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SCHWARZ TRIANGLE FUNCTIONS AND DUALITY FOR CERTAIN PARAMETERS OF THE GENERALISED CHAZY EQUATION

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Abstract. Schwarz triangle functions play a fundamental role in the solutions of the generalised Chazy equation. We determine the Schwarz triangle functions that appear in the solutions in the cases where $k = \frac{2}{3}, \frac{3}{2}, 2$ and 3. Chazy has shown that for the parameters $k = 2$ and 3, the equations can be linearised. Some of the Schwarz triangle functions that show up in the solutions to the generalised Chazy equation with these two parameters also show up in the dual cases where $k = \frac{2}{3}$ and $k = \frac{3}{2}$, suggesting an intriguing connection between the solutions for $k = 2$ and $k = 3$ with dihedral and tetrahedral symmetry respectively, and the solutions for $k = \frac{2}{3}, \frac{3}{2}$ with G_2 symmetry.

The generalised Chazy equation is a third order nonlinear autonomous ordinary differential equation (ODE) given by

$$y''' - 2y''y + 3(y')^2 - \frac{4}{36-k^2}(6y' - y^2)^2 = 0 \tag{0.1}$$

for $k \neq 6$. The equation (0.1) was introduced by Jean Chazy in the papers [**6**] and [**7**] when investigating the Painlevé property for third order ODEs.

Equation (0.1) is solved by Schwarz triangle functions [**1**]. Schwarz triangle functions determine through their inverse a map from the complex upper half plane to an open triangular domain with boundary given by the edges of the triangle. The angles of the triangles determined by the Schwarz functions depend on the parameter k in (0.1). When $k < 6$, the image is a spherical triangle and when $k > 6$, the triangle is hyperbolic. In the limiting case as k tends to ∞ we obtain Chazy's equation, which has a solution given by the Eisenstein series of weight 2. In this article we present the spherical Schwarz triangle functions corresponding to the solutions when k is given by $\frac{2}{3}$, $\frac{3}{2}$, 2 and 3 and determine them algebraically.

These four parameters are chosen because the equations also show up in the context of the geometry of differential equations. The problem of determining whether the solution set of a system of differential equation is equivalent to another, via for instance point or contact preserving transformation, can be solved using Cartan's method of equivalence.

The equations for the parameters $k = 2$ and $k = 3$ are shown to be linearisable by Chazy himself ([**7**], p. 346). The equation when $k = 2$ is linearisable to the ODE $y'''' = 0$. This is related to the third order Riccati equation as observed in

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[**16**]. Applying Cartan's method of equivalence, this 4th order ODE has vanishing Wilczynski invariants in the linear theory and also vanishing Bryant invariants in the non-linear theory [**4**], [**11**], [**14**], [**18**]. We discuss the general solution when $k = 2$ in Section 3.

The method of equivalence also applies to third order ODEs, as worked out by Chern [**8**]. The equation for the parameter $k = 3$ turns out to be the only equation of the form (0.1) that has vanishing Wünschmann invariant. A third order ODE with vanishing Wünschmann invariant defines a conformal structure of signature $(2, 1)$ on the space of its solutions. The conformal metric is obtained by quotienting out a degenerate split signature symmetric 2-tensor by the vector field that annihilates the distribution encoding the ODE $y''' = F(x, y, p, q)$. ODEs with vanishing Wünschmann invariant and $F_{qqqq} = 0$, satisfied for the $k = 3$ equation, are contact equivalent to the equation $y''' = 0$ [**13**]. The generalised Chazy equation for this parameter $k = 3$ is also linearisable and the general solution to this equation is described in Section 5.

The equations for the parameters $k = \pm\frac{3}{2}$ and $k = \pm\frac{2}{3}$ show up in the local equivalence problem for maximally non-integrable (or bracket generating) rank 2 distributions on 5-manifolds M that depend on a single function $F(x)$. Here the non-integrability condition implies that $F''(x) \neq 0$. For such non-integrable distributions, the bracket of the vector fields spanning the distribution $\mathcal{D}$ determines a filtration of the tangent bundle given by $\mathcal{D} \subset [\mathcal{D}, \mathcal{D}] \subset TM = [[\mathcal{D}, \mathcal{D}], \mathcal{D}]$ with the rank of $\mathcal{D} = 2$ and the rank of $[\mathcal{D}, \mathcal{D}] = 3$. Such distributions are therefore also known as $(2, 3, 5)$-distributions. Cartan solved the local equivalence problem for such geometries in [**5**] and constructed the fundamental curvature invariant. For distributions of the form $F(x)$ as described, the curvature invariant vanishes when $F''(x) = e^{\frac{1}{2}\int y(x)\mathrm{d}x}$ where $y(x)$ is a solution to the generalised Chazy equation (0.1) with parameter $k = \pm\frac{2}{3}$. In this case the distribution has split G_2 as its local group of symmetries. Furthermore An and Nurowski [**3**] showed that there is a duality that takes the solutions of this equation to the solutions of the 7th order ODE

$$10(y''')^3y^{(7)} - 70(y''')^2y^{(4)}y^{(6)} - 49(y''')^2(y^{(5)})^2 + 280y'''(y^{(4)})^2y^{(5)} - 175(y^{(4)})^4 = 0 \tag{0.2}$$

studied in [**19**] (where it appears in equation (6.64)), [**11**] and [**12**]. Historically, this ODE appeared already in the thesis of Noth [**17**] in 1904, who showed that this ODE admits the submaximal 10 dimensional group of contact symmetries on the plane.

This dual ODE (0.2) can also be reduced to a generalised Chazy equation but now the Chazy parameter is given by $k = \pm\frac{3}{2}$. In this fashion, the solutions with parameters $k = \frac{3}{2}$ and $k = \frac{2}{3}$ give rise to $(2, 3, 5)$-distributions with vanishing Cartan curvature invariants. We discuss the solutions for both these equations in Sections 4 and 6. We show in Appendix A that the Legendre duality property for the generalised Chazy equations with parameters $k = \pm\frac{3}{2}$ and $k = \pm\frac{2}{3}$ is unique only for these parameters.

Interestingly, one of the Schwarz triangle functions that solves the $k = 2$ equation also shows up in the solutions to the $k = \frac{2}{3}$ equation. Three of the Schwarz triangle functions that solve the $k = 3$ equation also show up in the solutions to the $k = \frac{3}{2}$ equation. For the $k = 2$ and $k = \frac{2}{3}$ cases, the Schwarz triangle functions are pullbacks through hypergeometric transformations of the Schwarz function

$s(\frac{1}{2}, \frac{1}{3}, \frac{1}{2}, x)$ that appears in Schwarz's list [**21**] with dihedral symmetry. For the $k = 3$ and $k = \frac{3}{2}$ cases, the functions are pullbacks of the function $s(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, x)$ that appears in Schwarz's list with tetrahedral symmetry. The result of Schwarz [**21**] and Klein tells us that the Schwarz functions we obtain in these cases are algebraic. The maps for the triangle functions that show up in the cases $k = \frac{2}{3}, \frac{3}{2}$, 2 and 3 are given in the Diagrams 1, 2 and 3. They show the hypergeometric transformations that are given by quadratic, cubic and quartic maps [**15**], [**22**].

Another motivation for the article is to work out the Schwarz functions that solve the $k = \frac{2}{3}$ and $k = \frac{3}{2}$ equations and determine examples of $(2, 3, 5)$-distributions with maximal symmetry group of split G_2. The solutions to the $k = \frac{2}{3}$ equation have appeared in [**20**], but we present them more explicitly here in the form of Table 2. We also present the solutions when $k = \frac{3}{2}$ here. To work out the distributions $D_{F(x)}$ with vanishing Cartan curvature invariant, we have to determine $F''(x)$ from the solutions of the $k = \frac{2}{3}$ Chazy equation and integrate twice further. This gives an algebraic relation involving (x, F). The dual curve of this plane algebraic curve gives us integral curves of equation (0.2).

The first two sections are background material. In Section 1 we set up the preliminaries and consider $SL_2(\mathbb{C})$ equivalent classes of solutions to (0.1). In Section 2 we review the definitions of Schwarz functions and in Sections 3, 4, 5, 6 we present the Schwarz triangle functions that appear in the solutions of the Chazy's equation for $k = 2$, $k = \frac{2}{3}$, $k = 3$ and $k = \frac{3}{2}$ respectively. The computations are done through MAPLE 17.

Contents

1. $SL_2(\mathbb{C})$ Action on the Space of Solutions

The material in the first two sections is collated from [**1**], [**2**], [**7**], [**9**] and [**10**]. We shall work over the complex field. Any element $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{C})$ acts on x by fractional linear transformations $g \cdot x = \tilde{x} = \frac{ax+b}{cx+d}$.

Proposition 1.1. *(See also* [**2**], [**9**]*) Under the action of $SL_2(\mathbb{C})$, for any solution $y(x)$ to the generalised Chazy equation (0.1) with $k \neq 6$, we obtain new solutions to (0.1) by*

$$\tilde{y}(x) = \frac{1}{(cx+d)^2} y\left(\frac{ax+b}{cx+d}\right) - \frac{6c}{cx+d}. \tag{1.1}$$

Proof. The action of SL_2 gives the differential relation $\mathrm{d}\tilde{x} = \frac{1}{(cx+d)^2}\mathrm{d}x$ and $\frac{\mathrm{d}}{\mathrm{d}\tilde{x}} = (cx+d)^2\frac{\mathrm{d}}{\mathrm{d}x}$. Differentiating equation (1.1), we find

$$(cx+d)^4(6\tilde{y}' - \tilde{y}^2) = 6y' - y^2$$

and

$$(cx+d)^6(9\tilde{y}'' - 9\tilde{y}\tilde{y}' + \tilde{y}^3) = 9y'' - 9yy' + y^3,$$

where prime on the left hand side denotes differentiation with respect to x while prime on the right hand side denotes differentiation with respect to $\tilde{x}$. Differentiating once more, we obtain

$$\begin{aligned}&(cx+d)^8\left(\tilde{y}''' - 2\tilde{y}''\tilde{y} + 3(\tilde{y}')^2 - \frac{4}{36-k^2}(6\tilde{y}' - \tilde{y}^2)^2\right)\\ &= y''' - 2y''y + 3(y')^2 - \frac{4}{36-k^2}(6y' - y^2)^2.\end{aligned}$$

We see that $\tilde{y}(x)$ is a solution to (0.1) if and only if $y(\tilde{x})$ is a solution as well. □

Let $f(x) = \exp(\frac{2}{k-6}\int y\mathrm{d}x)$. Chazy makes this substitution ([**7**], p. 321) and finds that f satisfies the 4th order differential equation

$$ff'''' - (k-2)f'f''' + \frac{3k(k-2)}{2(k+6)}(f'')^2 = 0. \tag{1.2}$$

It is immediate from (1.2) that when $k = 2$, the equation becomes linear, and we shall discuss this further in Section 3. When we integrate (1.1), we obtain

$$\begin{aligned}\int \tilde{y}(x)\mathrm{d}x &= \int \frac{1}{(cx+d)^2}y(\tilde{x})\mathrm{d}x - \int \frac{6c}{cx+d}\mathrm{d}x\\ &= \int y(\tilde{x})\mathrm{d}\tilde{x} - 6\log(cx+d) + c_0.\end{aligned}$$

We find that

$$\begin{aligned}\tilde{f}(x) &= \exp\left(\frac{2}{k-6}\int \tilde{y}(x)\mathrm{d}x\right)\\ &= \frac{\exp(\frac{2c_0}{k-6})}{(cx+d)^{\frac{12}{k-6}}}\exp\left(\frac{2}{k-6}\int y(\tilde{x})\mathrm{d}\tilde{x}\right) = \frac{\exp(\frac{2c_0}{k-6})}{(cx+d)^{\frac{12}{k-6}}}f(\tilde{x}).\end{aligned}$$

Absorbing constants (or normalising them so that $c_0 = 0$), we have

$$\tilde{f}(x) = \frac{1}{(cx+d)^{\frac{12}{k-6}}}f(\tilde{x}).$$

This motivates the following definition.

Definition 1.2. Suppose both functions $f(x)$ and $\tilde{f}(x) = (cx+d)^{-\frac{12}{k-6}}f(\tilde{x})$ satisfy the same differential equation (1.2). Then we say that the function $f(x)$ has **weight** $\frac{12}{k-6}$ since $f(\tilde{x}) = (cx+d)^{\frac{12}{k-6}}\tilde{f}(x)$ (following the convention in the literature about weights of modular forms).

Let us take $k = \frac{2}{3}$ and suppose $f(x) = (F''(x))^{-\frac{3}{4}}$ for some $F(x)$. Then $F''(x)$ has weight 3 under the action of $SL_2(\mathbb{C})$ and we find that $F(x)$ satisfies the 6th order ODE

$$\begin{aligned} 10F^{(6)}(F'')^3 - 80(F'')^2F^{(3)}F^{(5)} - 51(F'')^2(F^{(4)})^2 \\ + 336F''(F''')^2F^{(4)} - 224(F''')^4 = 0 \end{aligned} \tag{1.3}$$

in [**3**] upon substituting $f = (F'')^{-\frac{3}{4}}$ into (1.2).

Proposition 1.3. *If $F(x)$ is a solution to the 6th order ODE (1.3), then so is*

$$\tilde{F}(x) = (cx+d)F\left(\frac{ax+b}{cx+d}\right)$$

where $ad - bc = 1$.

According to the definition given above, the function $F(x)$ has weight -1. We have

Corollary 1.4. *The function $F(x) = x^2$ is a solution to the 6th order ODE (1.3), and therefore so is*

$$\tilde{F}(x) = (cx+d)\left(\frac{ax+b}{cx+d}\right)^2 = \frac{(ax+b)^2}{cx+d}$$

where $ad - bc = 1$.

Differentiating the above relation twice, we find that

$$\tilde{F}''(x) = \frac{1}{(cx+d)^3}F''(\tilde{x}).$$

Again the right hand side denotes differentiation with respect to $\tilde{x}$. For any $F(x)$ with weight -1 satisfying the 6th order ODE (1.3), we identify the solutions

$$\left(x, \tilde{F}(x)\right) = (x, (cx+d)F(g\cdot x)) \sim (g\cdot x, F(g\cdot x))\,.$$

We define $SL_2(\mathbb{C})$ equivalent solutions to the generalised Chazy's equation in the following fashion (see also [**9**]).

Definition 1.5. Two solutions $y(\tilde{x})$ and $\tilde{y}(x)$ to the generalised Chazy equation are said to be **equivalent** if there exists an element g of $SL_2(\mathbb{C})$ such that $\tilde{x} = g\cdot x$ and (1.1) holds for $y(g\cdot x)$ and $\tilde{y}(x)$.

From this we can identify the solutions

$$(x, \tilde{y}(x)) \sim \left(\tilde{x}, (cx+d)^2\tilde{y}(x) + 6c(cx+d)\right) = (\tilde{x}, y(\tilde{x})).$$

A direct calculation shows that $F(x) = x^m$ for $m \in \{-1, 0, \frac{1}{3}, \frac{2}{3}, 1, 2\}$ solves equation (1.3) (see [**5**]). We restrict to the values for $m \in \{-1, \frac{1}{3}, \frac{2}{3}, 2\}$ so that $F''(x) \neq 0$, and investigate how the solutions for $m \in \{-1, \frac{1}{3}, \frac{2}{3}, 2\}$ are equivalent solutions under $SL_2(\mathbb{C})$. Using that $y(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}\log(F''(x))$, this gives $y(x) = -\frac{6}{x}$, $y(x) = -\frac{10}{3x}$, $y(x) = -\frac{8}{3x}$ and $y(x) = 0$ as solutions to the $k = \frac{2}{3}$ equation. The solution $y = -\frac{8}{3(x+C)} - \frac{10}{3(x+B)}$ for constants B, C was further obtained following [**6**] and [**7**]. This corresponds to $F(x) = (x+B)^{\frac{1}{3}}(x+C)^{\frac{2}{3}}$.

Proposition 1.6. *The solutions to (0.1) for the parameter $k = \frac{2}{3}$ given by $y = -\frac{10}{3x}$, $y = -\frac{8}{3x}$ and $y = -\frac{8}{3(x+C)} - \frac{10}{3(x+B)}$ are equivalent in the sense of Definition 1.5.*

Proof. Applying an arbitrary $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C})$ to the solution given by $y = -\frac{8}{3x}$, we obtain $\tilde{y} = -\frac{8}{3(x+\frac{b}{a})} - \frac{10}{3(x+\frac{d}{c})}$. Applying $g_1 = \begin{pmatrix} -ec & \frac{f}{c(e-f)} \\ c & -\frac{1}{ec}(1+\frac{f}{e-f}) \end{pmatrix} \in SL_2(\mathbb{C})$ to $\tilde{y} = -\frac{8}{3(x+e)} - \frac{10}{3(x+f)}$ with $e \neq f$, we obtain $\tilde{\tilde{y}} = -\frac{10}{3x}$. To get back to $y = -\frac{8}{3x}$, we use the transformation given by $g_2 = \begin{pmatrix} 0 & b \\ -\frac{1}{b} & 0 \end{pmatrix} \in SL_2(\mathbb{C})$. □

As a consequence of Proposition 1.6, we see that the solutions given by $F(x) = (x+B)^{\frac{1}{3}}(x+C)^{\frac{2}{3}}$, $x^{\frac{2}{3}}$ and $x^{\frac{1}{3}}$ are equivalent to one another by this SL_2 action.

Proposition 1.7. *The solutions to (0.1) for the parameter $k = \frac{2}{3}$ given by $y = 0$ and $y = -\frac{6}{x}$ are equivalent in the sense of Definition 1.5.*

Proof. This is clear from applying $g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ to the zero solution and its inverse $g^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ to $y = -\frac{6}{x}$ to get the zero solution. □

2. Schwarz Functions and Equivalent Solutions under $SL_2(\mathbb{C})$

The solutions to the generalised Chazy equation (0.1) can be expressed in terms of logarithmic derivatives involving Schwarz triangle functions. This comes from the following observation (see [**1**] and [**2**]). The equation (0.1) can be written as a closed nonlinear system of first order autonomous differential equations, called the generalised Darboux-Halphen system. From the first order system, the equations can be transformed to a Schwarzian type equation with potential term $V(s)$. The solutions are then given precisely by Schwarz triangle functions. Let prime denote differentiation with respect to x.

Definition 2.1. A **Schwarz triangle function** $s(\alpha, \beta, \gamma, x)$ is a solution to the following third order non-linear differential equation

$$\{s, x\} + \frac{(s')^2}{2} V(s) = 0 \tag{2.1}$$

where

$$\{s, x\} = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{s''}{s'}\right) - \frac{1}{2}\left(\frac{s''}{s'}\right)^2$$

is the Schwarzian derivative and

$$V(s) = \frac{1-\beta^2}{s^2} + \frac{1-\gamma^2}{(s-1)^2} + \frac{\beta^2+\gamma^2-\alpha^2-1}{s(s-1)}$$

is the potential.

A Schwarz triangle function determines through its inverse a mapping from the complex upper half plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ to the interior of a spherical, planar or hyperbolic triangle Δ with angles between edges given by $(\alpha\pi, \beta\pi, \gamma\pi)$. The edges of the triangle are given by circular arcs. The inverse map $x : \mathbb{H} \to \Delta$ is single valued and meromorphic given by

$$x(s) = \frac{{}_2F_1(a-c+1, b-c+1; 2-c; s)}{{}_2F_1(a, b; c; s)} s^{1-c}. \tag{2.2}$$

The images of 0, ∞ and 1 under x are the vertices of the triangle with one vertex at the origin $x(0) = 0$ and the other vertex $x(1)$ connected to 0 by an edge that is real valued. The remaining vertex is $x(\infty)$. Here we have

$$a = \frac{1}{2}(1 - \alpha - \beta - \gamma),$$
$$b = \frac{1}{2}(1 + \alpha - \beta - \gamma),$$
$$c = 1 - \beta.$$

In (2.2), x is given by the quotient of linearly independent solutions to the hypergeometric differential equation

$$s(1 - s)z_{ss} + (c - (a + b + 1)s)z_s - abz = 0. \tag{2.3}$$

Here the subscript denotes differentiation with respect to s. The general solution to (2.3) is given by $\alpha z_1(s) + \beta z_2(s)$ where z_1, z_2 are linearly independent. Chazy finds the solutions to (0.1) in [**6**] and [**7**] by treating $x = \frac{z_2(s)}{z_1(s)}$ and taking $y = 6\frac{\mathrm{d}}{\mathrm{d}x}\log z_1(s)$.

We form the quotient $x = \frac{z_2}{z_1}$. If we take a different linear combination instead with

$$\tilde{x} = \frac{\beta z_1 - \delta z_2}{-\alpha z_1 + \gamma z_2} = \frac{\beta - \delta \frac{z_2}{z_1}}{-\alpha + \gamma \frac{z_2}{z_1}} = \frac{\beta - \delta x}{-\alpha + \gamma x},$$

then we find

$$x = \frac{\alpha \tilde{x} + \beta}{\gamma \tilde{x} + \delta}.$$

In other words, if we restrict to α, β, γ, δ such that $\alpha\delta - \beta\gamma = 1$, then $x = g \cdot \tilde{x}$ and $\tilde{x} = (g^{-1}) \cdot x$ for $g \in SL_2$. Hence SL_2 equivalent solutions to Chazy's equation are determined by the quotient $\frac{z_2}{z_1}$, and thus are completely determined by the Schwarz function s. Every distinct Schwarz function therefore gives rise to SL_2 equivalent solutions as in Definition 1.5. We will henceforth just consider the quotient $x = \frac{z_2}{z_1}$ in our computations, modulo constants that agree with the expression (2.2).

Our goal now is to present the various $(x(s), y(s))$, parametrised by the distinct Schwarz functions s that are found using the general method to solve Chazy's equation [**1**], [**2**]. Let us denote

$$\Omega_1 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log\frac{s'}{s(s-1)},$$
$$\Omega_2 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log\frac{s'}{s-1},$$
$$\Omega_3 = -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\log\frac{s'}{s}.$$

Each of these functions satisfies the generalised Darboux-Halphen system ([**1**], [**2**]). We have the following

Proposition 2.2. *The function $y = -2(\Omega_1 + \Omega_2 + \Omega_3)$ solves (0.1) when $(\alpha, \beta, \gamma) = (\frac{2}{k}, \frac{2}{k}, \frac{2}{k})$ or $(\frac{2}{k}, \frac{1}{3}, \frac{1}{3})$ and its cyclic permutations. In a similar way, we find that $y = -\Omega_1 - 2\Omega_2 - 3\Omega_3$ solves (0.1) when $(\alpha, \beta, \gamma) = (\frac{1}{k}, \frac{1}{3}, \frac{1}{2})$, $(\frac{1}{k}, \frac{2}{k}, \frac{1}{2})$ or $(\frac{1}{k}, \frac{1}{3}, \frac{3}{k})$. Also, $y = -4\Omega_1 - \Omega_2 - \Omega_3$ solves (0.1) when $(\alpha, \beta, \gamma) = (\frac{4}{k}, \frac{1}{k}, \frac{1}{k})$ or $(\frac{2}{3}, \frac{1}{k}, \frac{1}{k})$.*

Proof. For each of these combinations of y, we substitute it into equation (0.1) and using that s satisfies (2.1), we can solve for (α, β, γ) in terms of k for each of the combinations up to a permutation of the entries. □

Each of these values of (α, β, γ) determines the corresponding Schwarz triangle function and the values of (a, b, c) in (2.2). In the next four sections we compute the Schwarz triangle functions that arise in the cases where $k = 2$, $k = \frac{2}{3}$, $k = 3$ and $k = \frac{3}{2}$.

3. Generalised Chazy Equation with $k = 2$ and its Schwarz Functions

We first give the general solution to equation (0.1) for $k = 2$ and list the Schwarz functions that solve the equation in Table 1. The general solution to equation (0.1) for $k = 2$ has already been observed by Chazy in pages 346–347 of [**7**].

Theorem 3.1 ([**7**], p. 346-347)**.** *The general solution to the generalized Chazy equation with $k = 2$ over the Riemann surface $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is given by*

$$y(x) = -2\left(\frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3}\right)$$

and depends on 3 arbitrary (not necessarily distinct) points x_1, x_2, x_3 on the Riemann surface.

Proof. We make the following observation over the complex plane $\mathbb{C}$. Under the substitution $y = \frac{-2f'}{f}$ for f non-zero, the generalised Chazy equation with $k = 2$ is equivalent to the linear 4th order ODE $f'''' = 0$. The solution to $f'''' = 0$ is given by the cubic polynomial $f = ax^3 + 3bx^2 + 6cx + d$ and therefore the general solution to the $k = 2$ Chazy equation over $\mathbb{C}$ is given by its logarithmic derivative

$$y = -\frac{6ax^2 + 12bx + 12c}{ax^3 + 3bx^2 + 6cx + d}$$

where a, b, c, d are constants of integration. For $a \neq 0$, we can factorize $f = a(x - x_1)(x - x_2)(x - x_3)$ over $\mathbb{C}$, and we obtain $y = -2\left(\frac{1}{x-x_1} + \frac{1}{x-x_2} + \frac{1}{x-x_3}\right)$. The general solution for $a \neq 0$ therefore depends on three arbitrary points on $\mathbb{C}$. If we include the point at infinity, we allow solutions with $a = 0$ of the form $y = -2\left(\frac{1}{x-x_1} + \frac{1}{x-x_2}\right)$. □

From Proposition 2.2, when $k = 2$ the combination $y = -2(\Omega_1 + \Omega_2 + \Omega_3)$ that gives a solution to (0.1) occurs when $(\alpha, \beta, \gamma) = (1, 1, 1)$ or $(1, \frac{1}{3}, \frac{1}{3})$. The combination $y = -\Omega_1 - 2\Omega_2 - 3\Omega_3$ that gives a solution to (0.1) occurs when $(\alpha, \beta, \gamma) = (\frac{1}{2}, \frac{1}{3}, \frac{1}{2})$, $(\frac{1}{2}, 1, \frac{1}{2})$ or $(\frac{1}{2}, \frac{1}{3}, \frac{3}{2})$. The combination $y = -4\Omega_1 - \Omega_2 - \Omega_3$ that gives a solution to (0.1) occurs when $(\alpha, \beta, \gamma) = (2, \frac{1}{2}, \frac{1}{2})$ or $(\frac{2}{3}, \frac{1}{2}, \frac{1}{2})$.

For each of these values of (α, β, γ) we find $(a, b, c) = (\frac{1}{2}(1 - \alpha - \beta - \gamma), \frac{1}{2}(1 + \alpha + \beta - \gamma), 1 - \beta)$ and the corresponding parameterisation of x given by the ratio of linearly independent solutions to (2.3). Furthermore, for each of these values of (a, b, c), we can invert the map (2.2) given by

$$x = \frac{{}_2F_1(a - c + 1, b - c + 1; 2 - c; s)}{{}_2F_1(a, b; c; s)} s^{1-c}$$

to determine s as a function of x. We then use this function s to determine $y(x)$ through the above formulas.

We list the Schwarz triangle functions that show up in the solutions to the generalised Chazy equation with parameter $k=2$ in Table 1. We have the following

Theorem 3.2. *The first column in Table 1 below gives the values (α,β,γ) of the Schwarz triangle functions $s(\alpha,\beta,\gamma,t)$ that solve the generalised Chazy equation with parameter $k=2$. The second column gives the expression for $x(s)$ using (2.2) and the third column inverts to find $s(x)$. The fourth column gives $y(x)$ from the corresponding $s(x)$.*

(α,β,γ)	$x(s)$	$s(x)$	$y(x)$
$(1,1,1)$	$\frac{s}{1-s}$	$\frac{x}{x+1}$	$-\frac{2}{x}-\frac{2}{x+1}$
$(1,\frac{1}{3},\frac{1}{3})$	$\left(\frac{s}{1-s}\right)^{\frac{1}{3}}$	$\frac{x^3}{x^3+1}$	$-\frac{2}{x+1}-\frac{2}{x+w}-\frac{2}{x+w^2}$
$\left(\frac{1}{2},\frac{1}{3},\frac{1}{2}\right)$	$2^{\frac{2}{3}}\left(\frac{1-r}{1+r}\right)^{\frac{1}{3}}$ *and* $s=1-r^2$	$\frac{16x^3}{(x^3+4)^2}$	$-\frac{6x^2}{x^3+4}$
$\left(\frac{1}{2},1,\frac{1}{2}\right)$	$\frac{2}{\sqrt{1-s}}-2$	$1-\frac{4}{(x+2)^2}$	$-\frac{2}{x}-\frac{2}{x+2}-\frac{2}{x+4}$
$\left(\frac{1}{2},\frac{1}{3},\frac{3}{2}\right)$	$\frac{1}{2^{\frac{1}{3}}}\left(\frac{1-r}{1+r}\right)^{\frac{1}{3}}\left(\frac{3+r}{3-r}\right)$ *and* $s=1-r^2$	*Roots of the quartic polynomial* $\left(\frac{1}{2}-x^3\right)r^4+(8x^3+4)r^3$ $+(9-18x^3)r^2+27x^3-\frac{27}{2}=0$	$-\frac{3\cdot 2^{\frac{1}{3}}}{8}\left(\frac{r+1}{r-1}\right)^{\frac{1}{3}}\frac{r-3}{r^3}$ $\times(r-1)(r+3)^2$ $=-\frac{12x^2}{2x^3-1}$
$\left(2,\frac{1}{2},\frac{1}{2}\right)$	$\frac{\sqrt{s(1-s)}}{1-2s}$	$\frac{1}{2}(1\pm\frac{1}{\sqrt{4x^2+1}})$	$-\frac{2}{x}-\frac{16x}{4x^2+1}$
$\left(\frac{2}{3},\frac{1}{2},\frac{1}{2}\right)$	$\frac{3}{2}\frac{\sin(r)}{\cos(r)}$ *and* $s=\sin^2(\frac{3}{2}r)$	$\frac{1}{2}\left(1+\frac{9(4x^2-3)}{(4x^2+9)^{\frac{3}{2}}}\right)$	$-\frac{2}{x}-\frac{16x}{4x^2-27}$

TABLE 1. Schwarz functions for $k=2$

In the first column of Table 1, the values (α,β,γ) give the angles $(\alpha\pi,\beta\pi,\gamma\pi)$ of the spherical triangle. In the second column of Table 1 we find $x(s)$ using (2.2). The series expansion of $x(s)$ around a regular neighbourhood can be computed and can be checked to see if it agrees with entries in Table 1. In the third column, we invert the second column to present the Schwarz function, with a branch cut chosen for the functions in the last three rows. The entry in the third column of the fifth row requires the solution of a quartic polynomial determined by $x(r)$ on the second column. This same Schwarz function will show up again in the solutions to the $k=\frac{2}{3}$ equation. Finally in the last column we present $y(x)$ as determined by the formulas for $s(x)$ in the third column. Here $w=e^{\frac{2\pi i}{3}}$ that appears in the entry in the second row denotes the cube root of unity. The formulas that give y are discussed and presented in [**20**].

Up to fractional linear transformations, the identity map $x=s$ is given by $s(1,1,1,x)$. This function appears in the first row of Table 1. The upper half plane is mapped to the hemisphere of $\mathbb{S}^2$ bound by a great circle and the vertices of the triangle are three points lying on the great circle. Geometrically speaking then the spherical triangles make sense only when $(\alpha+\beta+\gamma)\pi \le 3\pi$. The occurrence of angles adding up to greater than 3π requires us to think of "triangles" overlapping onto itself (or a branched cover or folded triangle) to realise such exaggeratedly large angles.

The function appearing in the third row of Table 1 given by $s(\frac{1}{2},\frac{1}{3},\frac{1}{2},x)$ appears in Schwarz's list ([**21**] p. 323) and has dihedral symmetry.

We shall explain how the entry in the 5th row of the last column is obtained. The entry finds the solution y of equation (0.1) with parameter $k=2$ as a function of x when x is the inversion of the Schwarz triangle function $s(\frac{1}{2},\frac{1}{3},\frac{3}{2},x)$.

Proposition 3.3. *When $s = s(\frac{1}{2},\frac{1}{3},\frac{3}{2},x)$, the solution to (0.1) with $k=2$ is given by*

$$y = \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{(s')^3}{s^2(s-1)^{\frac{3}{2}}} = -\frac{12x^2}{2x^3-1}.$$

Proof. From the parametrisation of x given by

$$x = \frac{1}{2^{\frac{1}{3}}}\left(\frac{1-r}{1+r}\right)^{\frac{1}{3}}\left(\frac{3+r}{3-r}\right)$$

with $s = 1-r^2$, we obtain

$$\mathrm{d}x = -\frac{8}{3}\frac{4^{\frac{1}{3}}r^2}{(r-3)^2(r-1)^{\frac{2}{3}}(r+1)^{\frac{4}{3}}}\mathrm{d}r = \frac{1}{r'}\mathrm{d}r.$$

The corresponding formula for y that solves (0.1) then gives

$$\begin{aligned} y = -\Omega_1 - 2\Omega_2 - 3\Omega_3 &= \frac{\mathrm{d}}{\mathrm{d}x}\log\frac{(s')^3}{s^2(s-1)^{\frac{3}{2}}} = r'\frac{\mathrm{d}}{\mathrm{d}r}\log\frac{(-2r'r)^3}{(1-r^2)^2(-r^2)^{\frac{3}{2}}} \\ &= -\frac{3\cdot 2^{\frac{1}{3}}}{8}\frac{1}{r^3}(r-3)(r+3)^2(r-1)^{\frac{2}{3}}(r+1)^{\frac{1}{3}} \\ &= -\frac{3}{4}\frac{1}{2^{\frac{2}{3}}}\left(\frac{1-r}{1+r}\right)^{\frac{2}{3}}\left(\frac{3+r}{3-r}\right)^2\frac{(r-3)^3(r+1)}{r^3} \\ &= -\frac{3}{4}x^2\frac{(r-3)^3(r+1)}{r^3}. \end{aligned}$$

From the formula for x, we find

$$\frac{1}{2x^3-1} = \frac{(r-3)^3(r+1)}{16r^3}$$

and therefore

$$y = -\frac{12x^2}{2x^3-1}.$$

□

The maps between the Schwarz functions for the $k=2$ case are presented in Figure 1. We use the notation adopted in [**22**], where the number over the arrow denotes the degree of the algebraic transformation. These transformations are classified in [**22**].

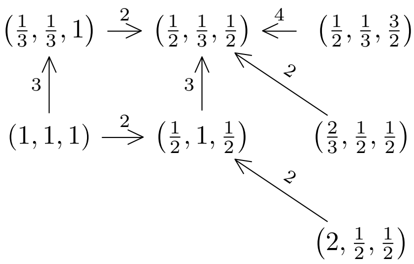

FIGURE 1. Mapping of Schwarz functions for $k = 2$

4. Generalised Chazy Equation with $k = \frac{2}{3}$ and its Schwarz Functions

In this section we determine the Schwarz functions for the values (α, β, γ) that show up in the solutions to the $k = \frac{2}{3}$ equation. When $k = \frac{2}{3}$, the combination $y = -2(\Omega_1 + \Omega_2 + \Omega_3)$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (3, 3, 3)$ or $(\frac{1}{3}, \frac{1}{3}, 3)$. The combination $y = -\Omega_1 - 2\Omega_2 - 3\Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{3}{2}, \frac{1}{3}, \frac{1}{2})$, $(\frac{3}{2}, 3, \frac{1}{2})$ or $(\frac{3}{2}, \frac{1}{3}, \frac{9}{2})$. The combination $y = -4\Omega_1 - \Omega_2 - \Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (6, \frac{3}{2}, \frac{3}{2})$ or $(\frac{2}{3}, \frac{3}{2}, \frac{3}{2})$.

For each of these Schwarz triangle functions, we present the flat or symmetric $(2, 3, 5)$-distribution that it determines by computing the anti-derivative

$$F(x) = \int \int e^{\frac{1}{2} \int y(x) \mathrm{d}x} \mathrm{d}x \mathrm{d}x.$$

The values for $F(x)$ are presented in the 3rd column of Table 2. In the table we take the constants of integration appearing in each integral of $F(x)$ to be 0.

(α, β, γ)	$x(s)$ or $x(r)$	$F(x(s))$ or $F(x(r))$
$(3, 3, 3)$	$\frac{1}{2} \frac{s-2}{2s-1} s^3$	$s + \frac{1}{2(2s-1)}$
$(\frac{1}{3}, \frac{1}{3}, 3)$	$\frac{1}{2} \left(\frac{s+2}{2s+1}\right) s^{\frac{1}{3}}$	$\frac{s^{\frac{2}{3}}}{2s+1}$
$(\frac{3}{2}, \frac{1}{3}, \frac{1}{2})$	$-\frac{1}{2^{\frac{1}{3}}} \left(\frac{1-r}{1+r}\right)^{\frac{1}{3}} \left(\frac{1+3r}{1-3r}\right)$ and $s = 1 - r^2$	$\frac{(r+1)^{\frac{1}{3}} (r-1)^{\frac{2}{3}}}{3r-1}$
$(\frac{3}{2}, 3, \frac{1}{2})$	$-2 \frac{s^2+4s-8}{\sqrt{1-s}} - 16$	$\frac{s-2}{\sqrt{s-1}}$
$(\frac{3}{2}, \frac{1}{3}, \frac{9}{2})$	$-2^{-\frac{4}{3}} \left(\frac{r+3}{r-3}\right) \left(\frac{1-r}{1+r}\right)^{\frac{1}{3}} \frac{3r^4-8r^3+54r^2-81}{3r^4+8r^3+54r^2-81}$ and $s = 1 - r^2$	$\left(\frac{r+1}{r-1}\right)^{\frac{1}{3}} \frac{(r+3)(r^2-9)(r-1)}{3r^4+8r^3+54r^2-81}$
$(6, \frac{3}{2}, \frac{3}{2})$	$\frac{(s(1-s))^{\frac{3}{2}} (2s-1)}{128s^4-256s^3+144s^2-16s-1}$	$\frac{1}{128s^4-256s^3+144s^2-16s-1}$
$(\frac{2}{3}, \frac{3}{2}, \frac{3}{2})$	$-\frac{81}{64} \frac{\sin(8r)-2\sin(4r)}{\cos(8r)+2\cos(4r)}$ and $s = \sin^2(3r)$	$\frac{1}{\cos(8r)+2\cos(4r)}$

TABLE 2. Parametrisations for $k = \frac{2}{3}$

We state the theorems concerning Table 2 as below.

Theorem 4.1. *The first column in Table 2 gives the values* (α, β, γ) *of the Schwarz triangle functions that solve the generalised Chazy equation with parameter* $k = \frac{2}{3}$*. The second column gives the expression for* $x(s)$ *using (2.2) and the third column finds* $F(x(s))$*.*

Inverting the function for s in terms of x requires the solution of a quartic equation in s except for two cases in the second and third last rows of Table 2, where the degrees of the polynomials involved are larger. We present $F(x)$ instead of the solutions $y(x)$ because the results take precedence in the theory of $(2, 3, 5)$-distributions.

There is a Legendre duality discovered in [**3**] that takes equation (1.3) to (0.2) given by the following:

$$(x, F) \mapsto (t, H) = (F', xF' - F).$$

We find that H as a function of t solves equation (0.2)

$$10(H'')^3 H^{(6)} - 70(H'')^2 H^{(3)} H^{(5)} - 49(H'')^2 (H^{(4)})^2 + 280 H''(H^{(3)})^2 H^{(4)} - 175(H^{(3)})^4 = 0 \quad (4.1)$$

whenever equation (1.3) holds. The prime in equation (4.1) refers to differentiation with respect to t. Equation (1.3) can be reduced to the generalised Chazy equation with $k = \frac{2}{3}$ while equation (4.1) can be reduced to the generalised Chazy equation with $k = \frac{3}{2}$. Since the entries in the last column of Table 2 give solutions to equation (1.3), we obtain

Theorem 4.2. *Each row of Table 2 determines a* $(2, 3, 5)$*-distribution* $D_{F(x)}$ *with split* G_2 *symmetry.*

To give an example of the Legendre transform, we consider $(x(s), F(x(s)))$ parametrised by $s(3, 3, 3, x)$ given in the first row of Table 2. Eliminating the parameter s between x and F gives an algebraic curve $P_1(x, F) = 0$ which is an integral curve of a solution to equation (1.3). For the family of curves parametrised by

$$(x, F) = \left(\frac{s^3(s-2)}{2(2s-1)}, c\left(s + \frac{1}{2(2s-1)} \right) \right)$$

where c is constant, we find that

$$P_1(x, F) = 3F^4 + 2c(8x-1)F^3 - c^2 \left(24x + \frac{15}{2} \right) F^2 - 9c^3 \left(4x + \frac{1}{2} \right) F - c^4 \left(64x^2 + 10x + \frac{13}{16} \right) = 0.$$

The Legendre dual $(x, F) \mapsto (t, H)$ is found by determining $t = F'$ and $H = xF' - F$. This gives

$$(t, H) = \left(\frac{4c}{3} \left(\frac{1}{s-1} - \frac{1}{s} \right), -c\left(\frac{1}{6} + \frac{2}{3}s + \frac{2}{3(s-1)} \right) \right).$$

The parameter s can again be eliminated to give the integral curves of the solutions to equation (4.1).

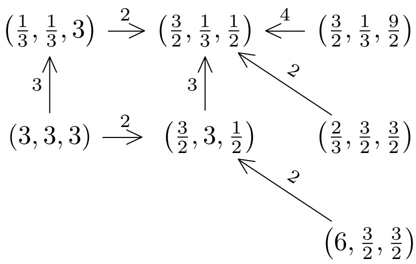

FIGURE 2. Mapping of Schwarz functions for $k = \frac{2}{3}$

We also see that the Schwarz triangle function $s(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}, x)$ appears in both the solutions to the $k = 2$ generalised Chazy equation and the $k = \frac{2}{3}$ equation. This gives an intriguing relationship between the solutions to (0.1) for $k = \frac{2}{3}$ and $k = 2$ determined by this Schwarz function.

Proposition 4.3. *Let $s = s(\frac{3}{2}, \frac{1}{3}, \frac{1}{2}, x)$. We find*

$$y_2 = \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{(s')^3}{(s-1)^{\frac{5}{2}} s^2}$$

is a solution to (0.1) with $k = 2$ while

$$y_{\frac{2}{3}} = \frac{\mathrm{d}}{\mathrm{d}x} \log \frac{(s')^3}{(s-1)^{\frac{3}{2}} s^2} = y_2 + \frac{\mathrm{d}}{\mathrm{d}x} \log(s-1)$$

is a solution to (0.1) with $k = \frac{2}{3}$.

The Schwarz functions for the $k = \frac{2}{3}$ equation are determined by the following pull-back maps in Figure 2, with the same notation from [**22**].

The domain of the spherical triangle corresponding to the Schwarz function $s(\frac{1}{2}, \frac{1}{3}, \frac{3}{2}, x)$ has angles $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{3}{2}\pi)$. This triangle is the complement of the triangle with angles $(\frac{1}{2}\pi, \frac{2}{3}\pi, \frac{1}{2}\pi)$ in a hemisphere with the edge of the hemisphere lying along the $\frac{1}{2}\pi$ and $\frac{2}{3}\pi$ edge of the triangle. Reflecting along the $\frac{3}{2}\pi$, $\frac{1}{2}\pi$ edge gives us the "triangle" with angles $(\frac{1}{3}\pi, \frac{1}{3}\pi, 3\pi)$ branched over the vertex with angle 3π. A reflection instead along the equatorial $\frac{1}{3}\pi$, $\frac{1}{2}\pi$ edge gives the triangle with angles $(\frac{2}{3}\pi, \frac{3}{2}\pi, \frac{3}{2}\pi)$, which is also the complement of the $(\frac{4}{3}\pi, \frac{1}{2}\pi, \frac{1}{2}\pi)$ triangle in the sphere.

5. Generalised Chazy Equation with $k = 3$ and its Schwarz Functions

In this section we determine the Schwarz triangle functions that solve equation (0.1) with $k = 3$. We let t denote now the independent variable. This is to distinguish the independent variable when we compute the Legendre dual curves later on.

Theorem 5.1. *The general solution to the generalised Chazy equation with $k = 3$ over the Riemann surface $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ is given by*

$$y(t) = -\frac{3}{2}\left(\frac{1}{t-t_1} + \frac{1}{t-t_2} + \frac{1}{t-t_3} + \frac{1}{t-t_4}\right) \tag{5.1}$$

where the 4 *points* t_1, t_2, t_3 t_4 *on the Riemann surface are subject to the constraint* $\mathcal{Q}$ *that* $12ae - 3bd + c^2 = 0$ *where*

$$a(t - t_1)(t - t_2)(t - t_3)(t - t_4) = at^4 + bt^3 + ct^2 + dt + e.$$

Proof. Under the substitution $y = \frac{-3f'}{2f}$ for f non-zero, the generalised Chazy equation with $k = 3$ is equivalent to the nonlinear 4th order ODE $2f''''f - 2f'''f' + (f'')^2 = 0$. Differentiating this ODE gives the linear ODE $f''''' = 0$ whose solutions are given by $f = at^4 + bt^3 + ct^2 + dt + e$. Substituting this back into the 4th order ODE, the coefficients are subject to the constraint $12ae - 3bd + c^2 = 0$ and therefore the general solution to the $k = 3$ Chazy equation over $\mathbb{C}$ is given by (5.1). □

This general solution has been mentioned in [**7**] and [**10**]. For this parameter, Chazy ([**7**] p. 347) makes the observation that assuming the four roots are distinct, the roots t_1, t_2, t_3 and t_4 can be chosen to lie on the *sommets* of the *tétraèdre régulier*. The condition $\mathcal{Q}$ defines a projective variety in $\mathbb{P}^3$. When one of the points is $t_4 = \infty$, the condition on the remaining three roots to be a solution is that they must lie on the conic $\mathcal{C} = \{t_1, t_2, t_3 \in \mathbb{C} | t_1^2 + t_2^2 + t_3^2 - t_1t_2 - t_1t_3 - t_2t_3 = 0\}$. In this situation,

$$y = -2\left(\frac{1}{t - t_1} + \frac{1}{t - t_2} + \frac{1}{t - t_3}\right)$$

satisfies the $k = 2$ equation while

$$y = -\frac{3}{2}\left(\frac{1}{t - t_1} + \frac{1}{t - t_2} + \frac{1}{t - t_3}\right)$$

satisfies the $k = 3$ equation. We have

Proposition 5.2. *The inclusion of varieties*

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{Q} \\ \cap & & \cap \\ \mathbb{P}^2 & \hookrightarrow & \mathbb{P}^3 \end{array}$$

corresponds to the following isomorphism

$$\left\{ \begin{array}{l} \textit{Solutions to } k = 3 \textit{ equation} \\ \textit{with one pole at } \{\infty\} \end{array} \right\} \cong \left\{ \begin{array}{l} \textit{Solutions to } k = 2 \textit{ equation} \\ \textit{with poles in } \mathcal{C} \end{array} \right\}$$

where the isomorphism is given by a constant rescaling.

When $k = 3$, the combination $y = -2(\Omega_1 + \Omega_2 + \Omega_3)$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ or $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. The combination $y = -\Omega_1 - 2\Omega_2 - 3\Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{2})$, $(\frac{1}{3}, \frac{2}{3}, \frac{1}{2})$ or $(\frac{1}{3}, \frac{1}{3}, 1)$. The combination $y = -4\Omega_1 - \Omega_2 - \Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$ or $(\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$. We present the Schwarz functions when $k = 3$ in Table 3. Observe that $s(1, \frac{1}{3}, \frac{1}{3}, t)$ appears both in the $k = 2$ and $k = 3$ equations. The Schwarz function $s(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, t)$ appears twice here, giving the two formulas for y with

$$y = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(s')^3}{s^2(s-1)^2} \text{ and } y = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(s')^3}{s^{\frac{5}{2}}(s-1)^{\frac{5}{2}}}$$

both satisfying the $k = 3$ Chazy equation. The functions here that appear in Schwarz's list are $s(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, t)$ and $s(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, t)$, both of which possess tetrahedral symmetry. Again $w = e^{\frac{2\pi i}{3}}$ denotes the cube root of unity.

We have the following

Theorem 5.3. *The first column in Table 3 below gives the values (α, β, γ) of the Schwarz triangle functions that solve the generalised Chazy equation with parameter $k = 3$. The second column gives the expression for $t(s)$ using (2.2) and the third column inverts to find $s(t)$. The fourth column finds $y(t)$ from the corresponding $s(t)$.*

(α,β,γ)	$t(s)$	$s(t)$	$y(t)$
$(\frac{2}{3},\frac{2}{3},\frac{2}{3})$	$\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};\frac{5}{3};s)}{{}_2F_1(\frac{1}{6},-\frac{1}{2};\frac{1}{3};s)}s^{\frac{2}{3}}$	$I(-2^{-\frac{2}{3}}w^2t) = \frac{s(t)^2}{4(s(t)-1)}$ *where* $I(t) = s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$	*Same as* $y(t)$ *for* $s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$
$(\frac{2}{3},\frac{1}{3},\frac{1}{3})$	$\frac{{}_2F_1(\frac{1}{6},\frac{5}{6};\frac{4}{3};s)}{{}_2F_1(-\frac{1}{6},\frac{1}{2};\frac{2}{3};s)}s^{\frac{1}{3}}$	$\frac{\sqrt{2}}{32}\frac{(\sqrt{t^3+2}+3\sqrt{2})^3(\sqrt{t^3+2}-\sqrt{2})}{(\sqrt{t^3+2})^3}$	$-\frac{9}{2}\frac{t^2}{t^3+2}$ *and* $-\frac{6(t^3-4)}{t(t^3-16)}$
$(\frac{1}{3},\frac{1}{3},\frac{1}{2})$	$\frac{{}_2F_1(\frac{7}{12},\frac{1}{4};\frac{4}{3};s)}{{}_2F_1(-\frac{1}{12},\frac{1}{4};\frac{2}{3};s)}s^{\frac{1}{3}}$	$-\frac{t^3(t^3-64)^3}{512(t^3+8)^3}$	$-\frac{9}{2}\frac{t^2}{t^3+8}$
$(\frac{2}{3},\frac{1}{3},\frac{1}{2})$	$\frac{{}_2F_1(\frac{1}{12},\frac{3}{4};\frac{4}{3};s)}{{}_2F_1(\frac{5}{12},-\frac{1}{4};\frac{2}{3};s)}s^{\frac{1}{3}}$	$\frac{s}{(4s-1)^3} = \frac{1}{512}\frac{t^3(27t^3+64)^3}{(27t^3-8)^3}$	$-\frac{6}{t}\left(\frac{27t^3+16}{27t^3+64}\right)$
$(\frac{1}{3},\frac{1}{3},1)$	$s^{\frac{1}{3}}$	t^3	$-\frac{9}{2}\frac{t^2}{t^3-1}$
$(\frac{4}{3},\frac{1}{3},\frac{1}{3})$	$\frac{{}_2F_1(\frac{7}{6},-\frac{1}{6};\frac{4}{3};s)}{{}_2F_1(-\frac{1}{2},\frac{5}{6};\frac{2}{3};s)}s^{\frac{1}{3}}$	$I(2^{\frac{2}{3}}t) = 4s(t)(1-s(t))$ *where* $I(t) = s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$	*Same as* $y(t)$ *for* $s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$

TABLE 3. Parametrisations for $k = 3$

Let us see how some of the entries in Table 3 are derived. Consider for example the entry for $s(t)$ in the 3rd column when s is the Schwarz triangle function $s(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, t)$.

Proposition 5.4. *When $s = s(\frac{2}{3}, \frac{1}{3}, \frac{1}{2}, t)$, we have*

$$\frac{s}{(4s-1)^3} = \frac{1}{512}\frac{t^3(27t^3+64)^3}{(27t^3-8)^3} \tag{5.2}$$

as written in the fourth row of Table 3.

Proof. This is obtained from considering the pull back map $(\frac{1}{3}, \frac{1}{3}, \frac{1}{2}) \overset{3}{\leftarrow} (\frac{2}{3}, \frac{1}{3}, \frac{1}{2})$ in Figure 3. This corresponds to the cubic transformation given in formula (121) of [**15**]. For $\alpha = -\frac{1}{12}$, we have

$${}_2F_1\left(\frac{5}{12}, -\frac{1}{4}; \frac{2}{3}; s\right) = (1-4s)^{\frac{1}{4}}\,{}_2F_1\left(-\frac{1}{12}, \frac{1}{4}; \frac{2}{3}; \frac{27s}{(4s-1)^3}\right)$$

while for $\alpha = \frac{1}{4}$, we have

$${}_2F_1\left(\frac{1}{12}, \frac{3}{4}; \frac{4}{3}; s\right) = (1-4s)^{-\frac{3}{4}}\,{}_2F_1\left(\frac{7}{12}, \frac{1}{4}; \frac{4}{3}; \frac{27s}{(4s-1)^3}\right).$$

Together this gives

$$\frac{{}_2F_1(\frac{1}{12},\frac{3}{4};\frac{4}{3};s)}{{}_2F_1(\frac{5}{12},-\frac{1}{4};\frac{2}{3};s)}s^{\frac{1}{3}}=-\frac{1}{3}\left(\frac{27s}{(4s-1)^3}\right)^{\frac{1}{3}}\frac{{}_2F_1(\frac{7}{12},\frac{1}{4};\frac{4}{3};\frac{27s}{(4s-1)^3})}{{}_2F_1(-\frac{1}{12},\frac{1}{4};\frac{2}{3};\frac{27s}{(4s-1)^3})}.$$

The formula for $s(\frac{1}{3},\frac{1}{3},\frac{1}{2},t)$ in the third row of Table 3 gives

$$\frac{27s}{(4s-1)^3}=-\frac{(-3t)^3}{512}\frac{((-3t)^3-64)^3}{((-3t)^3+8)^3},$$

which simplifies to

$$\frac{s}{(4s-1)^3}=\frac{1}{512}\frac{t^3(27t^3+64)^3}{(27t^3-8)^3}.$$

□

The Schwarz triangle function $s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$ gives the following expression for $y(t)$.

Proposition 5.5. *When $s=s\left(\frac{2}{3},\frac{1}{3},\frac{1}{2},t\right)$, the solution to (0.1) with $k=3$ is given by*

$$y=\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{(s')^3}{s^{\frac{5}{2}}(s-1)^{\frac{3}{2}}}=-\frac{6}{t}\left(\frac{27t^3+16}{27t^3+64}\right).$$

Proof. From (5.2), we obtain

$$\frac{s^{\frac{1}{3}}}{4s-1}=\frac{1}{8}\frac{t(27t^3+64)}{(27t^3-8)}.$$

Let Q denote the right hand side term of the above expression. We have

$$-\frac{1}{3}\frac{8s+1}{s^{\frac{2}{3}}(4s-1)^2}\mathrm{d}s=-\frac{1}{3}\frac{8s+1}{s(4s-1)}Q\mathrm{d}s=Q'\mathrm{d}t,$$

or alternatively,

$$s'=-3\frac{s(4s-1)}{8s+1}\frac{Q'}{Q}.$$

The formula for y gives

$$\begin{aligned}y=&\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{(s')^3}{s^{\frac{5}{2}}(s-1)^{\frac{3}{2}}}\\=&\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{\left(-3\frac{s(4s-1)}{8s+1}\frac{Q'}{Q}\right)^3}{s^{\frac{5}{2}}(s-1)^{\frac{3}{2}}}\\=&-3\frac{Q'}{Q}\frac{s(4s-1)}{8s+1}\frac{\mathrm{d}}{\mathrm{d}s}\log\frac{\left(-3\frac{s(4s-1)}{8s+1}\right)^3}{s^{\frac{5}{2}}(s-1)^{\frac{3}{2}}}+3\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{Q'}{Q}\\=&\frac{3}{2}\frac{Q'}{Q}\frac{64s^3-48s^2+66s-1}{(8s+1)^2(s-1)}+3\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{Q'}{Q}.\end{aligned}$$

Therefore we have

$$\frac{2Q}{3Q'}\left(y-3\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{Q'}{Q}\right)=\frac{64s^3-48s^2+66s-1}{(8s+1)^2(s-1)}. \tag{5.3}$$

Using the fact that $\frac{s}{(4s-1)^3}=\frac{1}{512}\frac{t^3(27t^3+64)^3}{(27t^3-8)^3}$ from Proposition 5.4, we eliminate s appearing in both this equation and (5.3) to obtain

$$y=-\frac{6}{t}\left(\frac{27t^3+16}{27t^3+64}\right).$$

□

If we make the substitution $t=\frac{-4\sigma^{\frac{1}{3}}}{\sigma+2}$ and $s=-\frac{1}{64}\frac{\sigma(\sigma+8)^3}{(1-\sigma)^3}$ into (5.2), we get an identity. This alternative parametrisation comes from the mapping $(\frac{2}{3},\frac{1}{3},\frac{1}{2})\overset{4}{\leftarrow}(\frac{2}{3},\frac{1}{3},2)$ in Figure 3 and σ agrees with $s(\frac{2}{3},\frac{1}{3},2,t)$ up to a constant reparametrisation of t. From this, we find that (t,y_3) given by

$$\begin{aligned}
t &= \frac{\sigma^{\frac{1}{3}}}{\sigma+2}\\
y_3 &= -\frac{3}{2}\frac{(\sigma+2)(\sigma^3+6\sigma^2-96\sigma+8)}{\sigma^{\frac{1}{3}}(\sigma+8)(\sigma-1)^2}\\
&= -\frac{3}{2t}\left(1-\frac{8}{\sigma-1}-\frac{9}{(\sigma-1)^2}+\frac{8}{\sigma+8}\right)=-\frac{3}{2t}\left(\frac{1-108t^3}{1-27t^3}\right)
\end{aligned}$$

solves the $k=3$ equation while $(t,y_{\frac{3}{2}})$ given by

$$\begin{aligned}
t &= \frac{\sigma^{\frac{1}{3}}}{\sigma+2}\\
y_{\frac{3}{2}} &= -\frac{9}{4}\frac{(\sigma+2)(\sigma-10)\sigma^{\frac{2}{3}}}{(\sigma-1)^2}\\
&= -\frac{9}{4t}\left(1-\frac{8}{\sigma-1}-\frac{9}{(\sigma-1)^2}\right)=\frac{3}{2}y_3+\frac{9}{4t}\frac{8}{\sigma+8}
\end{aligned}$$

solves the $k=\frac{3}{2}$ equation. The latter solution agrees with the solution given in Theorem 6.6, which means that we have reparametrised the solution for $y(t)$ given by $s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$ to those given in the subsequent Theorem 6.6.

For brevity we denote $I(t)=s(\frac{2}{3},\frac{1}{3},\frac{1}{2},t)$, $J(t)=s(\frac{4}{3},\frac{1}{3},\frac{1}{3},t)$ and $K(t)=s(\frac{2}{3},\frac{2}{3},\frac{2}{3},t)$. Using quadratic transformations (similar to those given in the next section), we find that $I(2^{\frac{2}{3}}t)=4J(t)(1-J(t))$ and $I(-2^{-\frac{2}{3}}w^2t)=\frac{K(t)^2}{4(K(t)-1)}$ where $w=e^{\frac{2}{3}\pi i}$.

Proposition 5.6. *The functions I, J, K determine the same solution to the generalised Chazy equation for $k=3$.*

Proof. We have $I(2^{\frac{2}{3}}t)=4J(t)(1-J(t))$ and $I(-2^{-\frac{2}{3}}w^2t)=\frac{K(t)^2}{4(K(t)-1)}$. A computation shows that

$$\frac{(I')^3}{I^{\frac{5}{2}}(I-1)^{\frac{3}{2}}}=c_1\frac{(J')^3}{J^{\frac{5}{2}}(J-1)^{\frac{3}{2}}}=c_2\frac{(K')^3}{K^2(K-1)^2}$$

for some possible complex constants c_1 and c_2. By choosing the same logarithmic branch, the logarithmic derivatives of these three functions give the same solution to the $k=3$ Chazy equation. □

6. Generalised Chazy Equation with $k = \frac{3}{2}$ and its Schwarz Functions

In this section we present the Schwarz triangle functions that show up in the solutions to the $k = \frac{3}{2}$ equation. When $k = \frac{3}{2}$, the combination $y = -2(\Omega_1+\Omega_2+\Omega_3)$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{4}{3}, \frac{4}{3}, \frac{4}{3})$ or $(\frac{4}{3}, \frac{1}{3}, \frac{1}{3})$. The combination $y = -\Omega_1 - 2\Omega_2 - 3\Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{2}{3}, \frac{1}{3}, \frac{1}{2})$, $(\frac{2}{3}, \frac{4}{3}, \frac{1}{2})$ or $(\frac{2}{3}, \frac{1}{3}, 2)$. The combination $y = -4\Omega_1 - \Omega_2 - \Omega_3$ gives a solution to (0.1) when $(\alpha, \beta, \gamma) = (\frac{8}{3}, \frac{2}{3}, \frac{2}{3})$ or $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$.

The Schwarz triangle functions that appear can be grouped into those that have already shown up in the the solutions to the $k = 3$ equation and those that have not. The additional ones are presented in Table 4. The Schwarz triangle functions I, J and K corresponding to the spherical triangles with angles $(\frac{2}{3}\pi, \frac{1}{3}\pi, \frac{1}{2}\pi)$, $(\frac{4}{3}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi)$ and $(\frac{2}{3}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi)$ respectively have already appeared in the solutions to the $k = 3$ generalised Chazy equation and they solve the $k = \frac{3}{2}$ equation (see the previous section) by deforming the $k = 3$ solutions along some function of K (or I, J).

Proposition 6.1. *Let $K(t) = s(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, t)$. We have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(K')^3}{K^{\frac{5}{2}}(K-1)}, \quad \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(K')^3}{K(K-1)^{\frac{5}{2}}} \text{ and } \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(K')^3}{K^{\frac{5}{2}}(K-1)^{\frac{5}{2}}}$$

all satisfying the generalised Chazy equation with $k = \frac{3}{2}$.

Proof. This comes from considering the different combinations $y = -4\Omega_1 - \Omega_2 - \Omega_3$, $y = -\Omega_1 - 4\Omega_2 - \Omega_3$ and $y = -\Omega_1 - \Omega_2 - 4\Omega_3$, discussed in [**20**]. In all three cases the function K remains the same since its triangular domain has the property of being equilateral. □

Let us denote $y_3 = \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{(K')^3}{K^2(K-1)^2}$. This is the solution to the $k = 3$ equation in the previous section.

Corollary 6.2. *The functions $y_3 + \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{K-1}{\sqrt{K}}$, $y_3 + \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{K}{\sqrt{K-1}}$ and $y_3 + \frac{\mathrm{d}}{\mathrm{d}t} \log \frac{1}{\sqrt{K(K-1)}}$ are all solutions to the generalised Chazy equation with parameter $k = \frac{3}{2}$.*

In the above expressions, K can be substituted for I or J as well.

The remaining Schwarz functions that have not shown up in the $k = 3$ case are given by $s(\frac{4}{3}, \frac{2}{3}, \frac{1}{2}, t)$, $s(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, t)$, $s(\frac{8}{3}, \frac{2}{3}, \frac{2}{3}, t)$ and $s(\frac{2}{3}, \frac{1}{3}, 2, t)$. We discuss $s(\frac{2}{3}, \frac{1}{3}, 2, t)$ in Theorem 6.6. Again for brevity let us denote $L(t) = s(\frac{4}{3}, \frac{2}{3}, \frac{1}{2}, t)$, $M(t) = s(\frac{4}{3}, \frac{4}{3}, \frac{4}{3}, t)$ and $N(t) = s(\frac{8}{3}, \frac{2}{3}, \frac{2}{3}, t)$. The relationship between L, M and N is presented in Table 4. We determine the relationship between L, M and N in the following fashion, using only quadratic transformations between hypergeometric functions. The inversion formula for L gives

$$t(L) = \frac{{}_2F_1(-\frac{1}{12}, \frac{5}{4}; \frac{5}{3}; L)}{{}_2F_1(-\frac{3}{4}, \frac{7}{12}; \frac{1}{3}; L)} L^{\frac{2}{3}}.$$

We find

$$\frac{{}_2F_1(-\frac{5}{6}, \frac{11}{6}; \frac{5}{3}; s)}{{}_2F_1(-\frac{3}{2}, \frac{7}{6}; \frac{1}{3}; s)} s^{\frac{2}{3}} = 2^{-\frac{4}{3}} \frac{{}_2F_1(-\frac{1}{12}, \frac{5}{4}; \frac{5}{3}; 4s(1-s))}{{}_2F_1(-\frac{3}{4}, \frac{7}{12}; \frac{1}{3}; 4s(1-s))} (4s(1-s))^{\frac{2}{3}}.$$

The left hand side is the inversion formula for $N(t)$, while the right hand side is $2^{-\frac{4}{3}}t(4N(1-N))$ and so

$$L(2^{\frac{4}{3}}t) = 4N(t)(1-N(t)).$$

Similarly, we find

$$\frac{{}_2F_1(-\frac{1}{6},\frac{7}{6};\frac{7}{3};s)}{{}_2F_1(-\frac{1}{6},-\frac{3}{2};-\frac{1}{3};s)}s^{\frac{4}{3}} = 2^{\frac{4}{3}}w^2\frac{{}_2F_1(-\frac{1}{12},\frac{5}{4};\frac{5}{3};\frac{s^2}{4(s-1)})}{{}_2F_1(-\frac{3}{4},\frac{7}{12};\frac{1}{3};\frac{s^2}{4(s-1)})}\left(\frac{s^2}{4(s-1)}\right)^{\frac{2}{3}},$$

where $w = e^{\frac{2}{3}\pi i}$. The left hand side is the inversion formula for $M(t)$, while the right hand side is $2^{\frac{4}{3}}w^2t(\frac{M^2}{4(M-1)})$ and so

$$L(2^{-\frac{4}{3}}wt) = \frac{M(t)^2}{4(M(t)-1)}.$$

We have the following

Theorem 6.3. *Along with the Schwarz triangle functions $I(t)$, $J(t)$ and $K(t)$ with values $(\alpha,\beta,\gamma) = (\frac{2}{3},\frac{1}{3},\frac{1}{2})$, $(\frac{4}{3},\frac{1}{3},\frac{1}{3})$ and $(\frac{2}{3},\frac{2}{3},\frac{2}{3})$ respectively, the first column in Table 4 below gives the values (α,β,γ) of the Schwarz triangle functions that solve the generalised Chazy equation with parameter $k=\frac{3}{2}$. The second column gives the expression for $t(s)$ using (2.2) and the third column relates their inverses $s(t)$.*

(α,β,γ)	$t(s)$	$s(t)$
$(\frac{4}{3},\frac{4}{3},\frac{4}{3})$	$\frac{{}_2F_1(-\frac{1}{6},\frac{7}{6};\frac{7}{3};s)}{{}_2F_1(-\frac{1}{6},-\frac{3}{2};-\frac{1}{3};s)}s^{\frac{4}{3}}$	$L(2^{-\frac{4}{3}}wt) = \frac{M(t)^2}{4(M(t)-1)}$
$(\frac{4}{3},\frac{2}{3},\frac{1}{2})$	$\frac{{}_2F_1(-\frac{1}{12},\frac{5}{4};\frac{5}{3};s)}{{}_2F_1(-\frac{3}{4},\frac{7}{12};\frac{1}{3};s)}s^{\frac{2}{3}}$	$L(t)$
$(\frac{8}{3},\frac{2}{3},\frac{2}{3})$	$\frac{{}_2F_1(-\frac{5}{6},\frac{11}{6};\frac{5}{3};s)}{{}_2F_1(-\frac{3}{2},\frac{7}{6};\frac{1}{3};s)}s^{\frac{2}{3}}$	$L(2^{\frac{4}{3}}t) = 4N(t)(1-N(t))$
$(\frac{2}{3},\frac{1}{3},2)$	$\frac{2s^{\frac{1}{3}}}{s+2}$	*Roots of the cubic polynomial* $s^3+6s^2+(12-\frac{8}{t^3})s+8=0$

TABLE 4. Parametrisations for $k=\frac{3}{2}$

We find that each of the Schwarz triangle functions $L(t)$, $M(t)$ and $N(t)$ gives the same solution to (0.1) with $k=\frac{3}{2}$.

Proposition 6.4. *The functions L, M, N determine the same solution to the generalised Chazy equation with $k=\frac{3}{2}$.*

Proof. Similar to Proposition 5.6, a computation shows that

$$\frac{(L')^3}{L^{\frac{5}{2}}(L-1)^{\frac{3}{2}}} = c_3\frac{(M')^3}{M^2(M-1)^2} = c_4\frac{(N')^3}{N^{\frac{5}{2}}(N-1)^{\frac{5}{2}}}$$

again for some possible complex constants c_3 and c_4. By taking the same logarithmic branch, the logarithmic derivatives of these three functions give the same solution to the $k=\frac{3}{2}$ Chazy equation. □

We are now left with the problem of determining what

$$y(t) = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{(L')^3}{L^{\frac{5}{2}}(L-1)^{\frac{3}{2}}}$$

is. We shall show that this can be reparametrised to give the solution (6.2) below.

Proposition 6.5. *Let* $\tau = s(\frac{1}{3}, \frac{2}{3}, 2, \frac{9t}{16})$ *and* $L = s(\frac{4}{3}, \frac{2}{3}, \frac{1}{2}, t)$. *We have*

$$\frac{L^{\frac{1}{3}}}{4L-1} = \frac{4}{3}\left(\frac{1-\tau}{1+8\tau}\right)\tau^{\frac{1}{3}}. \tag{6.1}$$

Proof. We first apply the cubic transformation $(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}) \overset{3}{\leftarrow} (\frac{4}{3}, \frac{2}{3}, \frac{1}{2})$. Let $u = \frac{27L}{(1+8L)^2(1-L)}$. We find $\frac{u}{u-1} = \frac{27L}{(4L-1)^3}$. The formulas (121) of [**15**] give

$${}_2F_1\left(-\frac{3}{4}, \frac{7}{12}; \frac{1}{3}; L\right) = (1-4L)^{\frac{3}{4}}{}_2F_1\left(-\frac{1}{4}, \frac{1}{12}; \frac{1}{3}; \frac{27L}{(4L-1)^3}\right)$$

for $\alpha = -\frac{1}{4}$ and

$${}_2F_1\left(-\frac{1}{12}, \frac{5}{4}; \frac{5}{3}; L\right) = (1-4L)^{-\frac{5}{4}}{}_2F_1\left(\frac{5}{12}, \frac{3}{4}; \frac{5}{3}; \frac{27L}{(4L-1)^3}\right)$$

for $\alpha = \frac{5}{12}$. Together they give the inversion formula for L

$$t = \frac{{}_2F_1(-\frac{1}{12}, \frac{5}{4}; \frac{5}{3}; L)}{{}_2F_1(-\frac{3}{4}, \frac{7}{12}; \frac{1}{3}; L)}L^{\frac{2}{3}} = \frac{1}{9}\frac{{}_2F_1(\frac{5}{12}, \frac{3}{4}; \frac{5}{3}; \frac{u}{u-1})}{{}_2F_1(-\frac{1}{4}, \frac{1}{12}; \frac{1}{3}; \frac{u}{u-1})}\left(\frac{u}{1-u}\right)^{\frac{2}{3}}.$$

Now we apply the degree 4 transformation $(\frac{1}{3}, \frac{2}{3}, \frac{1}{2}) \overset{4}{\leftarrow} (\frac{1}{3}, \frac{2}{3}, 2)$. Let $v = \frac{64\tau(1-\tau)^3}{(1+8\tau)^3}$. We find using equation (127) of [**15**] that the same parameter $\alpha = -\frac{1}{4}$ gives

$${}_2F_1\left(-\frac{1}{4}, \frac{1}{12}; \frac{1}{3}; v\right) = (1+8\tau)^{-\frac{3}{4}}{}_2F_1\left(-1, -\frac{2}{3}; \frac{1}{3}; \tau\right)$$

and $\alpha = \frac{5}{12}$ gives

$${}_2F_1\left(\frac{3}{4}, \frac{5}{12}; \frac{5}{3}; v\right) = (1+8\tau)^{\frac{5}{4}}{}_2F_1\left(\frac{5}{3}, 2; \frac{5}{3}; \tau\right).$$

We therefore obtain

$$\begin{aligned}\frac{{}_2F_1(\frac{3}{4}, \frac{5}{12}; \frac{5}{3}; v)}{{}_2F_1(-\frac{1}{4}, \frac{1}{12}; \frac{1}{3}; v)}v^{\frac{2}{3}} &= \frac{{}_2F_1(\frac{5}{3}, 2; \frac{5}{3}; \tau)(1+8\tau)^{\frac{5}{4}}}{{}_2F_1(-1, -\frac{2}{3}; \frac{1}{3}; \tau)(1+8\tau)^{-\frac{3}{4}}}\left(\frac{4\tau^{\frac{1}{3}}(1-\tau)}{1+8\tau}\right)^2 \\ &= \frac{16\tau^{\frac{2}{3}}}{1+2\tau}.\end{aligned}$$

Equating $v = \frac{u}{u-1}$, we obtain

$$\frac{64\tau(1-\tau)^3}{(1+8\tau)^3} = \frac{27L}{(4L-1)^3}$$

and we find that the parameters are related by

$$\frac{9t}{16} = \frac{\tau^{\frac{2}{3}}}{(1+2\tau)}$$

where the right hand side is the inverse $\tau = s(\frac{1}{3}, \frac{2}{3}, 2, \frac{9t}{16})$. □

We shall now use $L(t)$ to deduce the solution $y(t)$ to the generalised Chazy equation with parameter $k = \frac{3}{2}$. Let $R(\tau(t)) = \frac{4}{3}\frac{1-\tau}{1+8\tau}\tau^{\frac{1}{3}}$ denote the right hand side of (6.1). From the above Proposition 6.5 we have $t = \frac{16\tau^{\frac{2}{3}}}{9(1+2\tau)}$ and

$$L' = -3\frac{L(4L-1)}{8L+1}\frac{R'}{R}.$$

Analogous to Proposition 5.5, we find that

$$\frac{2R}{3R'}\left(y - 3\frac{\mathrm{d}}{\mathrm{d}t}\log\frac{R'}{R}\right) = \frac{64L^3 - 48L^2 + 66L - 1}{(8L+1)^2(L-1)}.$$

Eliminating L between this equation and (6.1) gives us the solution

$$(t, y) = \left(\frac{16\tau^{\frac{2}{3}}}{9(1+2\tau)}, \frac{81}{64}\frac{(1+2\tau)(10\tau-1)}{\tau^{\frac{2}{3}}(\tau-1)^2}\right). \tag{6.2}$$

Eliminating the parameter τ gives us an integral curve to the solution of (0.1) with $k = \frac{3}{2}$. Using the formula $H = \int\int e^{\frac{2}{3}\int y\mathrm{d}t}\mathrm{d}t\mathrm{d}t$, we further obtain

$$(t, H) = \left(\frac{16\tau^{\frac{2}{3}}}{9(1+2\tau)}, \frac{1024}{81}\frac{\tau^{\frac{1}{3}}}{1+2\tau}\right).$$

The corresponding dual curve found by $(t, H) \mapsto (x, F) = (H', tH' - H)$ is parametrised by

$$(x, F) = \left(\frac{32(4\tau-1)}{9\tau^{\frac{1}{3}}(\tau-1)}, \frac{512}{81}\frac{\tau^{\frac{1}{3}}}{\tau-1}\right).$$

We now determine the solution (t, H) to (4.1) when it is parametrised by the Schwarz function $s(\frac{2}{3}, \frac{1}{3}, 2, t)$.

Theorem 6.6. *For the Schwarz function $s(\frac{2}{3}, \frac{1}{3}, 2, t)$, the parametrisation for t is found to be given by*

$$t = \frac{2s^{\frac{1}{3}}}{(s+2)}$$

while the formula for $H(t(s))$ is found to be given by

$$H(t(s)) = -\frac{4s^{\frac{2}{3}}}{s+2}.$$

Proof. For the values of $(\alpha, \beta, \gamma) = (\frac{2}{3}, \frac{1}{3}, 2)$, we find $(a, b, c) = (-1, -\frac{1}{3}, \frac{2}{3})$. The general solution to $u_{ss} + \frac{1}{4}V(s)u = 0$ with these corresponding values is given by $u = \beta s^{\frac{1}{3}} + \alpha(s+2)$. Let $u_1 = (s+2)$ and $u_2 = 2s^{\frac{1}{3}}$. Then

$$t = \frac{2s^{\frac{1}{3}}}{(s+2)} = \frac{s^{\frac{1}{3}}}{{}_2F_1(-1, -\frac{1}{3}; \frac{2}{3}; s)}$$

agrees with the formula given by (2.2) and we find that

$$y = \frac{\mathrm{d}}{\mathrm{d}t}\log\frac{(s')^3}{s^2(s-1)^{\frac{3}{2}}} = -\frac{9}{8}\frac{(s+2)(s-10)}{(s-1)^2}s^{\frac{2}{3}}.$$

Also $s(t)$ is given by the root of the cubic equation

$$s^3+6s^2+\left(12-\frac{8}{t^3}\right)s+8=0.$$

We find that

$$\mathrm{d}t=-\frac{4}{3}\frac{s-1}{(s+2)^2s^{\frac{2}{3}}}\mathrm{d}s$$

and therefore

$$H=\int\int e^{\frac{2}{3}\int y\mathrm{d}t}\mathrm{d}t\mathrm{d}t=-\frac{4s^{\frac{2}{3}}}{s+2}.$$

□

Eliminating the variable s in $(t(s),y(t(s)))$, we obtain an algebraic curve $C(t,y)$ given by

$$\begin{aligned}C(t,y)&=t^2(3t-2)^2(9t^2+6t+4)^2y^3+18t(3t-2)(27t^3-2)(9t^2+6t+4)y^2\\&+324(243t^6-45t^3+1)y+1458t^2(108t^3-5)=0,\end{aligned}$$

which gives an integral curve of the solution to (0.1) with $k=\frac{3}{2}$. We can also eliminate s in $(t(s),H(t(s)))$ to get

$$H^3-16t^3-8Ht=0.$$

This gives an integral curve of the solution to Noth's equation (4.1). Using the formula $(x,F)=(H',tH'-H)$, we find that the dual curve is given by

$$(x,F)=\left(-\frac{s-4}{s-1}s^{\frac{1}{3}},\frac{2s^{\frac{2}{3}}}{s-1}\right).$$

To summarise the results of Section 6, we have the following

Theorem 6.7. *The functions I, J K give rise to the curve (t,H) parametrised (up to constants) by*

$$(t,H)=\left(\frac{2s^{\frac{1}{3}}}{s+2},-\frac{4s^{\frac{2}{3}}}{s+2}\right)\tag{6.3}$$

where $s=s(\frac{2}{3},\frac{1}{3},2,t)$ and the dual curve

$$(x,F)=\left(-\frac{s-4}{s-1}s^{\frac{1}{3}},\frac{2s^{\frac{2}{3}}}{s-1}\right)$$

determines a flat $(2,3,5)$-distribution $\mathcal{D}_{F(x)}$. The functions L, M N give rise to the curve (t,H) parametrised (up to constants) by

$$(t,H)=\left(\frac{2s^{\frac{2}{3}}}{1+2s},-\frac{4s^{\frac{1}{3}}}{1+2s}\right)$$

where this is obtained by inverting $s\mapsto\frac{1}{s}$ in (6.3). The dual curve given by

$$(x,F)=\left(-\frac{4s-1}{s^{\frac{1}{3}}(s-1)},\frac{2s^{\frac{1}{3}}}{1-s}\right)$$

also determines a flat $(2, 3, 5)$*-distribution* $\mathcal{D}_{F(x)}$.

Figure 3 shows the transformation maps between the Schwarz triangle functions that show up in the $k = \frac{3}{2}$ and $k = 3$ solutions. The Schwarz functions $s(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, x)$, $s(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, x)$ and $s(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}, x)$ appear in the solutions to the $k = 3$ and $k = \frac{3}{2}$ equations. The three diagrams 1, 2 and 3 can be combined at the nodes labelled by $(\frac{1}{3}, \frac{1}{3}, 1)$ and $(\frac{3}{2}, \frac{1}{3}, \frac{1}{2})$.

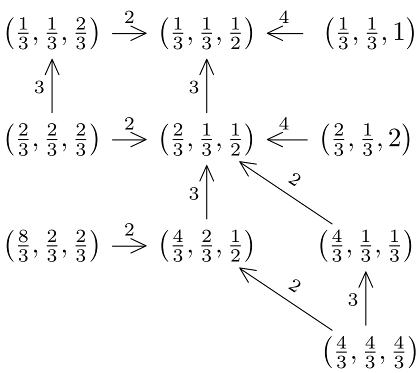

FIGURE 3. Mapping of Schwarz functions for $k = \frac{3}{2}$ and $k = 3$

We end by discussing the shapes of the spherical triangles that show up in the $k = \frac{3}{2}$ and $k = 3$ cases. The spherical triangle with angles $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi)$ corresponding to $s(\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, x)$ is given by the following. Divide the hemisphere equally into three lunes, with each end having angle $\frac{\pi}{3}$. The domain for the $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi)$ triangle is the complement of the fundamental domain of the $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{1}{3}\pi)$ triangle with tetrahedral symmetry in this lune. Eight of these triangles tile the whole sphere. The triangle with angles $(\frac{2}{3}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi)$ is generated by reflecting this domain along the long edge meeting the right angle, while the triangle with angles $(\frac{1}{3}\pi, \frac{1}{3}\pi, \frac{4}{3}\pi)$ is generated by reflection along the short edge meeting the right angle.

When $k = \frac{3}{2}$, the triangle with angles $(\frac{1}{2}\pi, \frac{2}{3}\pi, \frac{4}{3}\pi)$ also show up. This is the complement of the $(\frac{1}{2}\pi, \frac{1}{3}\pi, \frac{2}{3}\pi)$ triangle in the hemisphere with the edge between the $\frac{1}{2}\pi$ and $\frac{1}{3}\pi$ angles lying along the equator. The other triangle with angles $(\frac{4}{3}\pi, \frac{4}{3}\pi, \frac{4}{3}\pi)$ is generated by reflecting along the long edge meeting the right angle (or equivalently, the equator). This is also the complement of the equilateral triangle with angles $(\frac{2}{3}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi)$ in the whole sphere. Reflecting along the short edge meeting the right angle gives the "triangle" with angles $(\frac{8}{3}\pi, \frac{2}{3}\pi, \frac{2}{3}\pi)$, which overlaps at the vertex with angle $\frac{8}{3}\pi$.

Appendix A. Duality of Generalised Chazy Equations

There is a Legendre duality [**3**] that takes the generalised Chazy equation with parameter $k = \frac{2}{3}$ to its dual with parameter $k = \frac{3}{2}$. This is explained in [**3**] and [**20**]. We show that the generalised Chazy equation that is Legendre dual to another generalised Chazy equation has parameter k given by either $\pm\frac{2}{3}$ or $\pm\frac{3}{2}$.

Proposition A.1. *Let $m = \frac{4}{36-k^2}$ in equation (0.1). Any generalised Chazy equation can be put into the form*

$$f^3 f'''' - 2(\ell+2)f^2 f' f''' + 3(-12\ell m + \ell - 1)f^2(f'')^2 + 12(\ell^2 m + 6\ell m + 1)f(f')^2 f'' - (\ell^3 m + 12\ell^2 m + 36\ell m + \ell + 6)(f'')^4 = 0 \tag{A.1}$$

using the substitution $y = \ell \frac{f'}{f}$ for ℓ non-zero.

To pass to the dual equation, we use the substitution $f = \frac{1}{h}$ and $\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{h}\frac{\mathrm{d}}{\mathrm{d}t}$ to determine f, f', f'', f''' and f'''' in terms of h and its derivatives with respect to t and we obtain

$$\begin{aligned} h^3 h'''' &+ (2\ell - 11)h^2 h' h''' + (36\ell m - 3\ell - 7)h^2(h'')^2 \\ &+ (12\ell^2 m - 144\ell m - 2\ell + 59)h(h')^2 h'' \\ &+ (\ell^3 m - 24\ell^2 m + 144\ell m + 4\ell - 48)(h'')^4 = 0. \end{aligned}$$

Hence any equation of the form

$$\begin{aligned} h^3 h'''' &+ (2j - 11)h^2 h' h''' + (36jn - 3j - 7)h^2(h'')^2 \\ &+ (12j^2 n - 144jn - 2j + 59)h(h')^2 h'' \\ &+ (j^3 n - 24j^2 n + 144jn + 4j - 48)(h'')^4 = 0 \end{aligned}$$

is a Chazy equation only if the coefficients agree with that in equation (A.1). In this case, n and m is determined completely and is given by either $\frac{16}{135}$ or $\frac{9}{80}$. This gives $k = \pm\frac{2}{3}$ or $k = \pm\frac{3}{2}$.

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