\mathfrak{b}_{12,5}(n) - \frac{127}{1120}\,\mathfrak{b}_{12,6}(n) + \frac{21}{20}\,\mathfrak{b}_{12,7}(n), \tag{5.38}$$

$$W_{(1,7)}^{1,3}(n) = W_{(7,1)}^{3,1}(n) = \frac{7}{196080}\,\sigma_5(n) + \frac{1715}{19608}\,\sigma_5(\frac{n}{7}) + \frac{1}{24}\,\sigma_3(\frac{n}{7}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{7}) - \frac{1}{240}\,\sigma(n) + \frac{27}{6536}\,\mathfrak{b}_{21,1}(n) + \frac{1}{19}\,\mathfrak{b}_{21,2}(n) + \frac{1171}{6536}\,\mathfrak{b}_{21,3}(n), \tag{5.39}$$

$$W_{(7,1)}^{1,3}(n) = W_{(1,7)}^{3,1}(n) = \frac{35}{19608}\,\sigma_5(n) + \frac{16807}{196080}\,\sigma_5(\frac{n}{7}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{56}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{7}) - \frac{1171}{45752}\,\mathfrak{b}_{21,1}(n) - \frac{7}{19}\,\mathfrak{b}_{21,2}(n) - \frac{9261}{6536}\,\mathfrak{b}_{21,3}(n), \tag{5.40}$$

$$W_{(1,8)}^{1,3}(n) = W_{(8,1)}^{3,1}(n) = \frac{1}{61440}\,\sigma_5(n) + \frac{1}{4096}\,\sigma_5(\frac{n}{2}) + \frac{1}{256}\,\sigma_5(\frac{n}{4}) + \frac{1}{12}\,\sigma_5(\frac{n}{8}) + \frac{1}{24}\,\sigma_3(\frac{n}{8}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{8}) - \frac{1}{240}\,\sigma(n) + \frac{17}{4096}\,\mathfrak{b}_{32,1}(n) + \frac{3}{256}\,\mathfrak{b}_{32,2}(n) + \frac{1}{16}\,\mathfrak{b}_{32,3}(n) \tag{5.41}$$

$$W_{(8,1)}^{1,3}(n) = W_{(1,8)}^{3,1}(n) = \frac{1}{768}\,\sigma_5(n) + \frac{1}{256}\,\sigma_5(\frac{n}{2}) + \frac{1}{64}\,\sigma_5(\frac{n}{4}) + \frac{1}{15}\,\sigma_3(\frac{n}{8}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{64}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{8}) - \frac{7}{256}\,\mathfrak{b}_{32,1}(n) - \frac{9}{64}\,\mathfrak{b}_{32,2}(n) - \frac{1}{2}\,\mathfrak{b}_{32,3}(n) \tag{5.42}$$

$$W_{(1,14)}^{1,3}(n) = W_{(14,1)}^{3,1}(n) = \frac{1}{588240}\,\sigma_5(n) + \frac{1}{29412}\,\sigma_5(\frac{n}{2}) + \frac{245}{58824}\,\sigma_5(\frac{n}{7}) + \frac{1225}{14706}\,\sigma_5(\frac{n}{14}) + \frac{1}{24}\,\sigma_3(\frac{n}{14}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{14}) - \frac{1}{240}\,\sigma(n) + \frac{245}{58824}\,\mathfrak{b}_{14,1}(n) + \frac{265}{4902}\,\mathfrak{b}_{14,2}(n) + \frac{11461}{58824}\,\mathfrak{b}_{14,3}(n) + \frac{799}{14706}\,\mathfrak{b}_{14,4}(n) + \frac{818}{7353}\,\mathfrak{b}_{14,5}(n) + \frac{2276}{2451}\,\mathfrak{b}_{14,6}(n) - \frac{2995}{9804}\,\mathfrak{b}_{14,7}(n) - \frac{2}{7353}\,\mathfrak{b}_{14,8}(n), \tag{5.43}$$

$$\begin{aligned} W_{(14,1)}^{1,3}(n) \;=\; W_{(1,14)}^{3,1}(n) \;=& \frac{25}{58824}\,\sigma_5(n)+\frac{10}{7353}\,\sigma_5(\frac{n}{2})+\frac{2401}{117648}\,\sigma_5(\frac{n}{7})+\frac{2401}{36765}\,\sigma_5(\frac{n}{14}) \\ &+\frac{1}{24}\,\sigma_3(n)-\frac{1}{112}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{14})-\frac{27311}{823536}\,\mathfrak{b}_{14,1}(n) \\ &-\frac{5503}{9804}\,\mathfrak{b}_{14,2}(n)-\frac{320881}{117648}\,\mathfrak{b}_{14,3}(n)-\frac{34217}{14706}\,\mathfrak{b}_{14,4}(n) \\ &-\frac{11398}{7353}\,\mathfrak{b}_{14,5}(n)-\frac{56020}{2451}\,\mathfrak{b}_{14,6}(n)+\frac{20983}{4902}\,\mathfrak{b}_{14,7}(n) \\ &-\frac{80}{7353}\,\mathfrak{b}_{14,8}(n), \end{aligned} \tag{5.44}$$

$$\begin{aligned} W_{(1,15)}^{1,3}(n) \;=\; W_{(15,1)}^{3,1}(n) \;=& -\frac{137}{51054120}\,\sigma_5(n)-\frac{5457}{11345360}\,\sigma_5(\frac{n}{3})+\frac{8090239}{102108240}\,\sigma_5(\frac{n}{5}) \\ &+\frac{99291}{11345360}\,\sigma_5(\frac{n}{15})+\frac{1}{24}\,\sigma_3(\frac{n}{15})-\frac{1}{8}\,n\,\sigma_3(\frac{n}{15})-\frac{1}{240}\,\sigma(n) \\ &+\frac{85145}{20421648}\,\mathfrak{b}_{15,1}(n)+\frac{144869}{7854480}\,\mathfrak{b}_{15,2}(n)-\frac{551}{981810}\,\mathfrak{b}_{15,3}(n) \\ &+\frac{257647}{7854480}\,\mathfrak{b}_{15,4}(n)+\frac{22204441}{102108240}\,\mathfrak{b}_{15,5}(n)+\frac{4111}{109090}\,\mathfrak{b}_{15,6}(n) \\ &+\frac{59539}{109090}\,\mathfrak{b}_{15,7}(n)+\frac{9406}{490905}\,\mathfrak{b}_{15,8}(n), \end{aligned} \tag{5.45}$$

$$\begin{aligned} W_{(15,1)}^{1,3}(n) \;=\; W_{(1,15)}^{3,1}(n) \;=& \frac{14977}{34036080}\,\sigma_5(n)-\frac{97353}{11345360}\,\sigma_5(\frac{n}{3})-\frac{20972563}{34036080}\,\sigma_5(\frac{n}{5}) \\ &+\frac{4037967}{5672680}\,\sigma_5(\frac{n}{15})+\frac{1}{24}\,\sigma_3(n)-\frac{1}{120}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{15}) \\ &-\frac{383171}{11345360}\,\mathfrak{b}_{15,1}(n)-\frac{2102017}{2618160}\,\mathfrak{b}_{15,2}(n)+\frac{32467}{327270}\,\mathfrak{b}_{15,3}(n) \\ &-\frac{733187}{2618160}\,\mathfrak{b}_{15,4}(n)-\frac{84123719}{11345360}\,\mathfrak{b}_{15,5}(n)-\frac{32847}{109090}\,\mathfrak{b}_{15,6}(n) \\ &-\frac{364569}{109090}\,\mathfrak{b}_{15,7}(n)+\frac{59023}{163635}\,\mathfrak{b}_{15,8}(n), \end{aligned} \tag{5.46}$$

$$\begin{aligned} W_{(1,20)}^{1,3}(n) \;=\; W_{(20,1)}^{3,1}(n) \;=\; & -\frac{1567}{1342235520}\,\sigma_5(n)+\frac{110791}{1342235520}\,\sigma_5(\frac{n}{2})+\frac{17507}{4660540}\,\sigma_5(\frac{n}{4}) \\ & +\frac{140443}{536894208}\,\sigma_5(\frac{n}{5})+\frac{10264633}{2684471040}\,\sigma_5(\frac{n}{10}) \\ & +\frac{556307}{6990810}\,\sigma_5(\frac{n}{20})+\frac{1}{24}\,\sigma_3(\frac{n}{20})-\frac{1}{8}\,n\,\sigma_3(\frac{n}{20})-\frac{1}{240}\,\sigma(n) \\ & +\frac{1118843}{268447104}\,\mathfrak{b}_{20,1}(n)+\frac{1044929}{83889720}\,\mathfrak{b}_{20,2}(n) \\ & +\frac{140443}{2097243}\,\mathfrak{b}_{20,3}(n)+\frac{2006671}{83889720}\,\mathfrak{b}_{20,4}(n) \\ & -\frac{105593755}{536894208}\,\mathfrak{b}_{20,5}(n)+\frac{897229}{13981620}\,\mathfrak{b}_{20,6}(n) \\ & +\frac{1229331}{1165135}\,\mathfrak{b}_{20,7}(n)+\frac{103559}{902040}\,\mathfrak{b}_{20,8}(n) \\ & +\frac{2542442}{3495405}\,\mathfrak{b}_{20,9}(n)+\frac{536893}{1165135}\,\mathfrak{b}_{20,10}(n) \\ & -\frac{505232}{338265}\,\mathfrak{b}_{20,11}(n)-\frac{11008241}{3495405}\,\mathfrak{b}_{20,12}(n), \end{aligned} \tag{5.47}$$

$$\begin{aligned} W_{(20,1)}^{1,3}(n) \;=\; W_{(1,20)}^{3,1}(n) \;=\; & \frac{1020679}{2684471040}\,\sigma_5(n)-\frac{188714723}{13422355200}\,\sigma_5(\frac{n}{2})+\frac{32863}{5825675}\,\sigma_5(\frac{n}{4}) \\ & +\frac{12960941}{2684471040}\,\sigma_5(\frac{n}{5})+\frac{398439023}{13422355200}\,\sigma_5(\frac{n}{10}) \\ & +\frac{1066546}{17477025}\,\sigma_5(\frac{n}{20})+\frac{1}{24}\,\sigma_3(n)-\frac{1}{160}\,n\,\sigma_3(n)-\frac{1}{240}\,\sigma(\frac{n}{20}) \\ & -\frac{19219139}{536894208}\,\mathfrak{b}_{20,1}(n)-\frac{218941597}{838897200}\,\mathfrak{b}_{20,2}(n) \\ & -\frac{2441195}{2097243}\,\mathfrak{b}_{20,3}(n)-\frac{121663681}{104862150}\,\mathfrak{b}_{20,4}(n) \\ & -\frac{307560841}{536894208}\,\mathfrak{b}_{20,5}(n)-\frac{168184877}{139816200}\,\mathfrak{b}_{20,6}(n) \\ & -\frac{235637433}{11651350}\,\mathfrak{b}_{20,7}(n)-\frac{1612511}{4510200}\,\mathfrak{b}_{20,8}(n) \\ & -\frac{425220533}{17477025}\,\mathfrak{b}_{20,9}(n)+\frac{147217103}{5825675}\,\mathfrak{b}_{20,10}(n) \\ & +\frac{27183143}{1691325}\,\mathfrak{b}_{20,11}(n)+\frac{741227339}{17477025}\,\mathfrak{b}_{20,12}(n), \end{aligned} \tag{5.48}$$

$$\begin{aligned} W^{1,3}_{(1,21)}(n) \;=\; W^{3,1}_{(21,1)}(n) \;=& \frac{1}{2549040}\,\sigma_5(n) + \frac{245}{254904}\,\sigma_5(\frac{n}{7}) + \frac{9}{104}\,\sigma_5(\frac{n}{21}) \\ &+ \frac{1}{24}\,\sigma_3(\frac{n}{21}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{21}) - \frac{1}{240}\,\sigma(n) \\ &+ \frac{177}{42484}\,\mathfrak{b}_{21,1}(n) + \frac{107}{1976}\,\mathfrak{b}_{21,2}(n) + \frac{2078}{10621}\,\mathfrak{b}_{21,3}(n) \\ &+ \frac{3}{52}\,\mathfrak{b}_{21,4}(n) + \frac{3}{26}\,\mathfrak{b}_{21,5}(n) + \frac{27}{52}\,\mathfrak{b}_{21,6}(n) \\ &+ \frac{55}{312}\,\mathfrak{b}_{21,7}(n) + \frac{2}{13}\,\mathfrak{b}_{21,8}(n) + \frac{23}{13}\,\mathfrak{b}_{21,9}(n) \\ &- \frac{35}{52}\,\mathfrak{b}_{21,10}(n) - \frac{31}{26}\,\mathfrak{b}_{21,11}(n) + \frac{3}{52}\,\mathfrak{b}_{21,12}(n), \end{aligned} \tag{5.49}$$

$$\begin{aligned} W^{1,3}_{(21,1)}(n) \;=\; W^{3,1}_{(1,21)}(n) \;=& \frac{25}{127452}\,\sigma_5(n) + \frac{2401}{254904}\,\sigma_5(\frac{n}{7}) + \frac{81}{1040}\,\sigma_5(\frac{n}{21}) \\ &+ \frac{1}{24}\,\sigma_3(n) - \frac{1}{168}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{21}) - \frac{16019}{446082}\,\mathfrak{b}_{21,1}(n) \\ &- \frac{3755}{5928}\,\mathfrak{b}_{21,2}(n) - \frac{415273}{127452}\,\mathfrak{b}_{21,3}(n) - \frac{177}{52}\,\mathfrak{b}_{21,4}(n) \\ &- \frac{73}{26}\,\mathfrak{b}_{21,5}(n) - \frac{657}{52}\,\mathfrak{b}_{21,6}(n) - \frac{1713}{104}\,\mathfrak{b}_{21,7}(n) \\ &+ \frac{12}{13}\,\mathfrak{b}_{21,8}(n) - \frac{681}{13}\,\mathfrak{b}_{21,9}(n) + \frac{609}{52}\,\mathfrak{b}_{21,10}(n) \\ &+ \frac{165}{26}\,\mathfrak{b}_{21,11}(n) + \frac{135}{52}\,\mathfrak{b}_{21,12}(n), \end{aligned} \tag{5.50}$$

$$\begin{aligned} W^{1,3}_{(5,3)}(n) \;=\; W^{3,1}_{(3,5)}(n) \;=& \frac{5363}{102108240}\,\sigma_5(n) + \frac{33831}{11345360}\,\sigma_5(\frac{n}{3}) - \frac{10712659}{51054120}\,\sigma_5(\frac{n}{5}) \\ &+ \frac{3338883}{11345360}\,\sigma_5(\frac{n}{15}) + \frac{1}{24}\,\sigma_3(\frac{n}{3}) - \frac{1}{40}\,n\,\sigma_3(\frac{n}{3}) - \frac{1}{240}\,\sigma(\frac{n}{5}) \\ &- \frac{5363}{102108240}\,\mathfrak{b}_{15,1}(n) - \frac{128137}{7854480}\,\mathfrak{b}_{15,2}(n) + \frac{31687}{981810}\,\mathfrak{b}_{15,3}(n) \\ &+ \frac{438133}{7854480}\,\mathfrak{b}_{15,4}(n) + \frac{27841273}{102108240}\,\mathfrak{b}_{15,5}(n) - \frac{11069}{109090}\,\mathfrak{b}_{15,6}(n) \\ &- \frac{257003}{109090}\,\mathfrak{b}_{15,7}(n) + \frac{7003}{490905}\,\mathfrak{b}_{15,8}(n), \end{aligned} \tag{5.51}$$

$$\begin{aligned} W^{1,3}_{(5,4)}(n) \;=\; W^{3,1}_{(4,5)}(n) \;=& \frac{4591}{536894208}\,\sigma_5(n)+\frac{1460777}{13422355200}\,\sigma_5(\frac{n}{2})+\frac{573583}{34954050}\,\sigma_5(\frac{n}{4})\\ &+\frac{338063}{1342235520}\,\sigma_5(\frac{n}{5})+\frac{25485149}{6711177600}\,\sigma_5(\frac{n}{10})\\ &+\frac{1559503}{23302700}\,\sigma_5(\frac{n}{20})+\frac{1}{24}\,\sigma_3(\frac{n}{4})-\frac{1}{40}\,n\,\sigma_3(\frac{n}{4})-\frac{1}{240}\,\sigma(\frac{n}{5})\\ &-\frac{4591}{536894208}\,\mathfrak{b}_{20,1}(n)-\frac{164011}{419448600}\,\mathfrak{b}_{20,2}(n)\\ &-\frac{4591}{2097243}\,\mathfrak{b}_{20,3}(n)+\frac{12287251}{419448600}\,\mathfrak{b}_{20,4}(n)\\ &-\frac{6000859}{268447104}\,\mathfrak{b}_{20,5}(n)-\frac{6724451}{69908100}\,\mathfrak{b}_{20,6}(n)\\ &-\frac{961929}{5825675}\,\mathfrak{b}_{20,7}(n)+\frac{1822139}{4510200}\,\mathfrak{b}_{20,8}(n)\\ &+\frac{12696842}{17477025}\,\mathfrak{b}_{20,9}(n)+\frac{1733203}{5825675}\,\mathfrak{b}_{20,10}(n)\\ &-\frac{990632}{1691325}\,\mathfrak{b}_{20,11}(n)-\frac{47735111}{17477025}\,\mathfrak{b}_{20,12}(n), \end{aligned} \tag{5.52}$$

$$\begin{aligned} W^{1,3}_{(2,7)}(n) \;=\; W^{3,1}_{(7,2)}(n) \;=& \frac{1}{117648}\,\sigma_5(n)+\frac{1}{36765}\,\sigma_5(\frac{n}{2})+\frac{1225}{58824}\,\sigma_5(\frac{n}{7})\\ &+\frac{490}{7353}\,\sigma_5(\frac{n}{14})+\frac{1}{24}\,\sigma_3(\frac{n}{7})-\frac{1}{16}\,n\,\sigma_3(\frac{n}{7})-\frac{1}{240}\,\sigma(\frac{n}{2})\\ &-\frac{1}{117648}\,\mathfrak{b}_{14,1}(n)+\frac{37}{9804}\,\mathfrak{b}_{14,2}(n)+\frac{2455}{117648}\,\mathfrak{b}_{14,3}(n)\\ &+\frac{149}{14706}\,\mathfrak{b}_{14,4}(n)-\frac{326}{7353}\,\mathfrak{b}_{14,5}(n)-\frac{140}{2451}\,\mathfrak{b}_{14,6}(n)\\ &+\frac{1253}{4902}\,\mathfrak{b}_{14,7}(n)+\frac{3920}{7353}\,\mathfrak{b}_{14,8}(n), \end{aligned} \tag{5.53}$$

$$\begin{aligned} W^{1,3}_{(7,2)}(n) \;=\; W^{3,1}_{(2,7)}(n) \;=& \frac{5}{58824}\,\sigma_5(n)+\frac{25}{14706}\,\sigma_5(\frac{n}{2})+\frac{2401}{588240}\,\sigma_5(\frac{n}{7})\\ &+\frac{2401}{29412}\,\sigma_5(\frac{n}{14})+\frac{1}{24}\,\sigma_3(\frac{n}{2})-\frac{1}{56}\,n\,\sigma_3(\frac{n}{2})-\frac{1}{240}\,\sigma(\frac{n}{7})\\ &-\frac{5}{58824}\,\mathfrak{b}_{14,1}(n)-\frac{194}{17157}\,\mathfrak{b}_{14,2}(n)-\frac{35251}{411768}\,\mathfrak{b}_{14,3}(n)\\ &+\frac{367}{7353}\,\mathfrak{b}_{14,4}(n)-\frac{12722}{51471}\,\mathfrak{b}_{14,5}(n)+\frac{2476}{17157}\,\mathfrak{b}_{14,6}(n)\\ &-\frac{75203}{68628}\,\mathfrak{b}_{14,7}(n)-\frac{54622}{51471}\,\mathfrak{b}_{14,8}(n), \end{aligned} \tag{5.54}$$

$$\begin{aligned} W_{(3,7)}^{1,3}(n) \;=\; W_{(7,3)}^{3,1}(n) \;=& \frac{1}{254904}\,\sigma_5(n) + \frac{27}{80}\,\sigma_5(\frac{n}{3}) + \frac{1225}{127452}\,\sigma_5(\frac{n}{7}) \\ &- \frac{27}{104}\,\sigma_5(\frac{n}{21}) + \frac{1}{24}\,\sigma_3(\frac{n}{7}) - \frac{1}{24}\,n\,\sigma_3(\frac{n}{7}) - \frac{1}{240}\,\sigma(\frac{n}{3}) \\ &- \frac{1}{254904}\,\mathfrak{b}_{21,1}(n) - \frac{1}{5928}\,\mathfrak{b}_{21,2}(n) - \frac{85435}{254904}\,\mathfrak{b}_{21,3}(n) \\ &- \frac{35}{52}\,\mathfrak{b}_{21,4}(n) - \frac{9}{26}\,\mathfrak{b}_{21,5}(n) - \frac{1163}{104}\,\mathfrak{b}_{21,6}(n) - \frac{263}{104}\,\mathfrak{b}_{21,7}(n) \\ &+ \frac{7}{13}\,\mathfrak{b}_{21,8}(n) - \frac{7699}{52}\,\mathfrak{b}_{21,9}(n) + \frac{27}{52}\,\mathfrak{b}_{21,10}(n) + \frac{41}{26}\,\mathfrak{b}_{21,11}(n) \\ &- \frac{14337}{26}\,\mathfrak{b}_{21,12}(n), \end{aligned} \tag{5.55}$$

$$\begin{aligned} W_{(7,3)}^{1,3}(n) \;=\; W_{(3,7)}^{3,1}(n) \;=& \frac{5}{254904}\,\sigma_5(n) + \frac{75}{42484}\,\sigma_5(\frac{n}{3}) + \frac{2401}{2549040}\,\sigma_5(\frac{n}{7}) \\ &+ \frac{7203}{84968}\,\sigma_5(\frac{n}{21}) + \frac{1}{24}\,\sigma_3(\frac{n}{3}) - \frac{1}{56}\,n\,\sigma_3(\frac{n}{3}) - \frac{1}{240}\,\sigma(\frac{n}{7}) \\ &+ \frac{16019}{1338246}\,\mathfrak{b}_{21,1}(n) + \frac{9769}{137256}\,\mathfrak{b}_{21,2}(n) + \frac{1061}{3268}\,\mathfrak{b}_{21,3}(n) \\ &- \frac{5345}{34314}\,\mathfrak{b}_{21,4}(n) - \frac{1879}{3612}\,\mathfrak{b}_{21,5}(n) + \frac{201071}{137256}\,\mathfrak{b}_{21,6}(n) \\ &- \frac{319933}{297388}\,\mathfrak{b}_{21,7}(n) - \frac{71693}{17157}\,\mathfrak{b}_{21,8}(n) + \frac{277671}{45752}\,\mathfrak{b}_{21,9}(n) \\ &- \frac{116285}{68628}\,\mathfrak{b}_{21,10}(n) - \frac{305433}{22876}\,\mathfrak{b}_{21,11}(n) - \frac{4937}{411768}\,\mathfrak{b}_{21,12}(n). \end{aligned} \tag{5.56}$$

Proof. These identities follow from Theorem 3.6 when we set for example $(\alpha,\beta) = (1,14)$, $(3,4)$, $(1,20)$, $(5,4)$, $(1,21)$, $(3,7)$. □

6. Evaluation of the Convolution Sums $W_{(\alpha,\beta)}^{2i-1,2j-1}(n)$ for the Levels $\alpha\beta = 9, 16, 18, 27, 32 \in \mathbb{N}^* \setminus \mathfrak{N}$

Let $i, j \in \mathbb{N}^*$ be such that $i + j = 3$. Then we give explicit formulae for the convolution sums $W_{(\alpha,\beta)}^{2i-1,2j-1}(n)$ for $\alpha\beta = 9$, 16, 18, 27, 32. These levels belong to $\mathbb{N}^* \setminus \mathfrak{N}$. Hence, the primitive Dirichlet characters are non-trivial.

6.1. Bases for $E_6(\Gamma_0(\alpha\beta))$ and $S_6(\Gamma_0(\alpha\beta))$ when $\alpha\beta = 18, 27, 32$. By the inclusion relation (2.2), it is sufficient to consider the bases only for the levels 16, 18 18, 27, and 32.

The dimension formulae for the space of cusp forms as given in T. Miyake's book [**18**, Thrm 2.5.2, p. 60] and W. A. Stein's book [**25**, Prop. 6.1, p. 91] and (3.1) are applied to compute

$$\begin{aligned} \dim(E_6(\Gamma_0(9))) &= 4, & \dim(S_6(\Gamma_0(9))) &= 3, \\ \dim(E_6(\Gamma_0(16))) &= 6, & \dim(S_6(\Gamma_0(16))) &= 7, \\ \dim(E_6(\Gamma_0(18))) &= 8, & \dim(S_6(\Gamma_0(18))) &= 11, \\ \dim(E_6(\Gamma_0(27))) &= 6, & \dim(S_6(\Gamma_0(27))) &= 12, \\ \dim(E_6(\Gamma_0(32))) &= 8 & \text{and } \dim(S_6(\Gamma_0(32))) &= 16. \end{aligned}$$

We use Theorem 2.1 to determine many eta quotients which are elements of the spaces $S_6(\Gamma_0(18))$, $S_6(\Gamma_0(27))$ and $S_6(\Gamma_0(32))$, respectively.

Let $D(18)$, 27 and $D(32)$ denote the sets of positive divisors of 18, 27 and 32, respectively.

Corollary 6.1. *The following statement holds.*

(a) Let n be a positive integer, $\chi(n) = \left(\frac{-4}{n}\right)$ and $\psi(n) = \left(\frac{-3}{n}\right)$ be primitive Dirichlet characters such that χ is not an annihilator of $E_6(\Gamma_0(9))$ and ψ is not an annihilator of $E_6(\Gamma_0(16))$. Then the sets

$$\mathcal{B}_{E,18} = \{E_6(q^t) \mid t|18\} \cup \{\, E_{6,\left(\frac{-4}{n}\right)}(q^s) \mid s = 1, 3\,\},$$

$$\mathcal{B}_{E,27} = \{E_6(q^t) \mid t|27\} \cup \{\, E_{6,\left(\frac{-4}{n}\right)}(q^s) \mid s = 1, 3\} \text{ and}$$

$$\mathcal{B}_{E,32} = \{E_6(q^t) \mid t|32\} \cup \{\, E_{6,\left(\frac{-3}{n}\right)}(q^s) \mid s = 1, 2\,\}$$

constitute bases of $E_6(\Gamma_0(18))$, $E_6(\Gamma_0(27))$ and $E_6(\Gamma_0(32))$, respectively.

*(b) Let $1 \le i \le 11$, $1 \le j \le 12$ $1 \le k \le 16$ be positive integers.
Let $\delta_1 \in D(18)$ and $(r(i, \delta_1))_{i,\delta_1}$ be the Table 5 of the powers of $\eta(\delta_1 z)$.
Let $\delta_2 \in D(27)$ and $(r(j, \delta_2))_{j,\delta_2}$ be the Table 8 of the powers of $\eta(\delta_2 z)$.
Let $\delta_3 \in D(32)$ and $(r(k, \delta_3))_{k,\delta_3}$ be the Table 9 of the powers of $\eta(\delta_3 z)$.
Let furthermore*

$$\mathfrak{B}_{18,i}(q) = \prod_{\delta_1|18} \eta^{r(i,\delta_1)}(\delta_1 z), \qquad \mathfrak{B}_{27,j}(q) = \prod_{\delta_2|27} \eta^{r(j,\delta_2)}(\delta_2 z) \text{ and}$$

$$\mathfrak{B}_{32,k}(q) = \prod_{\delta_3|32} \eta^{r(k,\delta_3)}(\delta_3 z)$$

be selected elements of $S_6(\Gamma_0(18))$, $S_6(\Gamma_0(27))$ and $S_6(\Gamma_0(32))$, respectively. The sets

$$\mathcal{B}_{S,18} = \{\mathfrak{B}_{18,i}(q) \mid \ 1 \le i \le 11\}, \qquad \mathcal{B}_{S,27} = \{\mathfrak{B}_{27,j}(q) \mid \ 1 \le j \le 12\} \text{ and}$$

$$\mathcal{B}_{S,32} = \{\mathfrak{B}_{32,k}(q) \mid \ 1 \le k \le 16\}$$

are bases of $S_6(\Gamma_0(18))$, $S_6(\Gamma_0(27))$ and $S_6(\Gamma_0(32))$, respectively.

(c) The sets

$$\mathcal{B}_{M,18} = \mathcal{B}_{E,18} \cup \mathcal{B}_{S,18}, \mathcal{B}_{M,27} = \mathcal{B}_{E,27} \cup \mathcal{B}_{S,27} \text{ and } \mathcal{B}_{M,32} = \mathcal{B}_{E,32} \cup \mathcal{B}_{S,32}$$

constitute bases of $M_6(\Gamma_0(18))$ and $M_6(\Gamma_0(27))$, and $M_6(\Gamma_0(32))$, respectively.

By Remark 3.4 (r1), each $\mathfrak{B}_{\alpha\beta,i}(q)$ is expressible in the form $\sum_{n=1}^{\infty} \mathfrak{b}_{\alpha\beta,i}(n)q^n$.

Note that the basis elements $\mathfrak{B}_{27,12}(q)$ and $\mathfrak{B}_{32,16}(q)$ are elements of the cusp spaces $S_2(\Gamma_0(27))$ and $S_2(\Gamma_0(32))$, respectively.

Proof. It holds that $18 = 3^2 \times 2$. Since $\gcd(4, 3) = 1$, it holds that the primitive Dirichlet character $\chi(n) = \left(\frac{-4}{n}\right)$ is not an annihilator of $E_6(\Gamma_0(3^2))$. Hence, $\chi(n) = \left(\frac{-4}{n}\right)$ is not an annihilator of $E_6(\Gamma_0(9))$, $E_6(\Gamma_0(18))$ and $E_6(\Gamma_0(27))$. Similarly, since $\gcd(3, 4) = 1$, the primitive Dirichlet character $\psi(n) = \left(\frac{-3}{n}\right)$ is not an annihilator of the space $E_6(\Gamma_0(2^4))$. Therefore, $\psi(n) = \left(\frac{-3}{n}\right)$ is not an annihilator of $E_6(\Gamma_0(16))$ and $E_6(\Gamma_0(32))$.

We only give the proof for $\mathcal{B}_{M,18} = \mathcal{B}_{E,18} \cup \mathcal{B}_{S,18}$ since the other cases are proved similarly.

(a) Suppose that $x_\delta, z_1, z_3 \in \mathbb{C}$ with $\delta|18$. Let

$$\sum_{\delta|18} x_\delta\, E_6(q^\delta) + z_1\, E_{6,(\frac{-4}{n})}(q) + z_3\, E_{6,(\frac{-4}{n})}(q^3) = 0.$$

We observe that

$$\left(\frac{-4}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } \gcd(4,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 4). \end{cases} \tag{6.1}$$

and recall that for all $0 \neq a \in \mathbb{Z}$ it holds that $\left(\frac{a}{0}\right) = 0$. Since the conductor of the Dirichlet character $\left(\frac{-4}{n}\right)$ is 4, we infer from (2.1) that $C_0 = 0$. We then deduce

$$\sum_{\delta|18} x_\delta + \sum_{i=1}^{\infty}\Bigg(-504\sum_{\delta|18}\sigma_5(\frac{n}{\delta})x_\delta + \left(\frac{-4}{n}\right)\sigma_5(n)z_1 + \left(\frac{-4}{n}\right)\sigma_5(\frac{n}{3})z_3\Bigg)q^n = 0.$$

Then we equate the coefficients of q^n for $n \in D(18)$ plus for example $n = 5, 7$ to obtain a system of 8 linear equations whose unique solution is $x_\delta = z_1 = z_3 = 0$ with $\delta \in D(18)$. So, the set $\mathcal{B}_E$ is linearly independent. Hence, the set $\mathcal{B}_E$ is a basis of $E_6(\Gamma_0(18))$.

(b) Suppose that $x_i \in \mathbb{C}$ with $1 \leq i \leq 11$. Let $\sum_{i=1}^{11} x_i\, \mathfrak{B}_{18,i}(q) = 0$. Then

$$\sum_{i=1}^{11} x_i \sum_{n=1}^{\infty} \mathfrak{b}_{18,i}(n) q^n = \sum_{n=1}^{\infty}\Bigg(\sum_{i=1}^{11} \mathfrak{b}_{18,i}(n)\, x_i\Bigg)q^n = 0.$$

So, we equate the coefficients of q^n for $1 \leq n \leq 11$ to obtain a system of 11 linear equations whose unique solution is $x_i = 0$ for all $1 \leq i \leq 11$. It follows that the set $\mathcal{B}_S$ is linearly independent. Hence, the set $\mathcal{B}_S$ is a basis of $S_6(\Gamma_0(18))$.

(c) Since $M_6(\Gamma_0(18)) = E_6(\Gamma_0(18)) \oplus S_6(\Gamma_0(18))$, the result follows from (a) and (b).

□

6.2. Evaluation of $\mathbf{W^{2i-1,2j-1}_{(\alpha,\beta)}(n)}$ for $\boldsymbol{\alpha\beta = 9, 16, 18, 27, 32}$. In this section, the evaluation of the convolution sum $W^{2i-1,2j-1}_{(\alpha,\beta)}(n)$ is discussed for $\alpha\beta = 9$, 16, 18, 27 and 32.

In the following corollary we consider $(\,\alpha\, E_2(q^\alpha) - E_2(q)\,)\, E_4(q^\beta)$ and $E_4(q^\alpha)\,(\,E_2(q) - \beta\, E_2(q^\beta)\,)$ simultaneously. We will also make use of Theorem 3.9.

Corollary 6.2. *It holds that*

$$\begin{aligned}(\,9\, E_2(q^9) - E_2(q)\,)\, E_4(q^9) = 8 + \sum_{n=1}^{\infty}\Bigg(&-\frac{1}{7371}\,\sigma_5(n) - \frac{80}{7371}\,\sigma_5(\frac{n}{3}) \\ &+ \frac{729}{91}\,\sigma_5(\frac{n}{9}) + \frac{2800}{117}\,\mathfrak{b}_{18,1}(n) + \frac{640}{3}\,\mathfrak{b}_{18,2}(n) + \frac{6480}{13}\,\mathfrak{b}_{18,3}(n)\Bigg)\,q^n, \end{aligned} \tag{6.2}$$

$$E_4(q)\,(\,E_2(q)-9\,E_2(q^9)\,) = -8+\sum_{n=1}^{\infty}\bigg(\,\frac{81}{91}\,\sigma_5(n)-\frac{80}{91}\,\sigma_5(\frac{n}{3})$$
$$-\frac{729}{91}\,\sigma_5(\frac{n}{9})-\frac{19440}{13}\,\mathfrak{b}_{18,1}(n)-17280\,\mathfrak{b}_{18,2}(n)-\frac{680400}{13}\,\mathfrak{b}_{18,3}(n)\bigg)\,q^n, \quad (6.3)$$

$$(\,16\,E_2(q^{16})-E_2(q)\,)\,E_4(q^{16}) = 15+\sum_{n=1}^{\infty}\bigg(-\frac{1}{86016}\,\sigma_5(n)-\frac{5}{28672}\,\sigma_5(\frac{n}{2})$$
$$-\frac{5}{1792}\,\sigma_5(\frac{n}{4})-\frac{5}{112}\,\sigma_5(\frac{n}{8})+\frac{316}{21}\,\sigma_5(\frac{n}{16})+\frac{12285}{512}\,\mathfrak{b}_{32,1}(n)+\frac{2295}{32}\,\mathfrak{b}_{32,2}(n)$$
$$+\frac{765}{2}\,\mathfrak{b}_{32,3}(n)+\frac{315}{2}\,\mathfrak{b}_{32,4}(n)+360\,\mathfrak{b}_{32,5}(n)+1080\,\mathfrak{b}_{32,6}(n)+1440\,\mathfrak{b}_{32,7}(n)\bigg)\,q^n, \quad (6.4)$$

$$E_4(q)\,(\,E_2(q)-16\,E_2(q^{16})\,) = -15+\sum_{n=1}^{\infty}\bigg(\,\frac{79}{84}\,\sigma_5(n)-\frac{5}{28}\,\sigma_5(\frac{n}{2})$$
$$-\frac{5}{7}\,\sigma_5(\frac{n}{4})-\frac{20}{7}\,\sigma_5(\frac{n}{8})-\frac{256}{21}\,\sigma_5(\frac{n}{16})-3150\,\mathfrak{b}_{32,1}(n)-22680\,\mathfrak{b}_{32,2}(n)$$
$$-92160\,\mathfrak{b}_{32,3}(n)-105120\,\mathfrak{b}_{32,4}(n)-322560\,\mathfrak{b}_{32,5}(n)-414720\,\mathfrak{b}_{32,6}(n)$$
$$-184320\,\mathfrak{b}_{32,7}(n)\bigg)\,q^n, \quad (6.5)$$

$$(\,18\,E_2(q^{18})-E_2(q)\,)\,E_4(q^{18}) = 17+\sum_{n=1}^{\infty}\bigg(-\frac{1}{154791}\,\sigma_5(n)-\frac{20}{154791}\,\sigma_5(\frac{n}{2})$$
$$-\frac{80}{154791}\,\sigma_5(\frac{n}{3})-\frac{1600}{154791}\,\sigma_5(\frac{n}{6})-\frac{30}{637}\,\sigma_5(\frac{n}{9})+\frac{10866}{637}\,\sigma_5(\frac{n}{18})$$
$$+\frac{2792}{351}\,\mathfrak{b}_{18,1}(n)+\frac{293744}{2457}\,\mathfrak{b}_{18,2}(n)+\frac{97960}{819}\,\mathfrak{b}_{18,3}(n)+\frac{61504}{63}\,\mathfrak{b}_{18,4}(n)$$
$$-\frac{408}{7}\,\mathfrak{b}_{18,5}(n)+\frac{2016400}{819}\,\mathfrak{b}_{18,6}(n)-\frac{1216}{21}\,\mathfrak{b}_{18,7}(n)-\frac{10048}{21}\,\mathfrak{b}_{18,8}(n)$$
$$+\frac{6368}{7}\,\mathfrak{b}_{18,9}(n)+\frac{39104}{21}\,\mathfrak{b}_{18,10}(n)+\frac{3032}{189}\,\mathfrak{b}_{18,11}(n)\bigg)\,q^n, \quad (6.6)$$

$$E_4(q)\,(\,E_2(q)-18\,E_2(q^{18})\,) = -17+\sum_{n=1}^{\infty}\bigg(\,\frac{1811}{1911}\,\sigma_5(n)-\frac{320}{1911}\,\sigma_5(\frac{n}{2})$$
$$-\frac{800}{1911}\,\sigma_5(\frac{n}{3})-\frac{2560}{1911}\,\sigma_5(\frac{n}{6})-\frac{2430}{637}\,\sigma_5(\frac{n}{9})-\frac{7776}{637}\,\sigma_5(\frac{n}{18})$$
$$-\frac{21648}{13}\,\mathfrak{b}_{18,1}(n)-\frac{3354816}{91}\,\mathfrak{b}_{18,2}(n)-\frac{5815440}{91}\,\mathfrak{b}_{18,3}(n)-\frac{2562048}{7}\,\mathfrak{b}_{18,4}(n)$$
$$-\frac{1018656}{7}\,\mathfrak{b}_{18,5}(n)-\frac{126129600}{91}\,\mathfrak{b}_{18,6}(n)-\frac{1396224}{7}\,\mathfrak{b}_{18,7}(n)+\frac{1029888}{7}\,\mathfrak{b}_{18,8}(n)$$
$$+\frac{2871936}{7}\,\mathfrak{b}_{18,9}(n)-\frac{3573504}{7}\,\mathfrak{b}_{18,10}(n)-\frac{13728}{7}\,\mathfrak{b}_{18,11}(n)\bigg)\,q^n, \quad (6.7)$$

$$
\begin{aligned}
(\,2\,E_2(q^2) - E_2(q)\,)\,E_4(q^9) = 1 + \sum_{n=1}^{\infty}\Big(& -\frac{11}{154791}\,\sigma_5(n) + \frac{32}{154791}\,\sigma_5(\frac{n}{2}) \\
& - \frac{880}{154791}\,\sigma_5(\frac{n}{3}) + \frac{2560}{154791}\,\sigma_5(\frac{n}{6}) - \frac{330}{637}\,\sigma_5(\frac{n}{9}) + \frac{960}{637}\,\sigma_5(\frac{n}{18}) \\
& + \frac{14176}{351}\,\mathfrak{b}_{18,1}(n) + \frac{651712}{2457}\,\mathfrak{b}_{18,2}(n) + \frac{214880}{819}\,\mathfrak{b}_{18,3}(n) + \frac{512}{63}\,\mathfrak{b}_{18,4}(n) \\
& + \frac{1056}{7}\,\mathfrak{b}_{18,5}(n) + \frac{6080}{819}\,\mathfrak{b}_{18,6}(n) + \frac{36352}{21}\,\mathfrak{b}_{18,7}(n) - \frac{68864}{21}\,\mathfrak{b}_{18,8}(n) \\
& - \frac{35456}{7}\,\mathfrak{b}_{18,9}(n) + \frac{116992}{21}\,\mathfrak{b}_{18,10}(n) - \frac{3104}{189}\,\mathfrak{b}_{18,11}(n)\Big)\,q^n, \qquad (6.8)
\end{aligned}
$$

$$
\begin{aligned}
E_4(q^2)\,(\,E_2(q) - 9\,E_2(q^9)\,) = -8 + \sum_{n=1}^{\infty}\Big(& -\frac{27}{637}\,\sigma_5(n) - \frac{540}{637}\,\sigma_5(\frac{n}{2}) \\
& + \frac{80}{1911}\,\sigma_5(\frac{n}{3}) + \frac{1600}{1911}\,\sigma_5(\frac{n}{6}) + \frac{243}{637}\,\sigma_5(\frac{n}{9}) + \frac{4860}{637}\,\sigma_5(\frac{n}{18}) \\
& - \frac{7944}{13}\,\mathfrak{b}_{18,1}(n) - \frac{255408}{91}\,\mathfrak{b}_{18,2}(n) + \frac{896040}{91}\,\mathfrak{b}_{18,3}(n) - \frac{80064}{7}\,\mathfrak{b}_{18,4}(n) \\
& + \frac{44712}{7}\,\mathfrak{b}_{18,5}(n) - \frac{3429360}{91}\,\mathfrak{b}_{18,6}(n) + \frac{1728}{7}\,\mathfrak{b}_{18,7}(n) + \frac{281664}{7}\,\mathfrak{b}_{18,8}(n) \\
& + \frac{1552608}{7}\,\mathfrak{b}_{18,9}(n) - \frac{1086912}{7}\,\mathfrak{b}_{18,10}(n) + \frac{4296}{7}\,\mathfrak{b}_{18,11}(n)\Big)\,q^n, \qquad (6.9)
\end{aligned}
$$

$$
\begin{aligned}
(\,27\,E_2(q^{27}) - E_2(q)\,)\,E_4(q^{27}) = 26 + \sum_{n=1}^{\infty}\Big(& -\frac{1}{597051}\,\sigma_5(n) - \frac{80}{597051}\,\sigma_5(\frac{n}{3}) \\
& - \frac{80}{7371}\,\sigma_5(\frac{n}{9}) + \frac{2367}{91}\,\sigma_5(\frac{n}{27}) - \frac{17968400}{9477}\,\mathfrak{b}_{27,1}(n) - \frac{2746880}{243}\,\mathfrak{b}_{27,2}(n) \\
& - \frac{5880080}{351}\,\mathfrak{b}_{27,3}(n) + \frac{104320}{9}\,\mathfrak{b}_{27,4}(n) + \frac{640}{3}\,\mathfrak{b}_{27,5}(n) + \frac{2560}{3}\,\mathfrak{b}_{27,6}(n) \\
& - 50560\,\mathfrak{b}_{27,7}(n) + 1920\,\mathfrak{b}_{27,8}(n) + \frac{25200}{13}\,\mathfrak{b}_{27,9}(n) \\
& - 460800\,\mathfrak{b}_{27,10}(n) + 1920\,\mathfrak{b}_{27,12}(n)\Big)\,q^n, \qquad (6.10)
\end{aligned}
$$

$$
\begin{aligned}
E_4(q)\,(\,E_2(q) - 27\,E_2(q^{27})\,) = -26 + \sum_{n=1}^{\infty}\Big(& \frac{263}{273}\,\sigma_5(n) - \frac{80}{273}\,\sigma_5(\frac{n}{3}) \\
& - \frac{240}{91}\,\sigma_5(\frac{n}{9}) - \frac{2187}{91}\,\sigma_5(\frac{n}{27}) + \frac{54512400}{13}\,\mathfrak{b}_{27,1}(n) + 25113600\,\mathfrak{b}_{27,2}(n) \\
& + \frac{487185840}{13}\,\mathfrak{b}_{27,3}(n) - 25453440\,\mathfrak{b}_{27,4}(n) - 466560\,\mathfrak{b}_{27,5}(n) \\
& - 1451520\,\mathfrak{b}_{27,6}(n) + 111818880\,\mathfrak{b}_{27,7}(n) - 1399680\,\mathfrak{b}_{27,8}(n) \\
& - \frac{56628720}{13}\,\mathfrak{b}_{27,9}(n) + 1018967040\,\mathfrak{b}_{27,10}(n) - 4199040\,\mathfrak{b}_{27,12}(n)\Big)\,q^n, \qquad (6.11)
\end{aligned}
$$

$$(32\, E_2(q^{32}) - E_2(q))\, E_4(q^{32}) = 31 + \sum_{n=1}^{\infty}\Big(-\frac{1}{1376256}\,\sigma_5(n) - \frac{5}{458752}\,\sigma_5(\frac{n}{2}) - \frac{5}{28672}\,\sigma_5(\frac{n}{4}) - \frac{5}{1792}\,\sigma_5(\frac{n}{8}) - \frac{5}{112}\,\sigma_5(\frac{n}{16}) + \frac{652}{21}\,\sigma_5(\frac{n}{32}) - \frac{11599875}{8192}\,\mathfrak{b}_{32,1}(n) + \frac{36855}{512}\,\mathfrak{b}_{32,2}(n) - \frac{540675}{32}\,\mathfrak{b}_{32,3}(n) + \frac{5355}{32}\,\mathfrak{b}_{32,4}(n) + \frac{23805}{2}\,\mathfrak{b}_{32,5}(n) + \frac{2295}{2}\,\mathfrak{b}_{32,6}(n) - 90630\,\mathfrak{b}_{32,7}(n) + \frac{675}{2}\,\mathfrak{b}_{32,8}(n) + 23400\,\mathfrak{b}_{32,9}(n) + 1080\,\mathfrak{b}_{32,10}(n) + 1440\,\mathfrak{b}_{32,11}(n) + 2520\,\mathfrak{b}_{32,12}(n) + 96480\,\mathfrak{b}_{32,13}(n) + 4320\,\mathfrak{b}_{32,14}(n) + 11520\,\mathfrak{b}_{32,15}(n) + 1440\,\mathfrak{b}_{32,16}(n) \Big)\, q^n, \quad (6.12)$$

$$E_4(q)\, (E_2(q) - 32\, E_2(q^{32})) = -31 + \sum_{n=1}^{\infty}\Big(\frac{163}{168}\,\sigma_5(n) - \frac{5}{56}\,\sigma_5(\frac{n}{2}) - \frac{5}{14}\,\sigma_5(\frac{n}{4}) - \frac{10}{7}\,\sigma_5(\frac{n}{8}) - \frac{40}{7}\,\sigma_5(\frac{n}{16}) - \frac{512}{21}\,\sigma_5(\frac{n}{32}) + 5891265\,\mathfrak{b}_{32,1}(n) - 56700\,\mathfrak{b}_{32,2}(n) + 70536960\,\mathfrak{b}_{32,3}(n) - 367920\,\mathfrak{b}_{32,4}(n) - 48337920\,\mathfrak{b}_{32,5}(n) - 1658880\,\mathfrak{b}_{32,6}(n) + 376197120\,\mathfrak{b}_{32,7}(n) - 1684800\,\mathfrak{b}_{32,8}(n) - 97228800\,\mathfrak{b}_{32,9}(n) - 5806080\,\mathfrak{b}_{32,10}(n) - 9216000\,\mathfrak{b}_{32,11}(n) - 6727680\,\mathfrak{b}_{32,12}(n) - 400711680\,\mathfrak{b}_{32,13}(n) - 3317760\,\mathfrak{b}_{32,14}(n) - 47185920\,\mathfrak{b}_{32,15}(n) - 5898240\,\mathfrak{b}_{32,16}(n) \Big)\, q^n. \quad (6.13)$$

Proof. We give the proof for the case where $\alpha = 1$ and $\beta = 18$. The proof for the other cases can be done similarly.

This follows immediately when one sets $\alpha = 1$ and $\beta = 18$ in Lemma 3.5. However, we briefly show the proof for $(18\ E_2(q^{18}) - E_2(q))\, E_4(q^{18})$ as an example. One obtains

$$\begin{aligned}(18\, E_2(q^{18}) - E_2(q))\, E_4(q^{18}) &= \sum_{\delta|18} x_\delta E_6(q^\delta) + z_1\, E_{6,(\frac{-4}{n})}(q) + z_2\, E_{6,(\frac{-4}{n})}(q^2) + \sum_{j=1}^{11} y_j\, \mathfrak{B}_{18,j}(q) \\ &= \sum_{\delta|18} x_\delta + C_0\, z_1 + C_0\, z_3 + \sum_{i=1}^{\infty}\Big(\sum_{\delta|18} -504\,\sigma_5(\frac{n}{\delta})\, x_\delta \\ &\quad + \left(\frac{-4}{n}\right)\sigma_5(n)\, z_1 + \left(\frac{-4}{n}\right)\sigma_5(\frac{n}{2})\, z_2 \quad + \sum_{j=1}^{11} y_j\, \mathfrak{b}_{18,k}(n)\Big)\, q^n.\end{aligned} \quad (6.14)$$

We apply the primitive Dirichlet character

$$\left(\frac{-4}{n}\right) = \begin{cases} -1 & \text{if } n \equiv 3 \pmod 4, \\ 0 & \text{if } \gcd(4,n) \neq 1, \\ 1 & \text{if } n \equiv 1 \pmod 4. \end{cases}$$

Since the conductor of the Dirichlet character $\left(\frac{-4}{n}\right)$ is greater than zero, from (2.1) we have $C_0 = 0$. Now when we equate the right hand side of (6.14) with that of (2.13), and when we take the coefficients of q^n for which $1 \leq n \leq 11$ and $n = 12$, 13, 14, 15, 16, 17, 18, 36 for example, we obtain a system of 19 linear equations with a unique solution. Hence, we obtain the stated result. □

Now we state and prove our main result of this subsection.

Corollary 6.3. *Let n be a positive integer. Then*

$$\begin{aligned} W_{(1,9)}^{1,3}(n) = W_{(9,1)}^{3,1}(n) &= \frac{1}{84240}\,\sigma_5(n) + \frac{1}{1053}\,\sigma_5(\frac{n}{3}) + \frac{9}{104}\,\sigma_5(\frac{n}{9}) \\ &+ \frac{1}{24}\,\sigma_3(\frac{n}{9}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{9}) - \frac{1}{240}\,\sigma(n) \\ &+ \frac{35}{8424}\,\mathfrak{b}_{18,1}(n) + \frac{1}{27}\,\mathfrak{b}_{18,2}(n) + \frac{9}{104}\,\mathfrak{b}_{18,3}(n) \end{aligned} \tag{6.15}$$

$$\begin{aligned} W_{(1,9)}^{3,1}(n) = W_{(9,1)}^{1,3}(n) &= \frac{1}{936}\,\sigma_5(n) + \frac{1}{117}\,\sigma_5(\frac{n}{3}) + \frac{81}{1040}\,\sigma_5(\frac{n}{9}) \\ &+ \frac{1}{24}\,\sigma_3(n) - \frac{1}{72}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{9}) \\ &- \frac{3}{104}\,\mathfrak{b}_{18,1}(n) - \frac{1}{3}\,\mathfrak{b}_{18,2}(n) - \frac{105}{104}\,\mathfrak{b}_{18,3}(n), \end{aligned} \tag{6.16}$$

$$\begin{aligned} W_{(1,16)}^{1,3}(n) = W_{(16,1)}^{3,1}(n) &= \frac{1}{983040}\,\sigma_5(n) + \frac{1}{65536}\,\sigma_5(\frac{n}{2}) + \frac{1}{4096}\,\sigma_5(\frac{n}{4}) \\ &+ \frac{1}{256}\,\sigma_5(\frac{n}{8}) + \frac{1}{12}\,\sigma_5(\frac{n}{16}) + \frac{1}{24}\,\sigma_3(\frac{n}{16}) - \frac{1}{8}\,n\,\sigma_3(\frac{n}{16}) \\ &- \frac{1}{240}\,\sigma(n) + \frac{273}{65536}\,\mathfrak{b}_{32,1}(n) + \frac{51}{4096}\,\mathfrak{b}_{32,2}(n) + \frac{17}{256}\,\mathfrak{b}_{32,3}(n) \\ &+ \frac{7}{256}\,\mathfrak{b}_{32,4}(n) + \frac{1}{16}\,\mathfrak{b}_{32,5}(n) + \frac{3}{16}\,\mathfrak{b}_{32,6}(n) + \frac{1}{4}\,\mathfrak{b}_{32,7}(n), \end{aligned} \tag{6.17}$$

$$\begin{aligned} W_{(1,16)}^{3,1}(n) = W_{(16,1)}^{1,3}(n) &= \frac{1}{3072}\,\sigma_5(n) + \frac{1}{1024}\,\sigma_5(\frac{n}{2}) + \frac{1}{256}\,\sigma_5(\frac{n}{4}) + \frac{1}{64}\,\sigma_5(\frac{n}{8}) \\ &+ \frac{1}{15}\,\sigma_5(\frac{n}{16}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{128}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{16}) \\ &- \frac{35}{1024}\,\mathfrak{b}_{32,1}(n) - \frac{63}{256}\,\mathfrak{b}_{32,2}(n) - \mathfrak{b}_{32,3}(n) - \frac{73}{64}\,\mathfrak{b}_{32,4}(n) \\ &- \frac{7}{2}\,\mathfrak{b}_{32,5}(n) - \frac{9}{2}\,\mathfrak{b}_{32,6}(n) - 2\,\mathfrak{b}_{32,7}(n) \end{aligned} \tag{6.18}$$

$$\begin{aligned}
W_{(1,18)}^{1,3}(n) \;=\; W_{(18,1)}^{3,1}(n) \;=\; & \frac{1}{1769040}\,\sigma_5(n) + \frac{1}{88452}\,\sigma_5(\frac{n}{2}) + \frac{1}{22113}\,\sigma_5(\frac{n}{3}) \\
& + \frac{20}{22113}\,\sigma_5(\frac{n}{6}) + \frac{3}{728}\,\sigma_5(\frac{n}{9}) + \frac{15}{182}\,\sigma_5(\frac{n}{18}) + \frac{1}{24}\,\sigma_3(\frac{n}{18}) \\
& - \frac{1}{8}\,n\,\sigma_3(\frac{n}{18}) - \frac{1}{240}\,\sigma(n) + \frac{349}{252720}\,\mathfrak{b}_{18,1}(n) \\
& + \frac{18359}{884520}\,\mathfrak{b}_{18,2}(n) + \frac{2449}{117936}\,\mathfrak{b}_{18,3}(n) + \frac{961}{5670}\,\mathfrak{b}_{18,4}(n) \\
& - \frac{17}{1680}\,\mathfrak{b}_{18,5}(n) + \frac{25205}{58968}\,\mathfrak{b}_{18,6}(n) - \frac{19}{1890}\,\mathfrak{b}_{18,7}(n) \\
& - \frac{157}{1890}\,\mathfrak{b}_{18,8}(n) + \frac{199}{1260}\,\mathfrak{b}_{18,9}(n) + \frac{611}{1890}\,\mathfrak{b}_{18,10}(n) \\
& + \frac{379}{136080}\,\mathfrak{b}_{18,11}(n) \qquad (6.19)
\end{aligned}$$

$$\begin{aligned}
W_{(1,18)}^{3,1}(n) \;=\; W_{(18,1)}^{1,3}(n) \;=\; & \frac{5}{19656}\,\sigma_5(n) + \frac{2}{2457}\,\sigma_5(\frac{n}{2}) + \frac{5}{2457}\,\sigma_5(\frac{n}{3}) + \frac{16}{2457}\,\sigma_5(\frac{n}{6}) \\
& + \frac{27}{1456}\,\sigma_5(\frac{n}{9}) + \frac{27}{455}\,\sigma_5(\frac{n}{18}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{144}\,n\,\sigma_3(n) \\
& - \frac{1}{240}\,\sigma(\frac{n}{18}) - \frac{451}{28080}\,\mathfrak{b}_{18,1}(n) - \frac{17473}{49140}\,\mathfrak{b}_{18,2}(n) \\
& - \frac{8077}{13104}\,\mathfrak{b}_{18,3}(n) - \frac{1112}{315}\,\mathfrak{b}_{18,4}(n) - \frac{393}{280}\,\mathfrak{b}_{18,5}(n) \\
& - \frac{43795}{3276}\,\mathfrak{b}_{18,6}(n) - \frac{202}{105}\,\mathfrak{b}_{18,7}(n) + \frac{149}{105}\,\mathfrak{b}_{18,8}(n) \\
& + \frac{277}{70}\,\mathfrak{b}_{18,9}(n) - \frac{517}{105}\,\mathfrak{b}_{18,10}(n) - \frac{143}{7560}\,\mathfrak{b}_{18,11}(n), \\
& \qquad (6.20)
\end{aligned}$$

$$\begin{aligned}
W_{(2,9)}^{1,3}(n) \;=\; W_{(9,2)}^{3,1}(n) \;=\; & \frac{1}{353808}\,\sigma_5(n) + \frac{1}{110565}\,\sigma_5(\frac{n}{2}) + \frac{5}{22113}\,\sigma_5(\frac{n}{3}) \\
& + \frac{16}{22113}\,\sigma_5(\frac{n}{6}) + \frac{15}{728}\,\sigma_5(\frac{n}{9}) + \frac{6}{91}\,\sigma_5(\frac{n}{18}) + \frac{1}{24}\,\sigma_3(\frac{n}{9}) \\
& - \frac{1}{8}\,n\,\sigma_3(\frac{n}{9}) - \frac{1}{240}\,\sigma(\frac{n}{2}) - \frac{361}{252720}\,\mathfrak{b}_{18,1}(n) - \frac{1993}{442260}\,\mathfrak{b}_{18,2}(n) \\
& + \frac{2417}{117936}\,\mathfrak{b}_{18,3}(n) - \frac{2}{2835}\,\mathfrak{b}_{18,4}(n) - \frac{11}{840}\,\mathfrak{b}_{18,5}(n) \\
& - \frac{19}{29484}\,\mathfrak{b}_{18,6}(n) - \frac{142}{945}\,\mathfrak{b}_{18,7}(n) + \frac{269}{945}\,\mathfrak{b}_{18,8}(n) + \frac{277}{630}\,\mathfrak{b}_{18,9}(n) \\
& - \frac{457}{945}\,\mathfrak{b}_{18,10}(n) + \frac{97}{68040}\,\mathfrak{b}_{18,11}(n), \qquad (6.21)
\end{aligned}$$

$$\begin{aligned}
W_{(2,9)}^{3,1}(n) \;=\; W_{(9,2)}^{1,3}(n) \;=\; & \frac{1}{19656}\,\sigma_5(n) + \frac{5}{4914}\,\sigma_5(\frac{n}{2}) + \frac{1}{2457}\,\sigma_5(\frac{n}{3}) + \frac{20}{2457}\,\sigma_5(\frac{n}{6}) \\
& + \frac{27}{7280}\,\sigma_5(\frac{n}{9}) + \frac{27}{364}\,\sigma_5(\frac{n}{18}) + \frac{1}{24}\,\sigma_3(\frac{n}{2}) - \frac{1}{72}\,n\,\sigma_3(\frac{n}{2}) \\
& - \frac{1}{240}\,\sigma(\frac{n}{9}) + \frac{331}{28080}\,\mathfrak{b}_{18,1}(n) + \frac{5321}{98280}\,\mathfrak{b}_{18,2}(n) - \frac{2489}{13104}\,\mathfrak{b}_{18,3}(n) \\
& + \frac{139}{630}\,\mathfrak{b}_{18,4}(n) - \frac{69}{560}\,\mathfrak{b}_{18,5}(n) + \frac{4763}{6552}\,\mathfrak{b}_{18,6}(n) - \frac{1}{210}\,\mathfrak{b}_{18,7}(n) \\
& - \frac{163}{210}\,\mathfrak{b}_{18,8}(n) - \frac{599}{140}\,\mathfrak{b}_{18,9}(n) + \frac{629}{210}\,\mathfrak{b}_{18,10}(n) - \frac{179}{15120}\,\mathfrak{b}_{18,11}(n),
\end{aligned} \tag{6.22}$$

$$\begin{aligned}
W_{(1,27)}^{1,3}(n) \;=\; W_{(27,1)}^{3,1}(n) \;=\; & \frac{1}{6823440}\,\sigma_5(n) + \frac{1}{85293}\,\sigma_5(\frac{n}{3}) + \frac{1}{1053}\,\sigma_5(\frac{n}{9}) \\
& + \frac{9}{104}\,\sigma_5(\frac{n}{27}) + \frac{1}{24}\,\sigma_3(\frac{n}{27}) - \frac{3}{24}\,n\,\sigma_3(\frac{n}{27}) \\
& - \frac{1}{240}\,\sigma(n) - \frac{224605}{682344}\,\mathfrak{b}_{27,1}(n) - \frac{4292}{2187}\,\mathfrak{b}_{27,2}(n) \\
& - \frac{73501}{25272}\,\mathfrak{b}_{27,3}(n) + \frac{163}{81}\,\mathfrak{b}_{27,4}(n) + \frac{1}{27}\,\mathfrak{b}_{27,5}(n) \\
& + \frac{4}{27}\,\mathfrak{b}_{27,6}(n) - \frac{79}{9}\,\mathfrak{b}_{27,7}(n) + \frac{1}{3}\,\mathfrak{b}_{27,8}(n) + \frac{35}{104}\,\mathfrak{b}_{27,9}(n) \\
& - 80\,\mathfrak{b}_{27,10}(n) + \frac{1}{3}\,\mathfrak{b}_{27,12}(n),
\end{aligned} \tag{6.23}$$

$$\begin{aligned}
W_{(1,27)}^{3,1}(n) \;=\; W_{(27,1)}^{1,3}(n) \;=\; & \frac{1}{8424}\,\sigma_5(n) + \frac{1}{1053}\,\sigma_5(\frac{n}{3}) + \frac{1}{117}\,\sigma_5(\frac{n}{9}) \\
& + \frac{81}{1040}\,\sigma_5(\frac{n}{27}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{216}\,n\,\sigma_3(n) - \frac{1}{240}\,\sigma(\frac{n}{27}) \\
& + \frac{227135}{8424}\,\mathfrak{b}_{27,1}(n) + \frac{4360}{27}\,\mathfrak{b}_{27,2}(n) + \frac{25061}{104}\,\mathfrak{b}_{27,3}(n) \\
& - \frac{491}{3}\,\mathfrak{b}_{27,4}(n) - 3\,\mathfrak{b}_{27,5}(n) - \frac{28}{3}\,\mathfrak{b}_{27,6}(n) + 719\,\mathfrak{b}_{27,7}(n) \\
& - 9\,\mathfrak{b}_{27,8}(n) - \frac{2913}{104}\,\mathfrak{b}_{27,9}(n) + 6552\,\mathfrak{b}_{27,10}(n) - 27\,\mathfrak{b}_{27,12}(n),
\end{aligned} \tag{6.24}$$

$$\begin{aligned} W^{1,3}_{(1,32)}(n) \;=\; W^{3,1}_{(32,1)}(n) \;=& \frac{1}{15728640}\,\sigma_5(n) + \frac{1}{1048576}\,\sigma_5(\frac{n}{2}) + \frac{1}{65536}\,\sigma_5(\frac{n}{4}) \\ &+ \frac{1}{4096}\,\sigma_5(\frac{n}{8}) + \frac{1}{256}\,\sigma_5(\frac{n}{16}) + \frac{1}{12}\,\sigma_5(\frac{n}{32}) + \frac{1}{24}\,\sigma_3(\frac{n}{32}) \\ &- \frac{3}{24}\,n\,\sigma_3(\frac{n}{32}) - \frac{1}{240}\,\sigma(n) - \frac{257775}{1048576}\,\mathfrak{b}_{32,1}(n) \\ &+ \frac{819}{65536}\,\mathfrak{b}_{32,2}(n) - \frac{12015}{4096}\,\mathfrak{b}_{32,3}(n) + \frac{119}{4096}\,\mathfrak{b}_{32,4}(n) \\ &+ \frac{529}{256}\,\mathfrak{b}_{32,5}(n) + \frac{51}{256}\,\mathfrak{b}_{32,6}(n) - \frac{1007}{64}\,\mathfrak{b}_{32,7}(n) \\ &+ \frac{15}{256}\,\mathfrak{b}_{32,8}(n) + \frac{65}{16}\,\mathfrak{b}_{32,9}(n) + \frac{3}{16}\,\mathfrak{b}_{32,10}(n) + \frac{1}{4}\,\mathfrak{b}_{32,11}(n) \\ &+ \frac{7}{16}\,\mathfrak{b}_{32,12}(n) + \frac{67}{4}\,\mathfrak{b}_{32,13}(n) + \frac{3}{4}\,\mathfrak{b}_{32,14}(n) + 2\,\mathfrak{b}_{32,15}(n) \\ &+ \frac{1}{4}\,\mathfrak{b}_{32,16}(n), \end{aligned} \tag{6.25}$$

$$\begin{aligned} W^{3,1}_{(1,32)}(n) \;=\; W^{1,3}_{(32,1)}(n) \;=& \frac{1}{12288}\,\sigma_5(n) + \frac{1}{4096}\,\sigma_5(\frac{n}{2}) + \frac{1}{1024}\,\sigma_5(\frac{n}{4}) + \frac{1}{256}\,\sigma_5(\frac{n}{8}) \\ &+ \frac{1}{64}\,\sigma_5(\frac{n}{16}) + \frac{1}{15}\,\sigma_5(\frac{n}{32}) + \frac{1}{24}\,\sigma_3(n) - \frac{1}{256}\,n\,\sigma_3(n) \\ &- \frac{1}{240}\,\sigma(\frac{n}{32}) + \frac{130917}{4096}\,\mathfrak{b}_{32,1}(n) - \frac{315}{1024}\,\mathfrak{b}_{32,2}(n) \\ &+ \frac{6123}{16}\,\mathfrak{b}_{32,3}(n) - \frac{511}{256}\,\mathfrak{b}_{32,4}(n) - \frac{1049}{4}\,\mathfrak{b}_{32,5}(n) - 9\,\mathfrak{b}_{32,6}(n) \\ &+ 2041\,\mathfrak{b}_{32,7}(n) - \frac{585}{64}\,\mathfrak{b}_{32,8}(n) - \frac{1055}{2}\,\mathfrak{b}_{32,9}(n) - \frac{63}{2}\,\mathfrak{b}_{32,10}(n) \\ &- 50\,\mathfrak{b}_{32,11}(n) - \frac{73}{2}\,\mathfrak{b}_{32,12}(n) - 2174\,\mathfrak{b}_{32,13}(n) - 18\,\mathfrak{b}_{32,14}(n) \\ &- 256\,\mathfrak{b}_{32,15}(n) - 32\,\mathfrak{b}_{32,16}(n). \end{aligned} \tag{6.26}$$

Proof. Follows immediately when we set for example $(\alpha,\beta) = (1,9), (1,16), (1,18), (1,27), (1,32)$ in Theorem 3.6 and apply Theorem 3.9. □

Remark 6.4. In all explicit examples evaluated as yet, we notice that for all $\chi \in \mathcal{C}$ and for all $s \in D(\chi)$ the value of $Z(\chi)_s$ is zero; that means that the value of $Z(\chi)_s$ always vanish for all $\alpha\beta$ belonging to $\mathbb{N}^* \setminus \mathfrak{N}$.

7. Formulae for the Number of Representations of a Positive Integer

We make use of the convolution sums evaluated in Section 5 among others to determine explicit formulae for the number of representations of a positive integer n by the quadratic forms (1.4) – (1.7).

7.1. Representations by quadratic forms (1.4) and (1.5). We give formulae for the number of representations of a positive integer n by the Quadratic Form (1.4) and (1.5). We apply among others the evaluated convolution sums for the levels 4, 8, 12, 16. To achieve these results, we recall that for example $12 = 2^2 \cdot 3$ and $16 = 2^4$, which are of the restricted form in Section 4.1. Therefore, we apply

Proposition 4.2 to conclude that $\Omega_4 = \{(1,3)\}$ in case $\alpha\beta = 12$ and $\Omega_4 = \{(1,4)\}$ in case $\alpha\beta = 16$.

Corollary 7.1. *Let $n \in \mathbb{N}$ and $a,b) = (1,1)$, $(1,2)$, $(1,4)$. Then*

$$N^{4,8}_{(1,1)}(n) \;=\; N^{8,4}_{(1,1)}(n) = 8\,\sigma_5(n) - 512\,\sigma_5(\frac{n}{4}) + 16\,\mathfrak{b}_{32,1}(n) \;=\; r_{12}(n),$$

$$\begin{aligned} N^{4,8}_{(1,2)}(n) \;=\; N^{8,4}_{(2,1)}(n) \;=&\frac{1}{2}\,\sigma_5(n) - \frac{1}{2}\,\sigma_5(\frac{n}{2}) + 8\,\sigma_5(\frac{n}{4}) - 512\,\sigma_5(\frac{n}{8}) + \frac{15}{2}\,\mathfrak{b}_{32,1}(n) \\ &+ 24\,\mathfrak{b}_{32,2}(n) + 128\,\mathfrak{b}_{32,3}(n), \end{aligned}$$

$$\begin{aligned} N^{8,4}_{(1,2)}(n) \;=\; N^{4,8}_{(2,1)}(n) \;=&\,2\,\sigma_5(n) - 2\,\sigma_5(\frac{n}{2}) + 8\,\sigma_5(\frac{n}{4}) - 512\,\sigma_5(\frac{n}{8}) + 14\,\mathfrak{b}_{32,1}(n) \\ &- 16\,\mathfrak{b}_{32,1}(\frac{n}{2}) + 72\,\mathfrak{b}_{32,2}(n) + 256\,\mathfrak{b}_{32,3}(n), \end{aligned}$$

$$\begin{aligned} N^{4,8}_{(1,4)}(n) \;=\; N^{8,4}_{(4,1)}(n) \;=&\frac{1}{32}\,\sigma_5(n) + \frac{15}{32}\,\sigma_5(\frac{n}{2}) - \frac{65}{2}\,\sigma_5(\frac{n}{4}) + 40\,\sigma_5(\frac{n}{8}) - 512\,\sigma_5(\frac{n}{16}) \\ &+ \frac{255}{32}\,\mathfrak{b}_{32,1}(n) - 32\,\mathfrak{b}_{32,1}(\frac{n}{4}) + \frac{45}{2}\,\mathfrak{b}_{32,2}(n) + 120\,\mathfrak{b}_{32,3}(n) \\ &+ 56\,\mathfrak{b}_{32,4}(n) + 128\,\mathfrak{b}_{32,5}(n) + 384\,\mathfrak{b}_{32,6}(n) + 512\,\mathfrak{b}_{32,7}(n), \end{aligned}$$

$$\begin{aligned} N^{8,4}_{(1,4)}(n) \;=\; N^{4,8}_{(4,1)}(n) \;=&\frac{1}{2}\,\sigma_5(n) - \frac{5}{2}\,\sigma_5(\frac{n}{2}) + 130\,\sigma_5(\frac{n}{4}) - 120\,\sigma_5(\frac{n}{8}) - 512\,\sigma_5(\frac{n}{16}) \\ &+ \frac{31}{2}\,\mathfrak{b}_{32,1}(n) - 28\,\mathfrak{b}_{32,1}(\frac{n}{2}) - 128\,\mathfrak{b}_{32,1}(\frac{n}{4}) + 126\,\mathfrak{b}_{32,2}(n) \\ &- 144\,\mathfrak{b}_{32,2}(\frac{n}{2}) + 512\,\mathfrak{b}_{32,3}(n) - 512\,\mathfrak{b}_{32,3}(\frac{n}{2}) + 584\,\mathfrak{b}_{32,4}(n) \\ &+ 1792\,\mathfrak{b}_{32,5}(n) + 2304\,\mathfrak{b}_{32,6}(n) + 1024\,\mathfrak{b}_{32,7}(n). \end{aligned}$$

Proof. These formulae follow immediately from Theorem 4.1 when we set for example $(a,b) = (1,4)$.

$$\begin{aligned} N^{4,8}_{(1,4)}(n) =&\, 8\,\sigma(n) - 32\,\sigma(\frac{n}{4}) + 16\,\sigma_3(\frac{n}{4}) - 32\,\sigma_3(\frac{n}{8}) + 256\sigma_3(\frac{n}{16}) \\ &+ 128\,W^{1,3}_{(1,4)}(n) - 256\,W^{1,3}_{(1,8)}(n) + 2048\,W^{1,3}_{(1,16)}(n) - 512\,W^{1,3}_{(1,1)}(\frac{n}{4}) \\ &+ 1024\,W^{1,3}_{(1,2)}(\frac{n}{4}) - 8192\,W^{1,3}_{(1,4)}(\frac{n}{4}) \end{aligned}$$

$$\begin{aligned} N^{8,4}_{(1,4)}(n) =&\; 8\,\sigma(\frac{n}{4}) - 32\,\sigma(\frac{n}{16}) + 16\,\sigma_3(n) - 32\,\sigma_3(\frac{n}{2}) + 256\sigma_3(\frac{n}{4}) \\ &+ 128\,W^{3,1}_{(1,4)}(n) - 512\,W^{3,1}_{(1,16)}(n) - 256\,W^{3,1}_{(1,2)}(\frac{n}{2}) + 1024\,W^{3,1}_{(1,8)}(\frac{n}{2}) \\ &+ 2048\,W^{3,1}_{(1,1)}(\frac{n}{4}) - 8192\,W^{3,1}_{(1,4)}(\frac{n}{4}) \end{aligned}$$

One can then use the result of

- (2.11), J. G. Huard et al. [**5**, Thrm 6, p.22], (5.29), (5.41) and (6.17) for the sake of simplification of $N^{4,8}_{(1,4)}(n)$.
- (2.11), J. G. Huard et al. [**5**, Thrm 6, p.22], (5.30), (5.42) and (6.18) to simplify the formulae $N^{8,4}_{(1,4)}(n)$.

□

7.2. Representations by the quadratic forms (1.6) and (1.7). We determine formulae for the number of representations of a positive integer n by the Quadratic Form (1.6) and (1.7). We mainly apply the evaluation of the convolution sums for the levels 3, 6 and 9.

In order to achieve these results, we recall that for example $6 = 3\cdot 2$ and $9 = 3\cdot 3$, so that by Proposition 4.5 we get $\Omega_3 = \{(1,2),(1,3)\}$. We then deduce the following result:

Corollary 7.2. *Let* $n \in \mathbb{N}$ *and* $c,d) = (1,1),\ (1,2),\ (1,3),\ (1,7)$. *Then*

$$R^{4,8}_{(1,1)}(n) \;=\; R^{8,4}_{(1,1)}(n) \;= \frac{252}{13}\sigma_5(n) - \frac{6804}{13}\,\sigma_5(\frac{n}{3}) + \frac{216}{13}\,\mathfrak{b}_{18,1}(n) \;=\; s_{12}(n),$$

$$\begin{aligned} R^{4,8}_{(1,2)}(n) \;=\; R^{8,4}_{(2,1)}(n) \;=& \frac{12}{13}\sigma_5(n) + \frac{240}{13}\,\sigma_5(\frac{n}{2}) - \frac{324}{13}\,\sigma_5(\frac{n}{3}) - \frac{6480}{13}\,\sigma_5(\frac{n}{6}) \\ &+ \frac{144}{13}\,\mathfrak{b}_{12,1}(n) + \frac{1008}{13}\,\mathfrak{b}_{12,2}(n), \end{aligned}$$

$$\begin{aligned} R^{8,4}_{(1,2)}(n) \;=\; R^{4,8}_{(2,1)}(n) \;=& \frac{60}{13}\sigma_5(n) + \frac{192}{13}\,\sigma_5(\frac{n}{2}) - \frac{1620}{13}\,\sigma_5(\frac{n}{3}) - \frac{5184}{13}\,\sigma_5(\frac{n}{6}) \\ &+ \frac{252}{13}\,\mathfrak{b}_{12,1}(n) + \frac{2304}{13}\,\mathfrak{b}_{12,2}(n), \end{aligned}$$

$$\begin{aligned} R^{4,8}_{(1,3)}(n) \;=\; R^{8,4}_{(3,1)}(n) \;=& \frac{4}{13}\,\sigma_5(n) - \frac{724}{13}\,\sigma_5(\frac{n}{3}) - \frac{5832}{13}\,\sigma_5(\frac{n}{9}) + \frac{152}{13}\,\mathfrak{b}_{18,1}(n) \\ &- \frac{324}{13}\,\mathfrak{b}_{18,1}(\frac{n}{3}) + 96\,\mathfrak{b}_{18,2}(n) + \frac{2916}{13}\,\mathfrak{b}_{18,3}(n), \end{aligned}$$

$$\begin{aligned} R^{8,4}_{(1,3)}(n) \;=\; R^{4,8}_{(3,1)}(n) \;=& \frac{24}{13}\,\sigma_5(n) + \frac{2172}{13}\,\sigma_5(\frac{n}{3}) - \frac{8748}{13}\,\sigma_5(\frac{n}{9}) + \frac{288}{13}\,\mathfrak{b}_{18,1}(n) \\ &+ \frac{972}{13}\,\mathfrak{b}_{18,1}(\frac{n}{3}) + 288\,\mathfrak{b}_{18,2}(n) + \frac{11340}{13}\,\mathfrak{b}_{18,3}(n) \end{aligned}$$

If $n \not\equiv 0 \pmod 3$ *then*

$$\begin{aligned} R^{4,8}_{(1,7)}(n) \;=\; R^{8,4}_{(7,1)}(n) \;=& \frac{84}{10621}\,\sigma_5(n) - \frac{26244}{817}\,\sigma_5(\frac{n}{3}) + \frac{205800}{10621}\,\sigma_5(\frac{n}{7}) \\ &- \frac{2458836}{10621}\,\sigma_5(\frac{n}{21}) + \frac{127368}{10621}\,\mathfrak{b}_{21,1}(n) - \frac{26244}{817}\,\mathfrak{b}_{21,1}(\frac{n}{3}) \\ &+ \frac{38448}{247}\,\mathfrak{b}_{21,2}(n) - \frac{7776}{19}\,\mathfrak{b}_{21,2}(\frac{n}{3}) + \frac{9009864}{10621}\,\mathfrak{b}_{21,3}(n) \\ &- \frac{1138212}{817}\,\mathfrak{b}_{21,3}(\frac{n}{3}) + \frac{9504}{13}\,\mathfrak{b}_{21,4}(n) + \frac{7776}{13}\,\mathfrak{b}_{21,5}(n) \\ &+ \frac{143100}{13}\,\mathfrak{b}_{21,6}(n) + \frac{34344}{13}\,\mathfrak{b}_{21,7}(n) - \frac{864}{13}\,\mathfrak{b}_{21,8}(n) \\ &+ \frac{1722600}{13}\,\mathfrak{b}_{21,9}(n) - \frac{28512}{13}\,\mathfrak{b}_{21,10}(n) - \frac{57888}{13}\,\mathfrak{b}_{21,11}(n) \\ &+ \frac{6195528}{13}\,\mathfrak{b}_{21,12}(n), \end{aligned}$$

Proof. We only consider the case $(c,d) = (1,3)$ since the other cases can be proved in a similar way.

The proof follows immediately from Theorem 4.4 with $(c,d)=(1,3)$; that is,

$$\begin{aligned}R^{4,8}_{(1,3)}(n) =& 12\,\sigma(n)-36\,\sigma(\frac{n}{3})+24\,\sigma_3(\frac{n}{3})+216\,\sigma_3(\frac{n}{9})+288\,W^{1,3}_{(1,3)}(n)\\&+2592\,W^{1,3}_{(1,9)}(n)-864\,W^{1,3}_{(1,1)}(\frac{n}{3})-7776\,W^{1,3}_{(1,3)}(\frac{n}{3})\end{aligned}$$

$$\begin{aligned}R^{8,4}_{(1,3)}(n) =& 24\,\sigma_3(n)+216\,\sigma_3(\frac{n}{3})+12\,\sigma(\frac{n}{3})-36\,\sigma(\frac{n}{9})+288\,W^{3,1}_{(1,3)}(n)\\&-864\,W^{3,1}_{(1,9)}(n)+2592\,W^{3,1}_{(1,1)}(\frac{n}{3})-7776\,W^{3,1}_{(1,3)}(\frac{n}{3})\end{aligned}$$

One can then make use of

- (2.11), E. X. W. Xia and O. X. M. Yao [**28**] or (5.28) and (6.15) to simplify the formula $R^{4,8}_{(1,3)}(n)$ and then obtain the stated result.
- (2.11), (5.27) and (6.16) to simplify $R^{8,4}_{(1,3)}(n)$ and obtain the stated result.

□

Tables

	1	**2**	**3**	**4**	**6**	**12**
1	6	0	6	0	0	0
2	0	6	0	0	6	0
3	4	-2	-4	0	14	0
4	0	0	0	6	0	6
5	0	2	0	2	-2	10
6	0	4	0	-2	-4	14
7	-2	5	-2	-5	1	15

Table 2: Power of η-quotients being basis elements of $S_6(\Gamma_0(12))$

	1	**2**	**7**	**14**
1	10	0	2	0
2	6	0	6	0
3	2	0	10	0
4	0	6	0	6
5	3	-1	3	7
6	0	2	0	10
7	4	-2	-4	14
8	1	1	-7	17

Table 3: Power of η-quotients being basis elements of $S_6(\Gamma_0(14))$

	1	**3**	**5**	**15**
1	6	6	0	0
2	0	6	6	0
3	3	3	3	3
4	6	0	0	6
5	0	0	6	6
6	7	-1	-5	11
7	1	-1	1	11
8	1	-1	13	-1

Table 4: Power of η-quotients being basis elements of $S_6(\Gamma_0(15))$

	1	**2**	**3**	**6**	**9**	**18**
1	6	0	6	0	0	0
2	3	0	6	3	0	0
3	0	0	6	0	6	0
4	0	3	0	6	0	3
5	0	0	0	2	8	2
6	0	0	0	6	0	6
7	0	0	2	0	2	8
8	0	0	2	4	-6	12
9	0	0	4	-2	-4	14
10	0	0	5	-3	-7	17
11	0	6	8	-2	0	0

Table 5: Power of η-quotients being basis elements of $S_6(\Gamma_0(18))$

	1	**2**	**4**	**5**	**10**	**20**
1	0	12	0	0	0	0
2	0	0	2	8	4	-2
3	0	0	8	0	4	0
4	0	0	4	8	-4	4
5	0	0	0	0	12	0
6	0	2	0	-8	18	0
7	0	0	2	0	4	6
8	0	4	-4	-8	16	4
9	0	0	4	0	-4	12
10	1	-3	0	-5	11	8
11	0	2	0	0	-6	16
12	1	-2	-1	-5	6	13

Table 6: Power of η-quotients being basis elements of $S_6(\Gamma_0(20))$

	1	**3**	**7**	**21**
1	10	0	2	0
2	6	0	6	0
3	2	0	10	0
4	0	4	6	2
5	2	2	4	4
6	0	6	0	6
7	0	0	6	6
8	1	3	-1	9
9	0	2	0	10
10	2	0	-2	12
11	1	-1	-1	13
12	0	-2	0	14

Table 7: Power of η-quotients being basis elements of $S_6(\Gamma_0(21))$

	1	**3**	**9**	**27**
1	6	6	0	0
2	3	6	3	0
3	0	6	6	0
4	0	2	10	0
5	0	-2	14	0
6	0	3	6	3
7	0	8	-2	6
8	0	4	2	6
9	0	0	6	6
10	0	5	-2	9
11	0	1	2	9
12	0	2	2	0

Table 8: Power of η-functions being basis elements of $S_6(\Gamma_0(27))$

	1	**2**	**4**	**8**	**16**	**32**
1	0	12	0	0	0	0
2	0	0	12	0	0	0
3	0	4	0	8	0	0
4	0	0	0	12	0	0
5	0	0	2	6	4	0
6	0	0	4	0	8	0
7	0	0	6	-6	12	0
8	0	0	0	0	12	0
9	0	0	6	-4	6	4
10	0	0	0	2	6	4
11	0	0	6	-2	0	8
12	0	0	0	4	0	8
13	0	0	6	0	-6	12
14	0	0	0	6	-6	12
15	0	0	2	0	-2	12
16	0	0	2	2	0	0

Table 9: Power of η-functions being basis elements of $S_6(\Gamma_0(32))$

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Ebénézer Ntienjem
Centre for Research in Algebra
and Number Theory
School of Mathematics and
Statistics
Carleton University
1125 Colonel By Drive
Ottawa, Ontario, K1S 5B6,
Canada

ebenezer.ntienjem@carleton.ca
ntienjem@gmail.com